# Poof nets, coends and the Yoneda isomorphism

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Proof nets provide permutation-independent representations of proofs and are used to investigate coherence problems for monoidal categories. We investigate a coherence problem concerning Second Order Multiplicative Linear Logic *MLL*2, that is, the one of characterizing the equivalence over proofs generated by the interpretation of quantifiers by means of ends and coends.

By adapting the "rewiring approach" used in the proof net characterization of the free $^*$-autonomous category, we provide a compact representation of proof nets for a fragment of *MLL*2 related to the Yoneda isomorphism. We prove that the equivalence generated by coends over proofs in this fragment is fully characterized by the rewiring equivalence over proof nets.

## 1 Introduction

Proof nets are usually investigated as canonical representants of proofs. For the proof-theorist, the adjective "canonical" indicates a representation of proofs insensitive to admissible permutations of rules; for the category-theorist, it indicates a faithful representation of arrows in free monoidal categories (typically, $^*$-autonomous categories), by which coherence results can be obtained.

This twofold approach has been developed extensively in the case of Multiplicative Linear Logic (see for instance [4, 5]). The use of *MLL* proof nets to investigate coherence problems relies on the correspondence between proof nets and a particular class of dinatural transformations[1] (see [4]). As dinatural transformations provide a well-known interpretation of parametric polymorphism (see [1, 14]), it is natural to consider the extension of this correspondence to second order Multiplicative Linear Logic *MLL*2. This means investigating the "coherence problem" generated by the interpretation of quantifiers as ends/coends, that is, to look for a faithful proof net representation of coends within a $^*$-autonomous category.

The main difficulty of this extension is that, as is well-known, dinaturality does not scale to second order (e.g. System $F$, see [24]): the dinatural interpretation of proofs generates an equivalence over proofs which strictly extends the equivalence generated by $\beta$ and $\eta$ conversions. In particular, coends induce "generalized permutations" of rules ([33]) to which neither System $F$ proofs nor standard proof nets for *MLL*2 are insensitive. For instance, the interpretation of quantifiers as ends/coends (whose definition is recalled in appendix A) equates the distinct System $F$ derivations in fig. 1a as well as the distinct proof nets in fig. 1b. From these examples

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[1] *Extranatural transformations* form a special class of dinatural transformations for which the composition problem has been investigated in detail ([6]).

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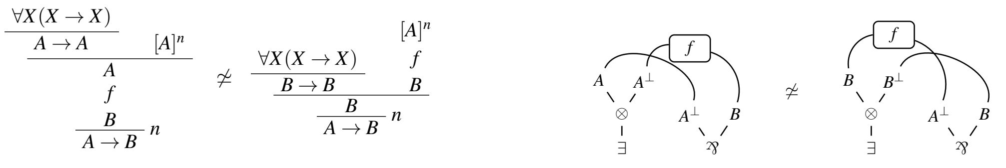

(a) Failure of dinaturality in System $F$

(b) Failure of dinaturality for proof nets

Figure 1

it can be seen that such generalized permutations do not preserve the witnesses of existential quantification (or, equivalently, of the elimination of universal quantification).

Several well-known failures in the System $F$ representation of categorial structures can be related to this phenomenon: (1) the failure of the “Yoneda isomorphism” $\forall X((A \multimap X) \multimap B[X])) \simeq B[A]$ as an isomorphism of types; (2) the failure of universality for the “Russell-Prawitz” translation of connectives (e.g. the failure of the isomorphism $A \otimes B \simeq \forall X((A \multimap B \multimap X) \multimap X))$; (3) the failure of initiality for the System $F$ representation of initial algebras. In all such cases, the failure is solved by considering proofs modulo the equivalence induced by dinaturality (see [31, 15]).

Some *a priori* limitations to the proof net representation of quantifiers as ends and coends can be deduced from the fact that, by the “Yoneda isomorphism” $\forall X(X \multimap X) \simeq \mathbf{1}$, it must include a faithful representation of multiplicative units. Now, it is well-known that no canonical representation of *MLL* with multiplicative units can have both a tractable correctness criterion and a tractable translation from sequent calculus ([16]). However, in usual approaches to multiplicative units proof nets are considered modulo an equivalence relation called *rewiring* ([34, 5, 20]), which provides a partial solution to this problem. The “rewiring approach” ([20]) allows to circumvent the complexity of checking arrows equivalence in the free $^*$-autonomous category by isolating the complex part into a geometrically intuitive equivalence relation.

In this paper we adapt the rewiring approach to define a compact representation of proof nets (called $\exists$-linkings) for the fragment of *MLL*2 involved in (1) and (2). More precisely, we consider the system $MLL2_{\mathscr{Y}}$, in which quantification $\forall XA$ is restricted to “Yoneda formulas”, i.e. formulas of the form $(\bigotimes_i^n C_i \multimap X) \multimap D[X]$. This fragment contains the multiplicative “Russell-Prawitz” formulas as well as the translation of multiplicative units. The approach presented is related to the rewiring approach in the sense that, when restricted to the translation of units, $\exists$-linkings are equivalent to the “lax linkings” in [20].

Our main result is that the equivalence over proofs generated by coends coincides exactly with the rewiring equivalence over $\exists$-linkings. More precisely, we define an equivalence $\simeq_\varepsilon$ over standard *MLL*2 proof nets, where two proof nets are equivalent when their dinatural interpretations coincide, and we show that, within the fragment $MLL2_{\mathscr{Y}}$, $\pi \simeq_\varepsilon \pi'$ holds iff the associated $\exists$-linkings $\ell_\pi$ and $\ell_{\pi'}$ are equivalent up to rewiring. These results imply that $\exists$-linkings form a $^*$-autonomous category in which $\forall X(X \multimap X)$ is the tensor unit and provide a faithful representation of coends.

$\exists$-linkings solve failures (1) and (2): the “Yoneda isomorphism” is an isomorphism of $\exists$-linkings, up to rewiring, and the “Russell-Prawitz” isomorphisms like $A \otimes B \simeq \forall X((A \multimap B \multimap$

$X) \multimap X)$ are consequences of Yoneda. Failure (3) falls outside the scope of the fragment $MLL_{\mathscr{Y}}$, as the latter does not include the formulas involved in the usual System $F$ representation of initial algebras. However, following the ideas in [35], a generalization of the approach here presented might yield similar results for the representation of initial algebras.

**Related work** Dinaturality is a well-investigated property of System $F$ and is usually related to parametric polymorphism (see [1, 31]). The connections between dinaturality, coherence and proof nets are well-investigated in the case of *MLL*, with or without units ([3, 4, 5, 22, 20, 17, 28, 18]). An extensive literature exists on coends in monoidal categories (see [25] for a survey). String diagram representations of some coends can be found in the literature on Hopf algebras and their application to quantum field theory ([21, 10]). Such coends are all of the restricted form considered in this paper and their representation seems comparable to the one here proposed. A different approach to quantifiers as ends/coends over a symmetric monoidal closed category appears in [29], through a bifibrational reformulation of the Lawvere's presheaf hyperdoctrine in the 2-category of distributors. It might be interesting to relate this approach with ours.

The universality problem for the "Russell-Prawitz" translation is related to the *instantiation overflow* property ([9]), by which one can transform the System $F$ proofs obtained by this translation into proofs in $F_{at}$ or *atomic System F*, which have the desired properties (see [8]). In [30] is shown that the atomized proofs are equivalent to the original ones modulo dinaturality. $\exists$-linkings provide a very simple approach to instantiation overflow, as the transformation from $F$ to $F_{at}$ consists in a unique rewiring.

The representation of proof nets here adopted is inspired from results on *MLL* with units ([34, 5, 20]) and on *MLL*1 ([19]). Proof nets for first-order and second order quantifers were first conceived by means of boxes ([11]). Later, Girard proposed two distinct boxes-free formalisms (in [12, 13] for *MLL*1 but extendable to *MLL*2, see [7]), the second of which is referred here as "Girard nets". Different refinements of proof nets for *MLL*1 and *MLL*2 have been proposed ([27, 19] for *MLL*1 and [32] for *MLL*2) to investigate variable dependency issues related to Herbrand theorem and unification, which are not considered here.

## 2 Preliminaries

We let $\mathscr{L}^2$ be the language generated by a countable set of variables $X, Y, Z, \dots \in \texttt{Var}$ and their negations $X^{\perp}, Y^{\perp}, Z^{\perp}, \dots$ and the connectives $\otimes, \parr, \forall, \exists$. Negation is obviously extended into an equivalence relation over formulas. By sequents $\Gamma, \Delta, \dots$ we indicate finite multisets of formulas. A sequent $\Gamma$ is *clean* when no variable occurs both free and bound in $\Gamma$ and any variable in $\Gamma$ is bound by at most one $\forall$ or $\exists$ connective.

By *MLL*2 we indicate the standard sequent calculus over $\mathscr{L}^2$. [13] describes proof nets for first-order *MLL*. Both the description of proof structures and the correctness criterion can be straightforwardly turned into a definition of proof nets for *MLL*2 (see for instance [7]). We indicate the latter as *Girard proof structures* and *Girard nets* (shortly, *G*-proof structures and *G*-nets[2]). We let $\mathbb{G}$ indicate the *category of G-nets*, whose objects are the types of *MLL*2 and

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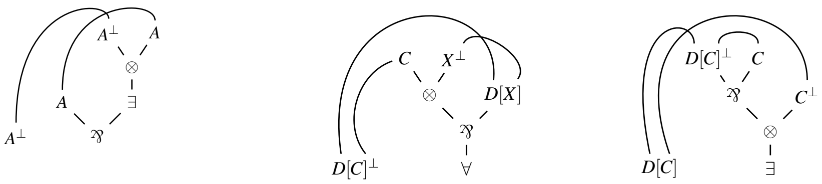
[2] In [13] the definition of proof structures is based on two conditions: (1) that any $\forall$ link has a distinct eigenvariable



(a) $G$-net $\pi_\perp^A$

(b) $G$-nets for the Yoneda isomorphism

Figure 2

where $\mathbb{G}(A,B)$ is the set of cut-free $G$-nets of conclusions $A^\perp, B$ (with composition given by cut-elimination).

We introduce Yoneda formulas:

**Definition 1** (Yoneda formula)**.** *Given a variable $X \in$ `Var` and a formula $A \in \mathscr{L}^2$, $A$ is* Yoneda in *$X$ (resp.* co-Yoneda in *$X$) if $A$ (resp. $A^\perp$) is of the form $(\bigotimes_i^n C_i \otimes X^\perp) \parr D[X]$*[3]*, where $X$ does not occur in any of the $C_i$ and $D[X]$ has a unique, positive, occurrence of $X$.*

We let $\mathscr{L}^2_{\mathscr{Y}} \subset \mathscr{L}^2$ be the language obtained by restricting $\forall$ quantification (resp. $\exists$ quantification) to Yoneda (resp. co-Yoneda) formulas. In other words $\forall XA \in \mathscr{L}^2_{\mathscr{Y}}$ (resp. $\exists XA \in \mathscr{L}^2_{\mathscr{Y}}$) only if $A$ is Yoneda in $X$ (resp. co-Yoneda in $X$). We indicate by $MLL2_{\mathscr{Y}}$ the restriction of $G$-nets to $\mathscr{L}^2_{\mathscr{Y}}$.

The *Yoneda translation $A_{\mathscr{Y}}$* of a formula $A \in \mathscr{L}^2_{\mathscr{Y}}$ is the multiplicative formula obtained by replacing systematically $\forall X((\bigotimes_i^n C_i \otimes X^\perp) \parr D[X])$ by $D[\bigotimes_i^n C_i \otimes \mathbf{1}]$ and $\exists X((\bigparr_i^n C_i \parr X) \otimes D[X^\perp])$ by $D[\bigparr_i^n C_i \parr \perp]$.

**Remark 1.** *For any Yoneda formula of the form $A = (C \otimes X^\perp) \parr D[X]$, there exist G-nets $Yo_1^A \in \mathbb{G}(D[C], \forall XA)$ and $Yo_2^A \in \mathbb{G}(\forall XA, D[C])$, illustrated in figure 2b.*

The formulas $\forall X(X^\perp \parr X)$ and $\exists X(X \otimes X^\perp)$ translate multiplicative units $\mathbf{1}, \perp$. In particular, there exists a unique $G$-net $\pi_{\mathbf{1}}$ of conclusion $\forall X(X^\perp \parr X)$ and a $G$-net $\pi_\perp^A \in \mathbb{G}(A, A \parr \perp)$, for any formula $A$ (see fig. 2a). We let $\mathscr{L}_{\mathbf{1},\perp} \subset \mathscr{L}^2_{\mathscr{Y}}$ be the language obtained by restricting $\forall XA$ to $A = X^\perp \parr X$ and $\exists XA$ to $A = X \otimes X^\perp$. We let $MLL2_{\mathbf{1},\perp}$ be the restriction of $G$-nets to $\mathscr{L}_{\mathbf{1},\perp}$.

## 3 Girard nets and $^*$-autonomous categories with coends

We show that any $G$-net can be interpreted as a morphism in any (strict) $^*$-autonomous category $\mathbb{C}$ in which coends exist. Any map $\varphi :$ `Var` $\to Ob_{\mathbb{C}}$ extends into a map $\varphi : \mathscr{L}^2 \to Ob_{\mathbb{C}}$ by letting

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and (2) that the conclusions of a proof structures have no free variable (in particular, new constants $\bar{x}$ are introduced to eliminate free variables). Moreover, in the definition of the correctness criterion any $\forall$-link of eigenvariable $X$ can *jump* on any formula in which $X$ occurs free. In [19] conditions (1) and (2) are replaced by the equivalent condition that the conclusions of the proof structure plus the witnesses of existential links must form a *clean* sequent and the correctness criterion is modified by demanding that a $\forall$-link of eigenvariable $X$ can *jump* on any $\exists$-link whose witness formula contains free occurrences of $X$. Here we will consider this formulation.

[3] Given a formula $A$ and a finite (possibly empty) sequence of formulas $C_1, \dots, C_n$, we indicate by $\bigotimes_i^n C_i \otimes A$ (resp. $\bigparr_i^n C_i \parr A$) the formula $C_1 \otimes \cdots \otimes C_n \otimes A$ (resp. $C_1 \parr \cdots \parr C_n \parr A$).

$(A \otimes B)^{\varphi} = A^{\varphi} \otimes B^{\varphi}$, $(\forall XA)^{\varphi} = \int_x A^{\varphi}(x,x)$ and $(A^{\perp})^{\varphi} = (A^{\varphi})^{\perp}$. We show (Prop. 1) that any such map $\varphi$ generates a (unique) functor $\Phi : \mathbb{G} \to \mathbb{C}$ such that, for all $A \in \mathscr{L}^2$, $\Phi(A) = A^{\varphi}$. Then we consider the equivalence relation $\simeq_{\varepsilon}$ over $G$-nets induced by such interpretations and show that it extends the equivalence relation generated by $\beta\eta$-equivalence.

Some useful definitions and properties of $^*$-autonomous categories and coends can be found in appendix A. It is well-known (see [23]) that, if we let $\mathbb{P}$ be the category of *MLL* proof nets and $\mathbb{C}$ be any (strict) $^*$-autonomous category, then any map $\varphi : \mathtt{Var} \to Ob_{\mathbb{C}}$ generates a (unique) functor $\Phi : \mathbb{P} \to \mathbb{C}$. In order to extend this result to *MLL*2 we must in addition (1) demand that coends exist in $\mathbb{C}$, in order to interpret quantifiers, and (2) show that $G$-nets correspond to *dinatural transformations* between multivariant functors over $\mathbb{C}$. In the following we will suppose $\mathbb{C}$ is a (strict) $^*$-autonomous category in which ends (hence, by duality, coends) exist.

Any formula $A \in \mathscr{L}^2$ whose free variables are within $X_1, \dots, X_n$ can be interpreted as a multivariant functor $A^{\mathbb{C}} : (\mathbb{C}^{op} \otimes \mathbb{C})^n \to \mathbb{C}$ by letting

$$X_i^{\mathbb{C}}(\vec{a}, \vec{b}) := b_i \qquad X_i^{\mathbb{C}}(\vec{f}, \vec{g}) := g_i$$
$$(A \otimes B)^{\mathbb{C}} := A^{\mathbb{C}} \otimes B^{\mathbb{C}} \quad (\forall YA)^{\mathbb{C}} := \int_y A^{\mathbb{C}}(y,y) \quad (A^{\perp})^{\mathbb{C}} := (A^{\mathbb{C}})^{\perp}$$

For a clean sequent $\Gamma = A_1, \dots, A_n$, whose free variables are within $X_1, \dots, X_n$, we let $\Gamma^{\mathbb{C}} := A_1^{\mathbb{C}} \parr \cdots \parr A_n^{\mathbb{C}}$ if $n \geq 1$ and $\Gamma^{\mathbb{C}} = \mathbf{1}_{\mathbb{C}}$ if $n = 0$.

**Lemma 1** (substitution lemma)**.** $(A[B/X])^{\mathbb{C}}(x,x) = A^{\mathbb{C}}(B^{\mathbb{C}}(x,x), B^{\mathbb{C}}(x,x))$.

Let $\pi$ be a cut-free $G$-net of conclusions $\Gamma, \Delta$, and let all formulas occurring in $\pi$ be within $X_1, \dots, X_n$. Then $\pi$ can be interpreted as a dinatural transformation $\pi^{\mathbb{C}} : (\Gamma^{\mathbb{C}})^{\perp} \to \Delta^{\mathbb{C}}$[4]. Similarly to [23], we can argue by induction on a sequentialization of $\pi$. We limit ourselves to the case of quantifiers:

- if $\Delta = \Sigma, \forall YA$ and $\pi$ is obtained from $\pi'$ of conclusions $\Sigma, A$, then $\pi^{\mathbb{C}}$ is obtained from $(\pi')_x^{\mathbb{C}}$[5] (which can be seen as a dinatural transformation from $\Gamma^{\mathbb{C}} \otimes (\Sigma^{\mathbb{C}})^{\perp}$ to $A^{\mathbb{C}}$) by the universality of ends, as shown by the diagram below:

$$\begin{tikzcd}
\Gamma^{\mathbb{C}} \otimes (\Sigma^{\mathbb{C}})^{\perp} \arrow[dr, dashed, "\pi^{\mathbb{C}}"] \arrow[drr, bend left, "(\pi')_a^{\mathbb{C}}"] \arrow[ddr, bend right, "(\pi')_b^{\mathbb{C}}"'] & & \\
& \int_y A^{\mathbb{C}}(y,y) \arrow[r, "\delta_a^{A^{\mathbb{C}}}"] \arrow[d, "\delta_b^{A^{\mathbb{C}}}"] & A^{\mathbb{C}}(a,a) \arrow[d, "A^{\mathbb{C}}(a,f)"] \\
& A^{\mathbb{C}}(b,b) \arrow[r, "A^{\mathbb{C}}(f,b)"'] & A^{\mathbb{C}}(a,b)
\end{tikzcd}$$

- if $\Delta = \Sigma, \exists YA$ and $\pi$ is obtained from $\pi'$ of conclusions $\Sigma, A[B/X]$, then $\pi^{\mathbb{C}}$ is obtained from $(\pi')^{\mathbb{C}}$ by the chain of arrows below (by exploiting lemma 1):

$$\Gamma^{\mathbb{C}} \xrightarrow{(\pi')^{\mathbb{C}}} \Sigma^{\mathbb{C}} \parr A^{\mathbb{C}}(B^{\mathbb{C}}, B^{\mathbb{C}}) \xrightarrow{\omega_{B^{\mathbb{C}}}^{\Sigma^{\mathbb{C}} \parr A^{\mathbb{C}}}} \int^x (\Sigma^{\mathbb{C}} \parr A^{\mathbb{C}}(x,x)) \xrightarrow{\nu} \Sigma^{\mathbb{C}} \parr \int^x A^{\mathbb{C}}(x,x)$$

  where $\nu$ is given by proposition 5.

The definition above can be extended to the case of a $G$-net with cuts: if $\pi$ has conclusions $\Gamma$ and cut-formulas $B_1, \dots, B_n$, then we can transform $\pi$ into a $G$-net $\pi_{cut}$ of conclusions $\Gamma, [B_1 \otimes B_1^{\perp}, \dots, B_n \otimes B_n^{\perp}]$. Then we can define $\pi^{\mathbb{C}}$ as $(id_{\Gamma} \parr \hat{\perp}_{B_1^{\mathbb{C}}} \parr \cdots \parr \hat{\perp}_{B_n^{\mathbb{C}}}) \circ \pi_{cut}^{\mathbb{C}}$. The following proposition shows that $\pi^{\mathbb{C}}$ is dinatural (this is not trivial, since the composition of dinaturals need not be dinatural) and invariant with respect to reduction.

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[4] As explained in appendix A, we omit for readability reference to variables $x_1, \dots, x_n$.

[5] More precisely, $\pi_{x_1,\dots,x_n}^{\mathbb{C}}$ is obtained from $(\pi')_{x_1,\dots,x_n,y}^{\mathbb{C}}$, where $\Gamma^{\mathbb{C}}$, $(\Sigma^{\mathbb{C}})^{\perp}$ do not depend on $y$.

**Proposition 1.** *Let $\pi$ be a G-net with cuts of conclusions $\Gamma$ and $\pi_0$ be the G-net obtained from $\pi$ by eliminating all cuts. Then $\pi^{\mathbb{C}} = (\pi_0)^{\mathbb{C}}$.*

**Theorem 1** (functor $\Phi : \mathbb{G} \to \mathbb{C}$). *Let $\varphi : \mathtt{Var} \to Ob_{\mathbb{C}}$ be any map from variables to objects of $\mathbb{C}$. Then there exists a (unique) functor $\Phi : \mathbb{G} \to \mathbb{C}$ such that, for all $A \in \mathbb{L}^2$, $\Phi(A) = A^{\varphi}$.*

We now consider the equivalence relation generated by the dinatural interpretation of *G*-nets:

**Definition 2** (equivalence $\simeq_{\varepsilon}$). *We let $\simeq_{\varepsilon}$ be the equivalence relation over G-nets given by $\pi \simeq_{\varepsilon} \pi'$ iff $\pi^{\mathbb{C}} = (\pi')^{\mathbb{C}}$, for any $^*$-autonomous category with coends $\mathbb{C}$.*

From proposition 1 it follows that $\simeq_{\varepsilon}$ includes $\beta\eta$-equivalence. The following examples show that $\simeq_{\varepsilon}$ strictly extends $\beta\eta$-equivalence.

**Example 1.** *The category $\mathbb{G}$ is not $^*$-autonomous. In particular, $\forall X(X^{\perp} \parr X)$ is not a tensor unit: by composing the G-net $\pi^{A}_{\perp} \in \mathbb{G}(A \otimes \forall X(X^{\perp} \parr X), A)$ with the unique G-net in $\mathbb{G}(A, A \otimes \forall X(X^{\perp} \parr X))$ one does not get $id_{A \otimes \forall X(X^{\perp} \parr X)}$.*

**Example 2.** *$\exists$ is not a coend: this can be seen from the two distinct G-nets in figure 1b, corresponding to the two sides of the diagram describing a coend.*

**Example 3.** *The "Yoneda isomorphism" is false in $\mathbb{G}$. For it suffices to remark that $Yo^{A}_{1} \circ Yo^{A}_{2} \neq id_{\forall XA}$.*

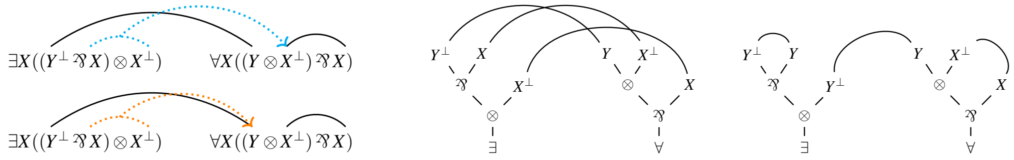
## 4 Linkings for *MLL2*$_{\mathscr{Y}}$

In this section we introduce a compact representation of proof nets for *MLL2*$_{\mathscr{Y}}$. We adopt a notion of *linking* inspired from [20, 19] and a notion of *rewiring* inspired from [5, 16, 20] (in which the role of thinning edges is given by *witness edges*). In particular, the restriction to $\mathscr{L}^{2}_{\mathbf{1},\perp}$ yields a formalism which is equivalent to the one in [20].

**Linkings** Given a formula $A$ (resp. a sequent $\Gamma$) we let $tA = (nA, eA)$ (resp. $t\Gamma = (n\Gamma, e\Gamma)$) be its parse tree (resp. parse forest). We will often confuse the nodes of $\Gamma$ with the associated formulas. Let $\Gamma$ be a clean sequent. An *edge* $e$ is a pair of leaves of $t\Gamma$ consisting in two occurrences of opposite polarity of the same variables. Any $\exists$-link in $t\Gamma$ has a distinguished eigenvariable. A variable is an *existential variable* if it occurs quantified existentially. Since in all formulas of the form $\exists XA$, $A$ is co-Yoneda in $X$, existential variables come in pairs, called *co-edges*. We let $\Gamma^{\exists}$ be the set of co-edges of $\Gamma$. Any co-edge $c$ is uniquely associated with an existential formula $A_c$. For any formula $B$ and co-edge $c$, we say that $B$ *depends on* $c$ when $c = (X, X^{\perp})$ and $X$ occurs either free or bound in $B$.

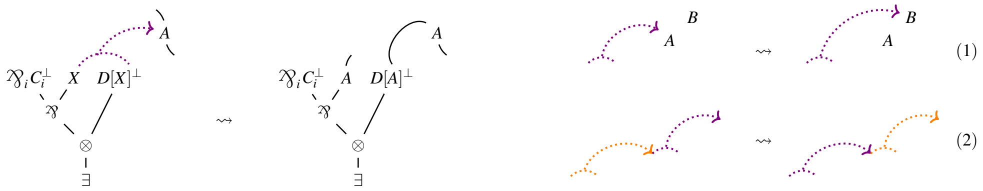
A *linking* of $\Gamma$ is a set of disjoint edges whose union contains all but the existential variables of $\Gamma$. A *witnessing function* over $\Gamma$ is an injective function $W : \Gamma^{\exists} \to n\Gamma$, associating any co-edge with a node of $\Gamma$. We will represent witnessing functions by using colored and dotted arrows, called *witness edges*, going from the two nodes of a co-edge $c$ to the formula $W(c)$. An $\exists$-*linking* over $\Gamma$ is a pair $\ell = (E, W)$, where $E$ is a linking over $\Gamma$ and $W$ is a witnessing function over $\Gamma$. Examples of $\exists$-linkings are shown in fig. 3a.

Given a witnessing function $W$, we let the *dependency graph of* $W$ be the directed graph $D_W$ with nodes the co-edges and arrows $c \to c'$ when $W(c)$ depends on $c'$. We call a witnessing



(a) $\sim$-equivalent $\exists$-linkings

(b) $\simeq_\varepsilon$-equivalent $G$-nets



(c) Expansion of a maximal co-edge

(d) Rewiring moves

Figure 3

function $W$ *acyclic* when the graph $D_W$ is directed acyclic. We call $\ell=(E,W)$ *acyclic* when $W$ is acyclic.

Acyclic $\exists$-linkings provide a compact representation of $G$-proof structures, since to an $\exists$-linking $\ell=(E,W)$ can be associated a unique $G$-proof structure $\pi(\ell)$. In particular, the acyclicity of $W$ allows to associate any $\exists$-link with a unique witness. $\pi(\ell)$ is constructed by repeatedly applying, to the graph $E\cup t\Gamma$, the co-edge expansion operation shown in fig. 3c, starting from co-edges which are maximal in $D_W$. An $\exists$-linking $\ell$ is *correct* when it is acyclic and $\pi(\ell)$ is a $G$-net.

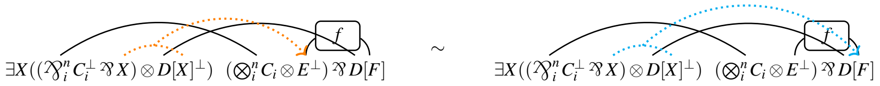
**Rewiring** We introduce an equivalence relation over correct $\exists$-linkings, called *rewiring* (as in [5, 16, 20]). Given a witnessing function $W$, a *simple rewiring of* $W$ is a witnessing function $W'$ obtained by either moving exactly one witness edge from one formula to another "free" one (i.e. to some formula $A$ such that $W^{-1}(A)=\emptyset$), or by switching two consecutive witness edges, i.e. two edges $c_1,c_2$ such that $W(c_1)\in c_2$, as shown in fig. 3d. We let $\ell\sim_1\ell'$ if $\ell=(E,W)$, $\ell'=(E,W')$ and $W'$ is a simple rewiring of $W$. We let $\sim$ be the reflexive and transitive closure of $\sim_1$.

In fig. 3a are shown all $\sim$-equivalent $\exists$-linkings over $\exists X((Y^\perp\parr X)\otimes X^\perp),\forall X((Y\otimes X^\perp)\parr X)$, corresponding to the two $\simeq_\varepsilon$-equivalent $G$-nets in fig. 3b. When $A$ is Yoneda in $X$, we let $ID_{\forall XA}$ denote the $\exists$-linking in figure 4a, Indeed, $\pi(ID_{\forall XA})$ is the $G$-net corresponding to the identity in $\mathbb{G}$.

We let $\mathbb{L}^\exists$ be the *category of $\exists$-linkings*, whose objects are the formulas of $MLL2_{\mathscr{Y}}$ and where $\mathbb{L}^\exists(A,B)$ is the set of $\sim$-equivalence classes of correct $\exists$-linkings of conclusions $A^\perp,B$, with composition given by cut-elimination (see appendix B). We let $\mathbb{L}^{\mathbf{1},\perp}$ be the restriction of $\mathbb{L}^\exists$ to $MLL2_{\mathbf{1},\perp}$ formulas.

**From $\exists$-linkings to *MLL* linkings** We extend the Yoneda translation of formulas into a translation $\ell\mapsto\ell_{\mathscr{Y}}$ from acyclic $\exists$-linkings over $\Gamma$ into "lax linkings" (in the sense of [20], p.22) over

(a) Identity ∃-linking (b) ∃-linking $\Omega^B_A$



(c) $\exists$ is a coend in $\mathbb{L}^\exists$

Figure 4

$\Gamma_{\mathscr{Y}}$. The linking $\ell_{\mathscr{Y}}$ is obtained in two steps: first, starting from co-edges which are minimal in $D_W$, replace $A_c = \exists X((⅋_i C_i ⅋ X)\otimes D[X^\perp])$ by $(A_c)_{\mathscr{Y}} = D[⅋_i C_i ⅋ \perp]$, add a lax thinning edge (in the sense of [20]) from the new occurrence of $\perp$ added to $W(c)$, and move all lax thinning edges pointing to $X, X^\perp$ onto $W(c)$; once all co-edges have been eliminated, replace any universal formula $\forall XA$ by $(\forall XA)_{\mathscr{Y}}$ and eliminate the unique edge $(X^\perp, X)$.

Observe that witness edges are replaced by lax thinning edges. In particular, the witness edges of the form $\exists X(X\otimes X^\perp)$ $A$ are replaced by thinning edges of the form $\perp$ $A$

By letting $\sim_{\mathscr{Y}}$ denote the rewiring equivalence over lax linkings, we have:

**Lemma 2.** *i. If $\ell$ is correct, then $\ell_{\mathscr{Y}}$ is correct.*
*ii. If $\ell \sim \ell' \Rightarrow \ell_{\mathscr{Y}} \sim_{\mathscr{Y}} \ell'_{\mathscr{Y}}$.*
*iii. If $\ell, \ell$ are in $MLL2_{\mathbf{1},\perp}$, then, $\ell_{\mathscr{Y}} \sim_{\mathscr{Y}} \ell'_{\mathscr{Y}} \Rightarrow \ell \sim \ell'$.*

By exploting lemma 2, proposition 6 and the results in [20] we get:

**Theorem 2.** *$\mathbb{L}^\exists$ is $^*$-autonomous. $\mathbb{L}^{\mathbf{1},\perp}$ is the free $^*$-autonomous category.*

## 5 Characterization of $\varepsilon$-equivalence

To any $G$-net $\pi$ we can associate an $\exists$-linking $\ell_\pi$ as follows: starting from the topmost $\exists$-links in $\pi$, if $A_c = \exists XA'$ with witness $B$, introduce a cut over $B, B^\perp$ and a witness edge $W(c) = B$. Now let $\ell_\pi$ be the normal form (see appendix B) of the resulting $\exists$-linking[6].

We let $\simeq_\ell$ be the equivalence relation over $G$-nets given by $\pi \simeq_\ell \pi'$ if $\ell_\pi \sim \ell_{\pi'}$.

**Theorem 3.** *$\pi \simeq_\varepsilon \pi'$ iff $\pi \simeq_\ell \pi'$.*

*Proof sketch:* ($\simeq_\varepsilon \subseteq \simeq_\ell$) As $\mathbb{L}^\exists$ is $^*$-autonomous, it suffices to show that $\exists$ is a coend in $\mathbb{L}^\exists$. For any $A = (⅋_i C_i ⅋ X)\otimes D[X^\perp]$ Yoneda in $X$ and any $B \in \mathscr{L}^2_{\mathscr{Y}}$, let $\Omega^B_A$ be the correct $\exists$-linking in fig. 4b. Given $A = (\bigotimes^n_i C_i \otimes X^\perp) ⅋ D[X]$, for any $E, F \in \mathscr{L}^2_{\mathscr{Y}}$ and $f \in \mathbb{L}^\exists(E,F)$, $\Omega^E_A \circ A(f,E)$ and $\Omega^F_A \circ A(F,f)$ differ by a unique rewiring, as shown in fig. 4c. We can then conclude:

**Proposition 2.** *For all $A$ Yoneda in $X$, the pair $(\exists XA^\perp, (\Omega^B_A)_{B\in\mathscr{L}^2_{\mathscr{Y}}})$ is a coend in $\mathbb{L}^\exists$.*

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[6] Since some rewirings might be needed to eliminate cuts, $\pi = \pi(\ell_\pi)$ does not hold general, but only $\pi \simeq_\varepsilon \pi(\ell_\pi)$ (as a consequence of prop. 3).

**Example 4.** *The "Yoneda isomorphism" of example 3 holds in* $\mathbb{L}^{\exists}$*, as the composition* $\ell_{Yo_1^A} \circ \ell_{Yo_2^A}$ *reduces to* $ID_{\forall XA}$ *(up to rewiring).*

$(\simeq_{\ell} \subseteq \simeq_{\varepsilon})$ Let $\mathbb{C}$ be $^*$-autonomous with coends. For each $\varphi : \mathtt{Var} \to \mathbb{C}$, $A^{\varphi}$ is isomorphic to $A^{\varphi}_{\mathscr{Y}}$ (by Yoneda). By lemma 2 and the bijection $\mathbb{C}(A^{\varphi}, B^{\varphi}) \simeq \mathbb{C}(A^{\varphi}_{\mathscr{Y}}, B^{\varphi}_{\mathscr{Y}})$ we get:

**Proposition 3.** *If* $\ell_{\pi} \sim \ell_{\pi'}$*, then* $\pi^{\mathbb{C}} = (\pi')^{\mathbb{C}}$.

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# A $^*$-autonomous categories and coends

We recall that a $^*$-autonomous category is a category $\mathbb{C}$ endowed with functors $\_\otimes\_ : \mathbb{C}^2 \to \mathbb{C}$ and $\_^\perp : \mathbb{C}^{op} \to \mathbb{C}$, an object $\mathbf{1}_{\mathbb{C}}$, the following natural isomorphisms:

$$\begin{aligned} \alpha_{a,b,c} &: a\otimes(b\otimes c) \to (a\otimes b)\otimes c \\ \lambda_a &: a\otimes \mathbf{1}_{\mathbb{C}} \to a \\ \rho_a &: \mathbf{1}_{\mathbb{C}}\otimes a \to a \\ \sigma_{a,b} &: a\otimes b \to b\otimes a \end{aligned}$$

and a natural bijection between $\mathbb{C}(a\otimes b, c)$ and $\mathbb{C}(a, c \parr b^\perp)$, where $a \parr b = (b^\perp \otimes a^\perp)^\perp$, satisfying certain coherence conditions (that we omit here, see [2]). In any $^*$-autonomous category $\mathbb{C}$ there is a natural isomorphism $A^{\perp\perp} \simeq A$. $\mathbb{C}$ is said *strict* when this isomorphism is an identity.

For the definition of multivariant functors and dinatural transformations the reader can look at [26]. When $F : (\mathbb{C}^{op}\otimes\mathbb{C})^{n+1} \to \mathbb{D}$ and the values $a_1,\dots,a_n \in Ob_{\mathbb{C}}$ are clear from the context, we will will often abbreviate $F((a_1,\dots,a_n,a),(a_1,\dots,a_n,b))$ as $F(a,b)$.

Given $\mathbb{C}$ $^*$-autonomous, for all $a \in Ob_{\mathbb{C}}$, there exist dinatural transformations $\hat{\mathbf{1}}_x : \mathbf{1}_{\mathbb{C}} \to x^\perp \parr x$ and $\hat{\perp}_x = \hat{\mathbf{1}}_x^\perp : x\otimes x^\perp \to \perp_{\mathbb{C}}$, where $\perp_{\mathbb{C}} := \mathbf{1}_{\mathbb{C}}^\perp$.

Given categories $\mathbb{C},\mathbb{D}$ and a multivariant functor $F : (\mathbb{C}^{op}\otimes\mathbb{C})^{n+1} \to \mathbb{D}$, an *end*[7] (dually, a *coend*, see [26]) is a pair $(\int_x F, \delta_{x_1,\dots,x_n,a})$ (resp. $\int^x F, \omega_{x_1,\dots,x_n,a})$[8] made of a functor $\int_x F : (\mathbb{C}^{op}\otimes\mathbb{C})^n \to \mathbb{D}$ and a universal dinatural transformation $\delta_a : \int_x F(x,x) \to F(a,a)$ (resp. $\omega_a : F(a,a) \to \int^x F(x,x)$) natural in $x_1,\dots,x_n$. This means that for any functor $G : (\mathbb{C}^{op}\otimes\mathbb{C})^n \to \mathbb{D}$ and dinatural transformation $\theta_a : G \to F(a,a)$ (resp. $\theta_a : F(a,a) \to G$) there exists a unique natural transformation $h : G \to \int_x F(x,x)$ (resp. $k : \int^x F(x,x) \to G$) such that the following diagrams commute for all $f \in \mathbb{C}(a,b)$:

$$\begin{tikzcd} G \arrow[dr, "h"] \arrow[drr, bend left, "\theta_a"] \arrow[ddr, bend right, "\theta_b"'] & & \\ & \int_x F(a,a) \arrow[r, "\delta_a"] \arrow[d, "\delta_b"] & F(a,a) \arrow[d, "F(a,f)"] \\ & F(b,b) \arrow[r, "F(f,b)"'] & F(a,b) \end{tikzcd} \qquad \begin{tikzcd} F(b,a) \arrow[r, "F(f,a)"] \arrow[d, "F(b,f)"] & F(a,a) \arrow[d, "\omega_a"] \arrow[ddr, bend left, "\theta_a"] & \\ F(b,b) \arrow[r, "\omega_b"] \arrow[drr, bend right, "\theta_b"] & \int^x F(x,x) \arrow[dr, "k"] & \\ & & G \end{tikzcd}$$

We let $\mathbb{C}$ be a $^*$-autonomous category in which ends (hence, by duality, coends) exist. We recall some basic facts about coends (see [26, 25]):

**Proposition 4** (Yoneda Lemma for coends). *Given $n \geq 0$, functors $F_1,\dots,F_n$ and a covariant functor $G(x)$, $\int_x((\bigotimes_i^n F_i \otimes x^\perp) \parr G(x))$ (resp. $\int^x((\bigparr_i^n F_i \parr x) \otimes G^\perp(x))$) is isomorphic to $G \circ (\bigotimes_i^n F \otimes \mathbf{1}_{\mathbb{C}})$ (resp. $G^\perp \circ (\bigparr_i^n F_i \parr \perp_{\mathbb{C}})$). In particular, $\int_x x^\perp \parr x \simeq \mathbf{1}_{\mathbb{C}}$ and $\int^x x\otimes x^\perp \simeq \perp_{\mathbb{C}}$.*

**Proposition 5** (commutation of $\int_x / \int^x$ and $\parr$). *Given a functor $F$ and a multivariant functor $G(x,y)$, there exist natural transformations $\mu : \int_x (F \parr G(x,x)) \to F \parr \int_x G(x,x)$ and $\nu : \int^x (F \parr G(x,x)) \to F \parr \int^x G(x,x)$.*

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[7] We give here a functorial definition of ends and coends which can be easily deduced from the usual definition (see [26]).

[8] We will abbreviate $\delta_{x_1,\dots,x_n,a}$ and $\omega_{x_1,\dots,x_n,a}$ simply as $\delta_a$ and $\omega_a$, respectively.

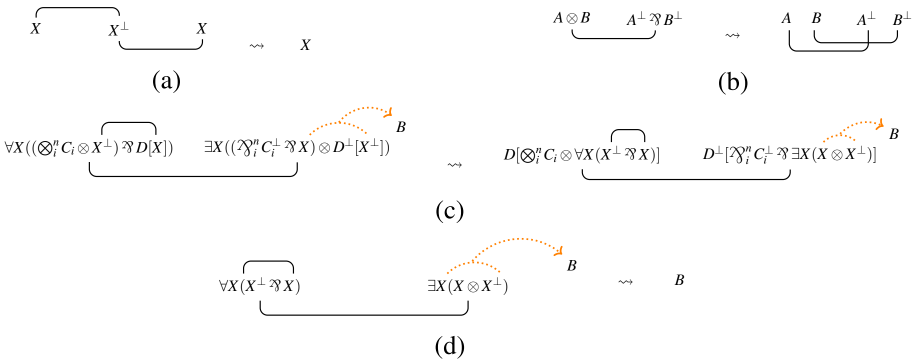

Figure 5: Cut elimination

## B Cut-elimination

We let a *cut sequent* be a sequent of the form $\Gamma,[\Delta]$, where $\Gamma,\Delta$ is a clean sequent and $\Delta$ is a multiset of formulas of the form $A\otimes A^{\perp}$ (corresponding to a configuration of the form $A \smile A^{\perp}$ in the parse forest).

By an $\exists$-linking (resp. a correct $\exists$-linking) over $\Gamma,[\Delta]$ we indicate an $\exists$-linking (resp., a correct $\exists$-linking) over $\Gamma,\Delta$. We call an $\exists$-linking $\ell=(E,W)$ *ready* when $W^{-1}(A)=\emptyset$ for all $A$ occurring in a cut-formula.

**Lemma 3.** *For any correct $\exists$-linking $\ell$ there exists a ready $\ell'$ such that $\ell'\sim\ell$.*

By lemma 3 it suffices to apply cut-elimination to ready $\exists$-linking. *Cut reduction* is the relation over ready $\exists$-linkings defined by the rewrite rules in figure 5, where in case 5c either $n\geq 1$ or $D[X]\neq X$, and, in case 5c and 5d the existence of the lefthand edge is forced by the fact that $\Gamma,\Delta$ is clean.

The Yoneda translation is extended straightforwardly to $\exists$-linkings with cuts. The following can be verified by inspecting the reduction steps.

**Proposition 6.** *Given acyclic $\exists$-linkings $\ell,\ell'$, if $\ell$ reduces to $\ell'$, then $\ell_{\mathscr{Y}}$ reduces to $\ell'_{\mathscr{Y}}$.*

We now verify usual properties of cut-elimination.

**Lemma 4** (confluence)**.** *Cut reduction is confluent.*

**Proposition 7** (stability)**.** *Let $\ell=(E,T)$ be a correct and ready $\exists$-linking over a sequent with cuts $\Gamma,[\Delta,A\otimes A^{\perp}]$. If $\ell\rightsquigarrow\ell'$, then $\ell'$ is correct.*

Strong normalization can be proved in a direct way, without reducibility candidates techniques.

**Proposition 8** (strong normalization)**.** *Let $\ell$ be a correct and ready $\exists$-linking over $\Gamma,[\Delta]$. Then all cut-reductions of $\ell$ terminate over a unique correct $\exists$-linking $nf(\ell)$ over $\Gamma$, called the* normal form *of $\ell$.*

By proposition 8 any correct $\exists$-linking has a unique normal form, up to rewiring.