On the Taylor expansion of λ-terms and the groupoid structure of their rigid approximants

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We show that the normal form of the Taylor expansion of a λ-term is isomorphic to its Böhm tree, improving Ehrhard and Regnier's original proof along three independent directions.

First, we simplify the final step of the proof by following the left reduction strategy directly in the resource calculus, avoiding to introduce an abstract machine *ad-hoc*.

We also introduce a groupoid of permutations of copies of arguments in a rigid variant of the resource calculus, and relate the coefficients of Taylor expansion with this structure, while Ehrhard and Regnier worked with groups of permutations of occurrences of variables.

Finally, we extend all the results to a non-deterministic setting: by contrast with previous attempts, we show that the uniformity property that was crucial in Ehrhard and Regnier's approach can be preserved in this setting.

1 Introduction

The field of quantitative semantics is currently very lively within the linear logic community and beyond. This momentum originates in the introduction by Ehrhard [1] of models where λ-terms could be interpreted by analytic maps, leading to the differential extensions of λ-calculus [3] and linear logic [5] by Ehrhard and Regnier. The keystone of this line of work is an analogue of the Taylor expansion formula, which allows to translate terms (or proofs) into infinite linear combinations of finite approximants [6]: in the case of λ-calculus, those approximants are the terms of a resource calculus, in which the copies of arguments of a function must be provided explicitly, and then consumed linearly, instead of duplicated or discarded during reduction.

By contrast with denotational semantics *à la* Scott, these approximants thus retain a dynamics, although a very simple, finitary one: the size of terms is strictly decreasing under reduction. In particular, Ehrhard and Regnier have shown that the Taylor expansion $\tau(M)$ of a λ-term M can always be normalized, and that its normal form is isomorphic to the Böhm tree $BT(M)$ of M [6, 4]: in particular, this normal form defines a proper denotational semantics.

The proof of Ehrhard and Regnier can be summed up as follows:

Step 1: The non-zero coefficients of resource terms in $\tau(M)$ do not depend on M. More precisely, we can write $\tau(M) = \sum_{s \in T(M)} \frac{1}{m(s)} s$, where $T(M)$ is the support set of Taylor expansion and $m(s)$ is an integer coefficient depending only on the resource term s.

Step 2: The set $T(M)$ is a clique, setting $s \frown s'$ if s and s' have the same syntactic tree, differing only by the multiplicity of arguments.

Step 3: Writing $NF(s)$ for the normal form of s, which is a finite sum of resource terms, the respective supports of $NF(s)$ and $NF(s')$ are disjoint cliques whenever $s \frown s'$ and $s \neq s'$. Then one can set $NF(\tau(M)) = \sum_{s \in T(M)} \frac{1}{m(s)} NF(s)$, the summands being pairwise disjoint.

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Step 4: If s is uniform, *i.e.* $s \frown s$, and t is in the support of $NF(s)$ then $NF(s)_t = \frac{m(s)}{m(t)}$

Step 5: By Step 1, $\tau(BT(M)) = \sum_{t \in T(BT(M))} \frac{1}{m(t)} t$. To conclude it is thus sufficient to prove that $t \in T(BT(M))$ iff there exists $s \in T(M)$ such that t is in the support of $NF(s)$.

The first two steps are easy consequences of the definitions. For Step 3, it is sufficient to observe that elementary resource reduction steps preserve coherence: this depends crucially on uniformity (Step 2). Step 4 relies on a careful investigation of the combinatorics of substitution in the resource calculus: this involves an elaborate argument about the structure of particular subgroups of the group of permutations of variable occurrences [6]. Finally, Ehrhard and Regnier establish Step 5 by relating Taylor expansion with execution in an abstract machine [4].

In the present work, we propose to revisit this seminal result, along three directions.

(i) We largely simplify Step 5, relying on a technique introduced by the second author [9]. We consider the parallel outmost reduction strategy (a slight variant of leftmost reduction, underlying the construction of Böhm trees) and show that it can be simulated directly in the resource calculus, through of Taylor expansion. We thus avoid the intricacies of an abstract machine with resource state.

(ii) We extend all the results to a model of non-determinism, introduced as a formal binary choice operator in the calculus. By contrast with previous proposals from Ehrhard [2], or Pagani, Tasson and Vaux Auclair [7, 9], we show that uniformity can still be relied upon, provided one keeps track of choices in the resource calculus: the coherence associated with non-deterministic choice is then that of the *with* connective (&) of linear logic.

(iii) We analyse coefficients in the Taylor expansion by introducing a groupoid of permutations on a rigid variant of resource terms, *i.e.* resource terms where multisets of arguments are replaced with lists, and the reduction is deterministic in the sense that it does not produce sums. This is more in accordance with the intuition that $m(s)$ is the number of permutations of arguments that leave s (or rather, any rigid representation of s) invariant: Ehrhard and Regnier rather worked on permutations of variable occurrences, which allowed to consider groups rather than a groupoid. Apart from this change of focus, the proof technique for Step 4 is essentially the same, but we believe our presentation gives a better understanding of the underlying combinatorics of substitution in resource terms.

Those three contributions are completely independent from each other.

It is natural to compare our proposals to the line of work of Tsukada, Asada and Ong [8]. Indeed, the resulting rigid calculus is very similar to theirs, but the viewpoints differ: whereas Tsukada, Asada and Ong thrive to develop an abstract understanding of reduction paths in non-deterministic λ-calculus, it follows from our results that Ehrhard and Regnier's technique can already be adapted to such non-determinism without introducing any new concept.

Also, our groupoid structure on rigid terms is reminiscent of morphisms in the interpretation of types in their generalized species of structures. In future work, we plan to investigate a possible refinement of our approach into a pure, untyped variant of their model: this will likely require us to study the interaction between our groupoid structure and permutations of free variable occurrences, which can be seen as a group action on contexts. We expect to obtain an explicit presentation of a reflexive object in the model of Tsukada, Asada and Ong.

2 Taylor expansion in a non-deterministic λ calculus

2.1 $\lambda_\oplus$-terms

We introduce a non-deterministic version of λ-calculus in a pure setting:[1]

$$\Lambda_\oplus \ni M, N, \ldots ::= x \mid \lambda x.M \mid MN \mid M \oplus N$$

As usual $\lambda_\oplus$-terms are considered up to renaming of bound variables. We write $M[N/x]$ for the capture avoiding substitution of N for x in M.

The reduction of $\lambda_\oplus$ is then the contextual extension of the following base cases:

$$(\lambda x.M)N \to M[N/x] \qquad \text{and} \qquad (M \oplus N)P \to MP \oplus NP.$$

We will moreover rely on the parallel left reduction strategy defined as follows:

$$L(P \oplus Q) = L(P) \oplus L(Q) \qquad L(\lambda x_1 \ldots \lambda x_l.xQ_1 \cdots Q_k) = \lambda x_1 \ldots \lambda x_l.xL(Q_1) \cdots L(Q_k)$$
$$L(\lambda x_1 \ldots \lambda x_l.(\lambda x.P)QQ_1 \cdots Q_k) = \lambda x_1 \ldots \lambda x_l.(P[Q/x])Q_1 \cdots Q_k$$
$$L(\lambda x_1 \ldots \lambda x_l.(P \oplus S)QQ_1 \cdots Q_k) = \lambda x_1 \ldots \lambda x_l.(PQ \oplus SQ)Q_1 \cdots Q_k$$

2.2 Resource terms

We define the set of *resource terms* $\Delta_\oplus$ and the set of *resource monomials* $\Delta^!_\oplus$ by mutual induction as follows:

$$\Delta_\oplus \ni s, t, u ::= x \mid \lambda x.s \mid \langle s \rangle \bar{t} \mid s \oplus \bullet \mid \bullet \oplus s \qquad \Delta^!_\oplus \ni \bar{s}, \bar{t}, \bar{u} ::= [] \mid [s] \cdot \bar{t}$$

We write $[s_1, \ldots, s_n]$ for $[s_1] \cdot \ldots \cdot [s_n] \cdot []$. Monomials are then considered up to permutations and resource terms up to renaming of bound variables. We call *resource expressions* the elements of $\Delta^{(!)}_\oplus = \Delta_\oplus \cup \Delta^!_\oplus$. For any resource expression e, we write $n_x(e)$ for the number of occurrences of variable x in e. If A is a set, we write $\mathbb{N}[A]$ for the set of finite formal sums of elements of A, or equivalently the set of finite linear combinations of elements of A with coefficients in $\mathbb{N}$. We extend the syntactical constructs of the resource calculus to finite sums of resource expressions by linearity: e.g., $[s+t] \cdot \bar{u} = s \cdot \bar{u} + t \cdot \bar{u}$.

Definition 2.1. *Let $e \in \Delta^{(!)}_\oplus$, $\bar{u} = [u_1, \ldots, u_n] \in \Delta^!_\oplus$ and $x \in \mathcal{V}$. We define the* n-linear substitution *$\partial_x e \cdot \bar{u}$ of $\bar{u}$ for x in e as follows:*

$$\partial_x e \cdot \bar{u} = \begin{cases} \sum_{\sigma \in \mathfrak{S}_n} e[u_{\sigma(1)}/x_1, \ldots, u_{\sigma(n)}/x_n] & \text{if } n_x(e) = n \\ 0 & \text{otherwise} \end{cases}$$

where $x_1, \ldots, x_{n_x(e)}$ enumerate the occurrences of x in e.

The reduction of resource terms is defined contextually from the following base cases:

$$\langle \lambda x.s \rangle \bar{t} \to_\partial \partial_x s \cdot \bar{t} \quad \langle s \oplus \bullet \rangle \bar{t} \to_\partial \langle s \rangle \bar{t} \oplus \bullet \quad \langle \bullet \oplus s \rangle \bar{t} \to_\partial \bullet \oplus \langle s \rangle \bar{t}.$$

As for the original resource calculus [6], the reduction relation $\to_\partial$ is confluent and strongly normalizing. We write $NF(s)$ for the unique normal form of s that is a formal sum of resource terms. We can extend the notion of left-parallel reduction to resource terms in the natural way. We write $L_\partial(e)$ for the left-parallel reduct of $e \in \Delta^{(!)}_\oplus$.

[1] Throughout the paper, we use a self explanatory if not standard variant of BNF notation for introducing syntactic objects: here we define the set $\Lambda_\oplus$ as that inductively generated by variables, λ-abstraction, application and sum, and we will denote terms using letters among M, N, P, Q, possibly with sub- and superscripts.

2.3 Taylor expansion of $\lambda_\oplus$-terms

If A is a set, we write $\mathbb{Q}^+\langle A\rangle$ for the set of possibly infinite linear combinations of elements of A with positive rational coefficients (in fact we could use any commutative semifield): equivalently, $\mathbb{Q}^+\langle A\rangle$ is the set of functions from A to $\mathbb{Q}^+$. We write $\alpha = \sum_{a\in A}\alpha_a.a \in \mathbb{Q}^+\langle A\rangle$. All the syntactic constructs we have introduced on resource expressions can be extended by linear-continuity [9].

Let $\sigma \in \mathbb{Q}^+\langle \Delta_\oplus\rangle$. We define σ^n by induction on n: if $n=0$ then $\sigma^n = []$; if $n = m+1$ then $\sigma^n = [\sigma]\cdot\sigma^m$. Then we define the *promotion* of σ as the series $\sigma^! = \sum_{n=0}^{\infty}\frac{1}{n!}\sigma^n$: because the supports of σ^n and σ^p are disjoint when $n \neq p$, this sum involves finite coefficients only. Let $\vec{a} = (a_1,\dots,a_n)$ be any finite sequence and $\sigma \in \mathfrak{S}_n$, we write $\vec{a}[\sigma] = (a_{\sigma^{-1}(1)},\dots,a_{\sigma^{-1}(n)})$ for the action of σ on $\vec{a}$. Then we write $St(\vec{a}) = \{\sigma\in\mathfrak{S}_n \mid \vec{a}[\sigma] = \vec{a}\}$ for the stabilizer subgroup.

Lemma 2.2. *Let $\sigma \in \mathbb{Q}^+\langle\Delta_\oplus\rangle$ and $\bar{s} \in supp(\sigma^!)$. If $\vec{s} = (s_1,\dots,s_n)$ is an enumeration of $\bar{s}$,* i.e. *$[s_1,\dots,s_n] = \bar{s}$, then $(\sigma^!)_{\bar{s}} = \dfrac{1}{Card(St(\vec{s}))}\cdot(\sigma)_{s_1}\cdots(\sigma)_{s_n}$.*

Proof. By definition and by linearity we have $\sigma^! = \sum_{n=0}^{\infty}\sum_{(s_1,\dots,s_n)\in\Delta^n}\frac{1}{n!}(\sigma)_{s_1}\cdots(\sigma)_{s_n}[s_1,\dots,s_n]$. In order to compute the coefficient of $\bar{s}$ in $\sigma^!$, we need to determine the number of possible enumerations of $\bar{s}$, *i.e.* the cardinality of $\{(s_1,\dots,s_n) \mid [s_1,\dots,s_n] = \bar{s}\}$: observe that, having fixed one enumeration $\vec{s} = (s_1,\dots,s_n)$, this set is nothing but the orbit of $\vec{s}$ by the action of the symetric group $\mathfrak{S}_n$. By a standard result of groups, $Card(Orb(\vec{s})) = \dfrac{Card(\mathfrak{S}_n)}{Card(St(\vec{s}))}$ and we conclude. □

Let M be a $\lambda_\oplus$-term. We inductively define $\tau(M) \in \mathbb{Q}^+\langle\Delta_\oplus\rangle$, the Taylor expansion of M:

$$\tau(x) = x \quad \tau(\lambda x.M') = \lambda x.\tau(M') \quad \tau(PQ) = \langle\tau(P)\rangle\tau(Q)^! \quad \tau(P\oplus Q) = (\tau(P)\oplus\bullet) + (\bullet\oplus\tau(Q))$$

Then we write $T(M)$ for the support of $\tau(M)$: $s \in T(M)$ iff $\tau(M)_s \neq 0$. Let $s \in \Delta_\oplus$. We inductively define $m(s)$, the *multiplicity* of s, as follows:

$$m(x) = 1 \qquad m(\lambda x.s) = m(s\oplus\bullet) = m(\bullet\oplus s) = m(s) \qquad m(\langle s\rangle\bar{t}) = m(s)m(\bar{t})$$

$$m([t_1^{n_1},\dots,t_n^{n_n}]) = \prod_{i=1}^{n} n_i!\; m(t_i)^{n_i}$$

assuming the t_i's are pairwise distinct in the case of a monomial. We also define the coherence relation $\frown \subseteq \Delta_\oplus^{(!)}\times\Delta_\oplus^{(!)}$ inductively by the following rules:

$$\frac{}{x\frown x} \qquad \frac{s\frown s'}{\lambda x.s\frown\lambda x.s'} \qquad \frac{s\frown s' \quad \bar{t}\frown\bar{t}'}{\langle s\rangle\bar{t}\frown\langle s'\rangle\bar{t}'} \qquad \frac{t_i\frown t_j \text{ for } 1\leq i,j\leq n+m}{[t_1,\dots,t_n]\frown[t_{n+1},\dots,t_{n+m}]}$$

$$\frac{s\frown s'}{s\oplus\bullet\frown s'\oplus\bullet} \qquad \frac{s\frown s'}{\bullet\oplus s\frown\bullet\oplus s'} \qquad \frac{}{s\oplus\bullet\frown\bullet\oplus s'}\ .$$

We can then reproduce Steps 1 and 2 of Ehrhard and Regnier's proof, establishing each of the following two lemmas by a straightforward induction on resource terms, also using Lemma 2.2 in the application case for Lemma 2.3:

Lemma 2.3. *Let $s\in T(M)$. Then $\tau(M)_s = \dfrac{1}{m(s)}$.*

Lemma 2.4. *The Taylor support $T(M)$ is a clique: $s\frown s'$ for all $s,s'\in T(M)$.*

We could as well obtain Step 3, following Ehrhard and Regnier again, but here we chose to present it in the framework of rigid terms and inductive permutations to be introduced later.

2.4 Böhm trees and resource normal forms

We rely on the presentation of Böhm trees by Ehrhard and Regnier [4], extended to $\Lambda_\oplus$ straightforwardly, no special meaning being given to the non-deterministic choice operator: e.g., $BT_n(M \oplus N) = BT_n(M) \oplus BT_n(N)$ where $BT_n(M)$ denotes the n-th Böhm tree approximant of M. The Taylor expansion of a Böhm tree is then obtained by setting $T(BT(M)) = \bigcup_{n\in\mathbb{N}} T(BT_n(M))$ and $\tau(BT(M)) = \sum_{s\in T(BT(M))} \frac{1}{m(s)} s$. Then we achieve Step 5 by showing that the parallel left strategy in $\Lambda_\oplus$ can be simulated in the support of Taylor expansion:

Lemma 2.5. *Let M be a $\lambda_\oplus$-term. Then $L_\partial(T(M)) = T(L(M))$.*

Observe that we only consider the action of L_∂ on support sets here. The desired equation then follows, using the fact that $BT(M)$ is the lub of the increasing sequence $(BT_n(M))$:

Theorem 2.6. *Let $M \in \Lambda_\oplus$. Then $T(BT(M)) = NF(T(M))$.*

Proof. Let us precise that $NF(T(M)) = \bigcup_{s\in T(M)} supp(NF(s))$. The proof is by double inclusion. ($\subseteq$) Let $s \in T(BT(M))$. Then there is $n \in \mathbb{N}$ such that $s \in T(BT_n(M))$ and the proof is by induction on n. ($\supseteq$) Let $s' \in \bigcup_{s\in T(M)} supp(NF(s))$. Then there is $m \in \mathbb{N}$ such that $s' \in supp(L_\partial^m(s))$ for some $s \in T(M)$, and the proof is by induction on m. In both directions, we use the previous lemma as well as the fact that $BT(M) = BT(L(M))$. □

3 The groupoid of permutations on rigid resource terms

3.1 Rigid resource terms

We introduce the set of rigid resource terms D and the set of rigid monomials $D^!$ by mutual induction as follows:

$$D \ni a,b,c,d ::= x \mid \lambda x.a \mid \langle a\rangle \vec{b} \mid \bullet \oplus a \mid a \oplus \bullet \qquad\qquad D^! \ni \vec{a},\vec{b},\vec{c},\vec{d} ::= () \mid a :: \vec{b}$$

Rigid resource terms are considered up to renaming of bound variables. We write $(a_1,\dots,a_n)$ for $a_1 :: \dots :: a_n :: ()$, $|(a_1,\dots,a_n)| = n$, and $(a_1,\dots,a_n) :: (a_{n+1},\dots,a_{n+m})$ for $(a_1,\dots,a_{n+m})$.

Definition 3.1. *We define $e\{\vec{b}/x\}$ for any $e \in D^{(!)}$ and $\vec{b} \in D^!$ such that $|\vec{b}| = n_x(e)$ inductively:*

$$x\{(b)/x\} = b \qquad y\{()/x\} = y \qquad (a \oplus \bullet)\{\vec{b}/x\} = a\{\vec{b}/x\} \oplus \bullet \qquad (\bullet \oplus a)\{\vec{b}/x\} = \bullet \oplus a\{\vec{b}/x\}$$
$$(\lambda y.a)\{\vec{b}/x\} = \lambda y.(a\{\vec{b}/x\}) \qquad (\langle c\rangle \vec{d})\{\vec{b}_0 :: \vec{b}_1/x\} = \langle c\{\vec{b}_0/x\}\rangle \vec{d}\{\vec{b}_1/x\}$$
$$(a_1,\dots,a_n)\{\vec{b}_1 :: \cdots :: \vec{b}_n/x\} = (a_1\{\vec{b}_1/x\},\dots,a_n\{\vec{b}_n/x\})$$

whenever $y \neq x$, $y \notin FV(\vec{b})$, $|\vec{b}| = n_x(a)$, $|\vec{b}_0| = n_x(c)$, $|\vec{b}_1| = n_x(\vec{d})$, and $|\vec{b}_i| = n_x(a_i)$ for $1 \le i \le n$.

Definition 3.2. *Let $e \in D^{(!)}$, $x \in \mathcal{V}$ and $\vec{b} \in D^!$. We define $a[\vec{b}/x]$ the* rigid substitution *of $\vec{b}$ for x in a, setting $a[\vec{b}/x] = a\{\vec{b}/x\}$ if $n_x(a) = |\vec{b}|$ and $a[\vec{b}/x] = 0$ otherwise.*

The reduction of rigid resource terms has the following base cases:

$$\langle \lambda x.a\rangle \vec{b} \to_r a[\vec{b}/x] \qquad \langle a \oplus \bullet\rangle \vec{b} \to_r \langle a\rangle \vec{b} \oplus \bullet \qquad \langle \bullet \oplus a\rangle \vec{b} \to_r \bullet \oplus \langle a\rangle \vec{b}.$$

extended contextually. Again, $\to_r$ is confluent and strongly normalizing. We write $NF(a)$ for the unique normal form of a that is a rigid term or 0. We can extend the coherence relation to rigid resource terms in the natural way.

3.2 Permutations of rigid monomials

Definition 3.3. *We define the set of permutations $\mathcal{G}$ of monomials in rigid resource terms by the following inference rules:*

$$\frac{}{id_x : x \cong x} \qquad \frac{\alpha : a \cong a'}{\lambda x.\alpha : \lambda x.a \cong \lambda x.a'} \qquad \frac{\gamma : c \cong c' \qquad \delta : \vec{d} \cong \vec{d'}}{\langle\gamma\rangle\delta : \langle c\rangle\vec{d} \cong \langle c'\rangle\vec{d'}}$$

$$\frac{\alpha : a \cong a'}{\alpha \oplus \bullet : a \oplus \bullet \cong a \oplus \bullet'} \qquad \frac{\alpha : a \cong a'}{\bullet \oplus \alpha : \bullet \oplus a \cong \bullet \oplus a'} \qquad \frac{\sigma \in \mathfrak{S}_n \quad \alpha_1 : a_1 \cong a'_{\sigma(1)} \quad \cdots \quad \alpha_n : a_n \cong a'_{\sigma(n)}}{(\sigma, \alpha_1, \ldots, \alpha_n) : (a_1, \ldots, a_n) \cong (a'_1, \ldots, a'_n)}$$

We write $a \cong a'$ if there exists some $f \in \mathcal{G}$ such that $f : a \cong a'$. It is easy to check that $\cong$ is an equivalence relation over rigid resource terms. In fact, we can organize the witnesses of this equivalence relation into a groupoid $\mathbb{G}$, with $D^{(!)}$ as collection of objects and for $a, a' \in D^{(!)}$, $\mathbb{G}(a,a') = \{\alpha \in \mathcal{G} \mid \alpha : a \cong a'\}$. The composition $\alpha'\alpha$ of $\alpha \in \mathbb{G}(a,a')$ and $\alpha' \in \mathbb{G}(a',a'')$ is defined by induction on the syntax of rigid resource terms in the obvious way: e.g., $(\sigma', \alpha'_1, \ldots, \alpha'_n)(\sigma, \alpha_1, \ldots, \alpha_n) = (\sigma'\sigma, \alpha'_{\sigma(1)}\alpha_1, \ldots, \alpha'_{\sigma(n)}\alpha_n)$; and the identity 1_a on a is the same as a, with each variable occurence x replaced with 1_x.

If $\vec{a} = (a_1, \ldots, a_n)$ and $\vec{a}' = (a'_1, \ldots, a'_n)$, we set $\vec{\mathbb{G}}(\vec{a}, \vec{a}') = \prod_{i=1}^n \mathbb{G}(a_i, a'_i)$. Observe that $\mathbb{G}(\vec{a}, \vec{a}') = \sum_{\sigma \in \mathfrak{S}_n} \vec{\mathbb{G}}(\vec{a}, \vec{a}'[\sigma])$. We call *quasi-stabilizer* of $\vec{a}$ the set $S(\vec{a}) = \{\sigma \in \mathfrak{S}_n \mid \text{for } 1 \leq i \leq n,\ a_i \cong a_{\sigma(i)}\}$.

Definition 3.4. *We define the approximation relation $\lhd \subseteq D^{(!)} \times \Delta_{\oplus}^{(!)}$ by the following rules:*

$$\frac{}{x \lhd x} \quad \frac{a \lhd s}{\lambda x.a \lhd \lambda x.s} \quad \frac{a \lhd s}{a \oplus \bullet \lhd s \oplus \bullet} \quad \frac{a \lhd s}{\bullet \oplus a \lhd \bullet \oplus s} \quad \frac{c \lhd s \qquad \vec{d} \lhd \bar{t}}{\langle c\rangle\vec{d} \lhd \langle s\rangle\bar{t}} \quad \frac{a_1 \lhd t_1 \quad \cdots \quad a_n \lhd t_n}{(a_1, \ldots, a_n) \lhd [t_1, \ldots, t_n]} \ .$$

We set the rigid expansion of s as $T_r(s) = \{a \in D^{(!)} \mid a \lhd s\}$. As a direct consequence of the definitions, we obtain a precise relationship between $\cong$ and $\lhd$:

Proposition 3.5. *Let $e \in \Delta_{\oplus}^{(!)}$ and let $a \lhd e$. For all $a' \in D^{(!)}$, $a \cong a'$ iff $a' \lhd e$.*

The rigid expansion of a resource term is thus the same as an equivalence class of rigid resource terms. In particular, this equivalence is compatible with coherence: if $a \frown b$ and $a \cong a'$ then $a' \frown b$. Indeed, coherence and uniformity are really properties of (non-rigid) resource terms.

Lemma 3.6. *If $\bar{t} = [t_1, \ldots, t_n] \in \Delta^!$ and $\vec{a} \in D^!$ with $\vec{a} \lhd \bar{t}$ then $St((t_1, \ldots, t_n)) = S(\vec{a}) = \pi_1(\mathbb{G}(\vec{a}, \vec{a}))$.*

From the former lemma we can in particular derive that $S(\vec{a})$ is a group.

Lemma 3.7. *Let $\vec{a} = (a_1, \ldots, a_n) \in D^!$ and let $\sigma \in \mathfrak{S}_n$. Then $Card(\mathbb{G}(\vec{a}, \vec{a})) = Card(S(\vec{a})) \times \prod_{i=1}^n Card(\mathbb{G}(a_i, a_i))$.*

We are then able to formalize the interpretation of the multiplicity of a resource term s as the number of permutations of monomials in s leaving any of its writings $a \lhd s$ unchanged:

Lemma 3.8. *Let $s \in \Delta^{(!)}$ and let $a \lhd s$. Then $m(s) = Card(\mathbb{G}(a,a))$.*

3.3 Permutations and substitution

Definition 3.9. *We define the* substitution of inductive permutations *as follows. Given $\alpha \in \mathbb{G}(a,a')$ and $\vec{\beta} \in \vec{\mathbb{G}}(\vec{b}, \vec{b'})$ with $|\vec{b}| = n_x(a)$, we construct $\alpha[\vec{\beta}/x]$ by induction on α:*

$$(id_x)[(\beta)/x] = \beta \qquad (id_y)[()/x] = id_y \qquad (\lambda y.\alpha)[\vec{\beta}/x] = \lambda y.(\alpha[\vec{\beta}/x])$$

$$(\alpha \oplus \bullet)[\vec{\beta}/x] = \alpha[\vec{\beta}/x] \oplus \bullet \qquad (\bullet \oplus \alpha)[\vec{\beta}/x] = \bullet \oplus \alpha[\vec{\beta}/x] \qquad (\langle\gamma\rangle\delta)[\vec{\beta}_0 :: \vec{\beta}_1/x] = (\langle\gamma[\vec{\beta}_0]\rangle\delta[\vec{\beta}_1/x])$$

$$(\sigma, (\alpha_1, \ldots, \alpha_n))[\beta_1 :: \cdots :: \beta_n/x] = (\sigma, (\alpha_1[\vec{\beta}_1/x], \ldots, \alpha_n[\vec{\beta}_n/x])) \ .$$

The action of $\alpha[\vec{\beta}/x]$ on $a[\vec{b}/x]$ is quite intricate: in general, $\alpha[\vec{\beta}/x] \notin \mathbb{G}(a[\vec{b}/x], a'[\vec{b}'/x])$.

Example 3.10. *Consider the following rigid monomials $a = (x,x)$ and $\vec{b} = (\langle z\rangle(), \langle z\rangle(z))$. Writing τ for the unique transposition of $\mathfrak{S}_2$, we obtain $\alpha = (\tau, id_x, id_x) \in \mathbb{G}(a,a)$. We also have that $\vec{\beta} = ((id_z, *), (id_z, (id_1, id_z))) \in \vec{\mathbb{G}}(\vec{b}, \vec{b})$. Then $\alpha[\vec{\beta}/x] = (\tau, (id_z, *), (id_z, (id_1, id_z)))$, hence $\alpha[\vec{\beta}/x] : a[\vec{b}/x] \cong (\langle z\rangle(z), \langle z\rangle()) \neq a[\vec{b}/x]$.*

To describe the image of $a[\vec{b}/x]$ through $\alpha[\vec{\beta}/x]$, we first introduce two operations on permutations. If $\sigma \in \mathfrak{S}_n$, $\tau \in \mathfrak{S}_p$ and $\tau_1 \in \mathfrak{S}_{k_1}, \ldots, \tau_n \in \mathfrak{S}_{k_n}$, we define the *concatenation* $\sigma \mid \tau \in \mathfrak{S}_{n+p}$ and the *multiplexing* $\sigma \cdot (\tau_1, \ldots, \tau_n) \in \mathfrak{S}_{k_1+\ldots+k_n}$:

$$(\sigma \mid \tau)(i) = \sigma(i) \qquad (\sigma \mid \tau)(n+j) = \tau(j) \qquad (\sigma \cdot (\tau_1, \ldots, \tau_n))\left(\sum_{r=1}^{i-1} k_r + l\right) = \sum_{r=1}^{i-1} k_{\sigma^{-1}(r)} + \tau_{\sigma^{-1}(i)}(l)$$

for $1 \leq i \leq n$, $1 \leq j \leq p$ and $1 \leq l \leq k_i$.

Then we can define the *restriction* $\alpha_{|x} \in \mathfrak{S}_{n_x(a)}$ of $\alpha \in \mathbb{G}(a, a')$ to the occurrences of x in a, by induction on α:

$$(id_x)_{|x} = id \in \mathfrak{S}_1 \qquad (id_y)_{|x} = * \in \mathfrak{S}_0 \qquad (\lambda y.\alpha)_{|x} = (\alpha \oplus \bullet)_{|x} = (\bullet \oplus \alpha)_{|x} = \alpha_{|x}$$
$$(\langle\gamma\rangle\delta)_{|x} = (\gamma_{|x}) \mid (\delta_{|x}) \qquad (\sigma, (\alpha_1, \ldots, \alpha_n))_{|x} = \sigma \cdot (\alpha_{1|x} \mid \cdots \mid \alpha_{n|x})$$

where we assume $x \neq y$. Then we write $\mathbb{G}(a,a')_{|x} = \{\alpha_{|x} \mid \alpha \in \mathbb{G}(a,a')\}$. Intuitively $\alpha_{|x}$ is the permutation induced by α on the occurrences $x_1, \ldots, x_{n_x(a)}$ of x in a. We obtain:

Lemma 3.11. *If $\alpha : a \cong a'$ and $\beta_i : b_i \cong b'_i$ for $1 \leq i \leq n_x(a)$ then $\alpha[\vec{\beta}/x] : a[\vec{b}/x] \cong a'[\vec{b}'[\alpha_{|x}]/x]$.*

3.4 Rigid resource terms and reduction

Proposition 3.12. *Let $u \in supp(\partial_x s \cdot \bar{t})$. If $c \lhd u$ then there are $a \lhd s$ and $\vec{b} \lhd \bar{t}$ with $c = a[\vec{b}/x]$.*

Lemma 3.13. *Let $a, a' \in D^{(!)}$ and $\vec{b}, \vec{b}' \in D^!$ with $a \frown a'$ and $\vec{b} \frown \vec{b}'$. If $a[\vec{b}/x] \cong a'[\vec{b}/x]$ then $a \cong a'$ and $\vec{b} \cong \vec{b}'$.*

Proposition 3.14. *Let $a, a' \in D^{(!)}$ with $a \frown a'$. If $NF(a) \cong NF(a')$ then $a \cong a'$.*

The coherence hypothesis is necessary as shown by the following example:

Example 3.15. *Let $a = \langle\langle y\rangle(x)\rangle\langle z\rangle(x)$, $a' = \langle\langle x\rangle(y)\rangle\langle z\rangle(x)$ and $\vec{b} = (y,z)$. We have that a and a' are not coherent and $\vec{b}$ is not uniform(i.e. coherent with itself). One has that $a[\vec{b}/x] \cong a'[\vec{b}/x]$ but a is not isomorphic to a'.*

We obtain Step 3 as a corollary:

Lemma 3.16. *Let $s, s' \in T(M)$ be such that $s \frown s'$. Then $supp(NF(s)) \cup supp(NF(s'))$ is a clique. If moreover $supp(NF(s)) \cap supp(NF(s')) \neq \emptyset$ then $s = s'$.*

We are now ready to establish preliminary results for Step 4. Let $\bar{\mathbb{G}}(a,a) = \{\alpha \in \mathbb{G}(a,a) \mid \alpha_{|x} \in S(\vec{b})\}$.

Lemma 3.17. *Let $a \in D^{(!)}$ and $\vec{b} \in D^!$ with $|b| = n_x(a)$. If $a \frown a$ then $Card(\mathbb{G}(a[\vec{b}/x], a[\vec{b}/x])) = Card(\bar{\mathbb{G}}(a,a))Card(\vec{G}(\vec{b}, \vec{b}))$.*

Proof. By induction on the definition of $a[\vec{b}/x]$ and by uniformity, applying Lemma 3.7 and Lemma 3.13. □

Let $\mathcal{S}_{a,x/\vec{b}} = \{\sigma \in \mathfrak{S}_n \mid a[\vec{b}/x] \cong a[\vec{b}[\sigma]/x])\}$ whenever $|\vec{b}| = n_x(a)$. A simple inspection of the definitions yields:

Proposition 3.18. *Let $a, a' \in D$ and $\vec{b}, \vec{b}' \in D^!$. then $\mathcal{S}_{a,x/\vec{b}} = \mathbb{G}(a,a)_{|x} S(\vec{b})$.*

4 Quantitative account of the Taylor expansion

In this section we shall utilise the results about the groupoid of rigid approximants to prove the commutation between Taylor expansion and Böhm trees.

Lemma 4.1. *Let $s \in \Delta_{\oplus}$ and $\bar{t} \in \Delta^{!}_{\oplus}$. If $u \in supp(\partial_x s \cdot \bar{t})$ then* $(\partial_x s \cdot \bar{t})_u = \dfrac{m(s)m(\bar{t})}{m(u)}$.

Proof. By definition of n-linear substitution we have that $\partial_x s \cdot \bar{t} = \sum_{\sigma \in \mathfrak{S}_k} s[t_{\sigma(1)}/x_1, \ldots, t_{\sigma(k)}/x_k]$. Let $c \lhd u$. By Proposition 3.12 there exists $a \lhd s$ and $\vec{b} \lhd \bar{t}$ such that $c = a[\vec{b}/x] \lhd u$. We consider c as a representative of the equivalence class $T_r(u)$. By Proposition 3.5, one has that $(\partial_x s \cdot \bar{t})_u = Card(\{\sigma \in \mathfrak{S}_k \mid s[t_{\sigma(1)}/x_1, \ldots, t_{\sigma(k)}/x_k] = u\}) = Card(\{\sigma \in \mathfrak{S}_k \mid a[\vec{b}[\sigma]/x] \in T_r(u)\}) = Card(\{\sigma \in \mathfrak{S}_k \mid \mathbb{G}(a[\vec{b}/x], a[\vec{b}[\sigma]/x]) \neq \emptyset\}) = Card(\mathcal{S}_{a,x/\vec{b}})$. The proof then follows a pattern similar to that of Ehrhard and Regnier [6, Section 4.2] *via* Lemma 3.17 and Proposition 3.18. □

We have thus achieved Step 4. It follows that if $u \in supp(L_{\partial}(s))$ then $(L_{\partial}(s))_u = \dfrac{m(s)}{m(u)}$, and then $(NF(s))_u = \dfrac{m(s)}{m(u)}$ by iterating left reduction.

Theorem 4.2. *Let $M \in \Lambda_{\oplus}$. Then $\tau(BT(M)) = NF(\tau(M))$.*

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