On invariance and reachability on semialgebraic sets for linear dynamics

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Abstract: Reachability analysis is a powerful tool which is being used extensively and efficiently for the analysis and control of dynamical systems, especially when linear systems and convex sets are involved. In this note, we investigate whether exact or approximate reachability operations can be performed efficiently for the affine–semialgebraic setting, that is when we are dealing with general affine dynamics and basic semialgebraic sets. We show that it is partially true, we pinpoint the underlying challenges when this is not possible and indicate some directions for this case.

Keywords: Invariance, Reachability, Discrete-time systems, Liftings, Semialgebraic sets.

1. INTRODUCTION

We study the recursive application of reachability maps to construct invariant sets for discrete-time systems whose dynamics has a linear dependence on the state, input and disturbances. We consider the system Σ to have the general form

$$x(t+1) = A(\lambda(t))x(t) + B(\lambda(t))u(t) + w(t). \qquad (1)$$

In (1), $x(t) \in \mathbb{R}^n$ is the state variable, $u(t) \in \mathcal{U} \subseteq \mathbb{R}^m$ is the input signal, $w(t) \in \mathcal{W} \subset \mathbb{R}^n$ is a disturbance signal, where both $\mathcal{U}$ and $\mathcal{W}$ are compact sets. The vector $\lambda \in \mathcal{L}$ corresponds to the uncertainties of the system; when λ takes values from a discrete, finite set, system (1) becomes a switching system, whose switching pattern may be arbitrary, state-dependent or it may be constrained to follow paths in an automaton, see, e.g., Liberzon (2003), Shorten et al. (2007). If $\lambda \in \mathcal{L} \subset \mathbb{R}^{n_\lambda}$, then the system (1) becomes a parameter varying system, see, e.g., Toth (2010).

Reachability analysis generates knowledge about the behaviour of a system in a subset of the state space. Forward reachability provides information on where a set of initial states will be in the future: given the system Σ and a set $\mathcal{S} \subset \mathbb{R}^n$, the one-step forward reachability map is

$$\mathcal{F}(\Sigma, \mathcal{S}) = \{A(\lambda)x+B(\lambda)u+w : (x, u, \lambda, w) \in \mathcal{S}\times\mathcal{U}\times\mathcal{L}\times\mathcal{W}\}. \qquad (2)$$

On the other hand backward reachability proceeds inversely in time. Given the system Σ and a set $\mathcal{S} \subset \mathbb{R}^n$, the one-step backward reachability map is

$$\mathcal{B}(\Sigma, \mathcal{S}) = \{x : (\exists u \in \mathcal{U} : \forall(\lambda, w) \in \mathcal{L} \times \mathcal{W}, \; A(\lambda)x + B(\lambda)u + w \in \mathcal{S})\}. \qquad (3)$$

Among others, reachability analysis has been used to construct invariant sets, assess stability, safety and design stabilizing controllers, see e.g. Blanchini and Miani (2008) for the linear case and Aubin et al. (2011) for general nonlinear dynamics. When the involved sets $\mathcal{S}, \mathcal{U}, \mathcal{L}, \mathcal{W}$ are polytopic (or unions of polytopes) and the dependence of the pair (A, B) on λ is linear, there exist algorithmic procedures to calculate the reachability mappings efficiently, at least in moderate state space dimensions. Consequently, one can compute efficiently invariant sets as fixed points of appropriately initialized set sequences, that are generated by iterative application of forward and backward reachability maps. See, for example, the viability kernel algorithm Aubin et al. (2011), the backward reachability algorithm Blanchini and Miani (2008) for the largest controlled invariant set, the forward reachability algorithm for the minimal invariant set and its approximations Rakovic et al. (2005), Kolmanovsky and Gilbert (1998)).

Most available practicable approaches consider either systems of a linear type and polytopes (e.g., Kvasnica et al. (2004), Herceg et al. (2013)) or nonlinear systems and/or non polytopic sets using approximative, however scalable, approaches (a non-exhaustive list of relevant works includes Asarin et al. (2003), Frehse et al. (2011), Kurzhanski and Varaiya (2007), Althoff et al. (2008), Mitchell et al. (2005) Sloth et al. (2012)). In this paper, we consider systems of the type (1), and we explore whether we can still compute efficiently reachability maps when the involved sets are basic [1] semialgebraic sets. These sets can be seen as a generalization of polyhedral sets, and appear in several settings in control. For example, quadratic control Lyapunov functions and sliding mode control induce state-dependent switching on semialgebraic regions Geromel and Colaneri (2006), Buisson and Richard (2005). Additionally, constraint sets in power conversion are also semialgebraic, while such sets can be used to describe nonconvex, even non-connected environments in obstacle avoidance and path planning formulations, see e.g., Belta et al. (2005).

Recently (Athanasopoulos and Jungers (2016a), Athanasopoulos and Jungers (2016b)) we investigated whether reachability operations, and consequently invariant set constructions, were efficient for special cases of (1), namely, arbitrarily switching linear and switching affine systems. The approach involves the lifting of the system and constraints in a higher dimensional space, induced by the so-called Veronese embedding. This lift has been used extensively in stability analysis Parrilo and Jadbabaie (2008), Zelentsovsky (1994), Papachristodoulou and Prajna (2005) and more generally in Sum of Squares pro-

[1] For convenience, we drop the term basic hereafter.

gramming, e.g., Parrilo (2000), Prajna et al. (2002). It allows, among others, to propose polynomial Lyapunov functions and solve effectively the corresponding algebraic decrease conditions, thus, leading to less conservative stability criteria and approximations of the domain of attraction.

Our approach is illustrated in Figure 1 and consists of three steps. First, the system is lifted in an N-dimensional space, $N > n$, where the transformed constraint set can be described by the intersection of the Veronese surface and a polyhedron. Second, the reachability sequences of the lifted system are constructed, which under suitable assumptions converge to a polytopic invariant set $\mathcal{S}_{\mathcal{Y}}$. Last, the computed set is projected by a simple operation to the original state space. The projected set retains the invariance and maximality/minimality properties.

$$\begin{tikzcd} (\Sigma, \mathbb{R}^n) \arrow[d, "\text{Lift}"'] & \mathcal{S} \subset \mathbb{R}^n \\ (\Sigma_{\mathcal{Y}}, \mathbb{R}^N) \arrow[r, "\text{Reachability}"] & \mathcal{S}_{\mathcal{Y}} \subset \mathbb{R}^N \arrow[u, "\text{Lower}"'] \end{tikzcd}$$

Fig. 1. The idea behind the proposed approach.

Contributions: The paper is divided in three parts; first, the results obtained in Athanasopoulos and Jungers (2016b) are extended (i) by dealing with *state-dependent switching on semialgebraic surfaces* and (ii) by *characterizing the relation between the reachability maps* of the system and its lifted counterpart. The second part studies the case when the uncertainty vector λ belongs to a set $\mathcal{L} \subset \mathbb{R}^{n_\lambda}$. The challenge in this case is the *nonlinear dependence* of the lifted system on the uncertainty vector, making reachability analysis intractable. We propose overapproximations of the uncertainty set and establish the relationship between the reachability mappings of the system and the approximating lifted system. The third part studies systems with inputs that take values from a compact set $\mathcal{U} \subset \mathbb{R}^m$. The *lifted system dynamics becomes polynomial with respect to the input vector*, however it remains linear with respect to the states. We propose to construct large controlled invariant sets, borrowing the idea of sequential, directional enlargement of property-preserving invariant sets of, e.g., Athanasopoulos et al. (2014a), Athanasopoulos et al. (2014b).

2. PRELIMINARIES

First, we provide the definitions of an invariant set.

Definition 1. A set $\mathcal{S} \subseteq \mathcal{X}$ is called an admissible robust controlled invariant with respect to the system (1) and the constraint sets $\mathcal{X} \subset \mathbb{R}^n, \mathcal{U} \subset \mathbb{R}^m$, if $x(0) \in \mathcal{S}$ implies that for all admissible $\lambda(0) \in \mathcal{L}$ and $w(0) \in \mathcal{W}$ there exists an admissible input $u(0) \in \mathcal{U} \subset \mathbb{R}^m$ such that $x(1) \in \mathcal{S}$.

In the absence of input signals, we can define the corresponding notion of *admissible robust positively invariant* sets. For simplicity, we refer to the aforementioned sets as controlled invariant and invariant respectively. Of special importance are the minimal and maximal invariant sets.

Definition 2. The set $\mathcal{S}_{\max}$ is called the maximal (controlled) invariant set with respect to the system (1) and the constraint sets $\mathcal{X} \subset \mathbb{R}^n$, $\mathcal{U} \subseteq \mathbb{R}^m$, if it is (controlled) invariant and contains any other (controlled) invariant set. The set $\mathcal{S}_{\min}$ is called the minimal (controlled) invariant set with respect to the system (1) and the constraint sets $\mathcal{X} \subset \mathbb{R}^n, \mathcal{U} \subseteq \mathbb{R}^m$ if it is (controlled) invariant and is included in any other (controlled) invariant set.

Assumption 1. The sets $\mathcal{X} \subset \mathbb{R}^n, \mathcal{U} \subset \mathbb{R}^m$, $\mathcal{W} \subseteq \mathbb{R}^n$ contain the origin.

We can characterize the minimal and maximal (controlled) invariant sets with the set sequences generated by the forward and backward reachability maps respectively, see, e.g., Blanchini and Miani (2008) for systems of the type (1).

Theorem 1. Consider the system Σ (1), the constraint sets $\mathcal{X}, \mathcal{U}$ and the set sequences $\{\mathcal{R}_i\}$, $\{\mathcal{C}_i\}$, generated by $\mathcal{R}_0 = \{0\}$, $\mathcal{R}_{i+1} = \mathcal{F}(\mathcal{R}_i, \Sigma)$ and $\mathcal{C}_0 = \mathcal{X}$ $\mathcal{C}_{i+1} = \mathcal{B}(\mathcal{C}_i, \Sigma) \cap \mathcal{X}$. Then, (i) $\mathcal{S}_{\min} = \lim\limits_{j\to\infty} \cup_{i=0}^{j} \mathcal{R}_i$, (ii) $\mathcal{S}_{\max} = \lim\limits_{j\to\infty} \cap_{i=0}^{j} \mathcal{C}_i$.

For special types of (1), the above results can be further refined with sufficient (and sometimes necessary) finite termination conditions. In the remaining of the section, we consider general affine dynamics and omit the possible dependencies on the time variable or another signal to present in a simple manner the subsequent algebraic manipulations.

2.1 Lifting of affine systems

Let us consider systems updated by the rule

$$x^+ = Cx + e, \tag{4}$$

where $x \in \mathbb{R}^n$, $C \in \mathbb{R}^{n\times n}$ and $e \in \mathbb{R}^n$. We note that the matrix C and vector e can be constant, functions of a discrete switching signal or a continuous signal, and they can depend on a time-varying parameter. We construct the lifted system dynamics, induced by the state-space transformation induced by the monomials of x of a maximum degree $d > 1$. To this purpose, given a n-tuple $\alpha \in \mathbb{N}^n$, the α monomial of a vector $x \in \mathbb{R}^n$ is $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$. The degree of the monomial is $d = \sum_{i=1}^n \alpha_i$. We denote by $\alpha!$ the multinomial coefficient $\alpha! = \frac{d!}{\alpha_1! \dots \alpha_n!}$.

Definition 3. (Vector and matrix d-lift, Parrilo and Jadbabaie (2008), Jungers (2009)). Given a vector $x \in \mathbb{R}^n$ and an integer $d \geq 1$, the d-lift of x, denoted by $x^{[d]}$, is the vector in $\mathbb{R}^{\binom{n+d-1}{d}}$, having as elements all the exponents α of degree d, i.e,. $x_\alpha = \sqrt{\alpha!}x^\alpha$. Given $C \in \mathbb{R}^{n\times n}$ and an integer $d \geq 1$, the d-lift of the matrix C is $C^{[d]} \in \mathbb{R}^{\binom{n+d-1}{d}\times\binom{n+d-1}{d}}$, associated to the linear map $C^{[d]} : x^{[d]} \to (Cx)^{[d]}$.

We can obtain a numerical expression of the entries of $C^{[d]}$ (Parrilo and Jadbabaie (2008)), with the formula $C^{[d]}_{\alpha\beta} = \frac{\mathrm{per}(C(\alpha,\beta))}{\sqrt{\mu(\alpha)\mu(\beta)}}$,, where the indices α, β are all the d-element multisets of $\{1, ..., n\}$, $\mu(\alpha)$ is the product of the factorials of the multiplicities of the elements of the multiset α and $\mathrm{per}(C)$ is the permanent of the matrix $C \in \mathbb{R}^{n\times n}$.

Lemma 1. For a matrix $A \in \mathbb{R}^{n\times n}$, a vector $x \in \mathbb{R}^n$, it holds (i) $(Cx)^{[d]} = C^{[d]}x^{[d]}$ and (ii) $\begin{bmatrix} x \\ 1 \end{bmatrix}^{[d]} = \begin{bmatrix} x^{[d!]} \\ 1 \end{bmatrix}$, where $x^{[d!]} = T \left[x^{[d]\top} \; x^{[d-1]\top} \; \cdots \; x^{[1]\top}\right]^\top$, with $T = \mathrm{diag}\{T_d, ..., T_1, 1\}$, $T_i = \mathrm{diag}\{t_i, ..., t_i\}$, $t_i = \sqrt{\frac{\prod_{j=i+1}^d j}{(d-i)!}}$, and $x^{[1]} = x$.

Definition 4. (Lifted system). Consider the discrete-time system (1) and an integer d. The system

$$y^+ = h(y, u, \lambda, w),$$

$$A := \begin{bmatrix} C^{[d]} & a_{1,2}(C^{[d-1]}, e) & \dots & a_{1,d-1}(C^{[2]}, e^{[d-2]}) & a_{1,d}(C, e^{[d-1]}) \\ 0 & C^{[d-1]} & \dots & a_{2,d-1}(C^{[2]}, e^{[d-3]}) & a_{2,d}(C, e^{[d-2]}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & C^{[2]} & a_{d-1,d}(C, e) \\ 0 & 0 & \dots & 0 & C \end{bmatrix}, \quad b := e^{[d!]}. \tag{5}$$

$h : \mathbb{R}^{\binom{n+d}{d}-1} \times \mathbb{R}^m \times \mathbb{R}^{n_\lambda} \times \mathbb{R}^n \to \mathbb{R}^{\binom{n+d}{d}-1}$ is called the lifted system of (1) if for any $x(0) \in \mathbb{R}^n$ and $y(0) = (x(0))^{[d!]}$ and any set of sequences $\{u(0), u(1), ...\}$, $\{\lambda(0), \lambda(1), ...\}$, and $\{w(0), w(1), ...\}$ it holds that $y(t) = (x(t))^{[d!]}$, for all $t \geq 0$.

Lemma 2. Consider the system (4) and an integer $d > 1$. Then, the lifted system of (4) is

$$y^+ = Ay + b, \tag{6}$$

where A, b are given in (5) and the elements $a_{ij}(C^{[l]}, e^{[k]})$, $(i, j, k, l) \in \{1, d-1\} \times \{2, d\} \times \{1, d-1\} \times \{1, d-1\}$, are blocks in A depending on the elements of $C^{[l]}$ and $e^{[k]}$.

Proof We bring the dynamics of system (4) in the augmented form $z := \begin{bmatrix} Cx + e \\ 1 \end{bmatrix} = \begin{bmatrix} C & e \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$. From Lemma 1, we have $z^{[d]} = \begin{bmatrix} (Cx + d)^{[d!]} \\ 1 \end{bmatrix} = \begin{bmatrix} C & e \\ 0 & 1 \end{bmatrix}^{[d]} \begin{bmatrix} x^{[d!]} \\ 1 \end{bmatrix}$. The form (5) follows by taking the first $\binom{n+d}{d} - 1$ rows of $z^{[d]}$ and arranging accordingly. ■

2.2 *Lifting and lowering*

In general, for any set $\mathcal{S} \subseteq \mathbb{R}^n$ and integer $d > 1$, its representation in the lifted space is $\mathcal{S}^{[d!]} \subset \mathbb{R}^N$, $N = \binom{n+d}{d} - 1$, where

$$\mathcal{S}^{[d!]} = \{x^{[d!]} \in \mathbb{R}^N : x \in \mathcal{S}\}.$$

We consider sets $\mathcal{X} \subseteq \mathbb{R}^n$ of the form

$$\mathcal{X} = \{x \in \mathbb{R}^n : g_i(x) \geq 0, i = 1, ..., p\}, \tag{7}$$

where $g_i(\cdot) : \mathbb{R}^n \to \mathbb{R}$, $i = 1, ..., p$ are functions of degree $d \geq 1$. The set (7) in the space of monomials of x is

$$\mathcal{X}^{[d!]} = \{x^{[d!]} \in \mathbb{R}^N : \overline{g}_i^\top x^{[d!]} \leq \overline{w}_i, i = 1, ..., p\},$$

where $\overline{g}_i \in \mathbb{R}^N$, are vectors having as elements the coefficients of the monomials appearing in $-g_i(x)$, while $\overline{w}_i$, $i = 1, ..., p$, corresponds to the constant term of $-g_i(x)$, $i = 1, ..., p$.

Definition 5. (Lifted Set). Consider the semialgbraic set $\mathcal{X}$, defined in (7). The *lifted set* of $\mathcal{X}$ is

$$\mathcal{Y} := \{y \in \mathbb{R}^N : \overline{g}_i^\top y \leq \overline{w}_i, i = 1, ..., p\}. \tag{8}$$

It is clear that $\mathcal{Y}$ is a polyhedron. We denote the Veronese surface, i.e., the manifold of the monomials of x in $\mathbb{R}^N$ by

$$\mathcal{V} = \{y \in \mathbb{R}^N : \exists x \in \mathbb{R}^n : y = x^{[d!]}\}.$$

Last, we define the reverse operation. To this purpose, given an integer $d > 1$ and a set $\mathcal{Y} \subset \mathbb{R}^{\binom{n+d}{d}-1}$, the lowering operation is given by

$$\mathrm{lower}(\mathcal{Y}) = \{x \in \mathbb{R}^n : (\exists y \in \mathcal{Y} : y = x^{[d!]})\}.$$

When $\mathcal{Y}$ is a polyhedron, lowering requires only to match the coefficient of each linear inequality that defines $\mathcal{Y}$ with the monomials of x. We summarize several elementary properties of the lifting and lowering operations below.

Fact 1. Consider any two semialgebraic sets $\mathcal{X}_1 \subseteq \mathbb{R}^n, \mathcal{X}_2 \subseteq \mathbb{R}^n$, an integer $d > 1$ and the corresponding lifted sets $\mathcal{Y}_1 \subseteq \mathbb{R}^N$, $\mathcal{Y}_2 \subseteq \mathbb{R}^N$, $N = \binom{n+d}{d} - 1$. Let $\star$ denote the operation of either union or intersection, i.e., $\star \in \{\cup, \cap\}$. The following hold.

(i) $\mathrm{lower}(\mathcal{Y}_1 \star \mathcal{Y}_2) = \mathrm{lower}(\mathcal{Y}_1) \star \mathrm{lower}(\mathcal{Y}_2)$,
(ii) $\mathrm{lower}(\mathcal{Y}_1) = \mathrm{lower}(\mathcal{Y}_1 \cap \mathcal{V})$,
(iii) $(\mathcal{X}_1 \star \mathcal{X}_2)^{[d!]} = \mathcal{X}_1^{[d!]} \star \mathcal{X}_2^{[d!]}$,
(iv) $\mathcal{X}_1 \subseteq \mathcal{X}_2 \Rightarrow \mathcal{Y}_1 \subseteq \mathcal{Y}_2$.
(v) Consider a system Σ (1) and the corresponding lifted system $\Sigma_\mathcal{Y}$. If $\mathcal{Y}_1$ is (controlled) invariant with respect to $\Sigma_\mathcal{Y}$ then $\mathcal{X}_1$ is (controlled) invariant with respect to Σ.

Proof We show only item (v). We observe that $\mathrm{lower}(\mathcal{Y}_1) = \mathrm{lower}(\mathcal{Y}_1 \cap \mathcal{V})$. Since $\mathcal{V}$ is (controlled) invariant by construction, it follows that $\mathcal{Y}_1 \cap \mathcal{V}$ is (controlled) invariant as well with respect to $\Sigma_\mathcal{Y}$. The result follows directly.

3. SYSTEMS THAT PRESERVE LINEARITY THROUGH LIFTING

In this section, we extend the setting of Athanasopoulos and Jungers (2016a) and Athanasopoulos and Jungers (2016b) by allowing state-dependent switching, and also by investigating the relations between reachability operations of the original and the lifted system. To this purpose we consider the system Σ^{sd} subject to the state constraints $\mathcal{X}$ (7), of the form

$$x^+ = C_i x + e_i, \quad x \in \mathcal{C}_i, \tag{9}$$

where $C_i \in \mathbb{R}^{n \times n}$, $e_i \in \mathbb{R}^n$, $i = 1, ..., M$ and $\mathcal{C}_i$ are semialgebraic sets

$$\mathcal{C}_i = \{x \in \mathbb{R}^n : f_{i,j}(x) \leq 0, j = 1, ..., p_i\},$$

such that $\cup_{i=1}^M \mathcal{C}_i = \mathcal{X}$. This setting can be found in the literature, for instance in Lyapunov based state-dependent switching control Geromel and Colaneri (2006) and in sliding mode control Buisson and Richard (2005). Given an integer $d > 1$, we define the lifted system $\Sigma_\mathcal{Y}^{\mathrm{sd}}$ to be

$$y^+ = A_i y + b_i, \quad y \in \mathcal{Z}_i, \tag{10}$$

where A_i, b_i are generated as in (5) for each $i = 1, ..., M$. The sets $\mathcal{Z}_i = \{y \in \mathbb{R}^N : \overline{f}_{i,j}^\top y \leq w_{i,j}, j = 1, ..., p_i\}$ are the corresponding lifted sets generated by the coefficients of the monomials of $f_{i,j}(x)$ of $\mathcal{C}_i$, $i = 1, ..., M$, $j = 1, ..., p_i$. We observe that $\mathcal{Z}_i$, $i = 1, ..., M$ are polyhedra.

Proposition 1. Consider the system (9), the state constraint set $\mathcal{X} \subset \mathbb{R}^n$ (7), a set $\mathcal{S} \subset \mathbb{R}^n$, an integer $d > 1$ and the corresponding lifted set $\mathcal{S}_\mathcal{Y} \subset \mathbb{R}^N$, $N = \binom{n+d}{d} - 1$. The following hold.

(i) Assume C_i, $i = 1, ..., M$ are nonsingular. Then, it holds that $\mathcal{F}(\Sigma^{\mathrm{sd}}, \mathcal{S}) = \mathrm{lower}(\mathcal{F}(\Sigma_\mathcal{Y}^{\mathrm{sd}}, \mathcal{S}_\mathcal{Y}))$.
(ii) $\mathcal{B}(\Sigma^{\mathrm{sd}}, \mathcal{S}) = \mathrm{lower}(\mathcal{B}(\Sigma_\mathcal{Y}^{\mathrm{sd}}, \mathcal{S}_\mathcal{Y}))$.

Proof **(i)** We have by Fact 1(i), (ii), that

$$\mathrm{lower}(\mathcal{F}(\Sigma_\mathcal{Y}^{\mathrm{sd}}, \mathcal{S}_\mathcal{Y})) = \mathrm{lower}(\cup_{i=1}^M \{A_i y + b_i : y \in \mathcal{S}_\mathcal{Y} \cap \mathcal{Z}_i\})$$
$$= \cup_{i=1}^M \mathrm{lower}(\{A_i y + b_i \in \mathcal{V} : y \in \mathcal{S}_\mathcal{Y} \cap \mathcal{Z}_i\}).$$

We show that for any $i \in \{1,..,M\}$ and any $y \in \mathbb{R}^N$ such that $A_i y + b_i \in \mathcal{V}$ it necessarily follows that[2] $y \in \mathcal{V}$. Since $A_i y + b_i \in \mathcal{V}$, there is a vector $z \in \mathbb{R}^n$ such that $A_i y + b_i = z^{[d!]}$. Let $x \in \mathbb{R}^n$ such that $C_i x + e_i = z$. Consequently, $(C_i x + e_i)^{[d!]} = A_i y + b_i$, or, $A_i(x^{[d!]} - y) = 0$, or $x^{[d!]} - y$ belongs to the nullspace of A_i. Since A_i is block diagonal, its eigenvalues are equal to the union of the sets of eigenvalues of the matrices $C_i^{[j]}$, $j = 1, ..., d$. Since all $C_i^{[j]}$ are invertible, see e.g., Parillo and Jadbabaie (2008), A_i is invertible as well. Consequently, $y = x^{[d!]}$ and $y \in \mathcal{V}$. Thus, we have

$$
\begin{aligned}
&\mathrm{lower}(\mathcal{F}(\Sigma^{\mathrm{sd}}_{\mathcal{Y}}, \mathcal{S}_{\mathcal{Y}})) \\
&= \cup_{i=1}^{M} \mathrm{lower}(\{A_i y + b_i \in \mathcal{V} : y \in (\mathcal{S}_{\mathcal{Y}} \cap \mathcal{V}) \cap (\mathcal{Z}_i \cap \mathcal{V})\}) \\
&= \cup_{i=1}^{M} \mathrm{lower}(\{A_i y + b_i : (\exists x \in \mathcal{S} \cap \mathcal{C}_i : y = x^{[d!]}\}) \\
&= \cup_{i=1}^{M} \mathcal{F}(\Sigma_i, \mathcal{S} \cap \mathcal{C}_i)) = \mathcal{F}(\Sigma^{\mathrm{sd}}, \mathcal{S}),
\end{aligned}
$$

where Σ_i accounts for the system $x^+ = C_i x + e_i$.

(ii) Fom Fact 1(i), (ii) and the fact that $\mathcal{V}$ is by construction invariant with respect to (10), we have that

$$
\begin{aligned}
&\mathrm{lower}(\mathcal{B}(\Sigma^{\mathrm{sd}}_{\mathcal{Y}}, \mathcal{S}_{\mathcal{Y}})) = \mathrm{lower}(\cup_{i=1}^{M} \{y \in \mathcal{Z}_i : A_i y + b_i \in \mathcal{S}_{\mathcal{Y}}\}) \\
&= \mathrm{lower}(\cup_{i=1}^{M} \{y \in \mathcal{Z}_i \cap \mathcal{V} : A_i y + b_i \in \mathcal{S}_{\mathcal{Y}} \cap \mathcal{V}\}) \\
&= \mathrm{lower}(\cup_{i=1}^{M} (\mathcal{B}(\Sigma_i, \mathcal{S}) \cap \mathcal{C}_i)) = \mathcal{B}(\Sigma^{\mathrm{sd}}, \mathcal{S}). \blacksquare
\end{aligned}
$$

An immediate corollary from Proposition1 and Theorem 1 is that $\mathcal{S}_{\max} = \mathrm{lower}(\mathcal{S}_{\max,\mathcal{Y}})$, where $\mathcal{S}_{\max}$ and $\mathcal{S}_{\max,\mathcal{Y}}$ are the maximal invariant sets of (9) and (10) respectively.

Proposition 1 suggests that reachability analysis is possible by set operations involving (unions of) polyhedral sets. This is a significant advantage since one can perform efficiently reachability operations on piecewise affine systems using standard tools, e.g., Rakovic et al. (2004). Dealing with the case of switching affine systems under constrained switching is also possible by extending Proposition 1 in the direction of multi-set invariance (Athanasopoulos et al. (2017)).

4. UNCERTAIN SYSTEMS

In this section we focus on the system Σ^λ of the form

$$
x^+ = C(\lambda)x + e(\lambda), \tag{11}
$$

where

$$
C(\lambda) = \sum_{i=1}^{n_\lambda} \lambda_i C_i, \quad e(\lambda) = \sum_{i=1}^{n_\lambda} \lambda_i e_i,
$$

and $C_i \in \mathbb{R}^{n \times n}$, $e_i \in \mathbb{R}^n$, $i = 1, ..., n_\lambda$. The variable λ takes values from a compact semialgebraic set, i.e., $\lambda \in \mathcal{L}$, where

$$
\mathcal{L} = \{\lambda \in \mathbb{R}^{n_\lambda} : l_{i,j}(\lambda) \geq 0, i = 1, ..., p_j, j = 1, .., d_\lambda\}, \tag{12}
$$

where d_λ denotes the maximum degree of monomials appearing in the description of $\mathcal{L}$. For example, there are p_1 linear inequalities $l_{i,1}(\lambda) \geq 0$. This setting covers several uncertainty types, e.g., polytopic uncertainties with $\mathcal{L} = \{\lambda \in \mathbb{R}^{n_\lambda} : \lambda \geq 0, \sum_{i=1}^{n_\lambda} \lambda_i = 1\}$ and ellipsoidal uncertainties with $\mathcal{L} = \left\{\lambda \in \mathbb{R}^n : \|\lambda\|_\infty \leq 1, \sum_{i=1}^{n} \lambda_i^2 = 1\right\}$. We consider an integer $d > 1$ and the corresponding lifted system $\Sigma^\lambda_{\mathcal{Y}}$

$$
y^+ = A(\lambda)x + b(\lambda). \tag{13}
$$

The pair $(A(\lambda), b(\lambda))$ has a polynomial dependence on the uncertainty vector λ. Specifically, it is a function of monomials of λ up to a maximum degree d. First, we identify the interrelation between reachability operations on the system and the lifted system.

Proposition 2. Consider the system Σ^λ (11), a set $\mathcal{S} \subset \mathbb{R}^n$, an integer $d > 1$ and the corresponding lifted set $\mathcal{S}_{\mathcal{Y}} \subset \mathbb{R}^N$, $N = \binom{n+d}{d} - 1$. The following hold.

(i) $\mathcal{F}(\Sigma^\lambda, \mathcal{S}) \subseteq \mathrm{lower}(\mathcal{F}(\Sigma^\lambda_{\mathcal{Y}}, \mathcal{S}_{\mathcal{Y}}))$. The relation holds with equality if $C(\lambda)$ is invertible for all $\lambda \in \mathcal{L}$.
(ii) $\mathcal{B}(\Sigma^{\mathrm{sd}}, \mathcal{S}) = \mathrm{lower}(\mathcal{B}(\Sigma^{\mathrm{sd}}_{\mathcal{Y}}, \mathcal{S}_{\mathcal{Y}}))$.

Proof (i) We have

$$
\begin{aligned}
&\mathrm{lower}(\mathcal{F}(\Sigma^\lambda_{\mathcal{Y}}, \mathcal{S}_{\mathcal{Y}})) = \mathrm{lower}(\cup_{\lambda \in \mathcal{L}} \{A(\lambda)y + b(\lambda) : y \in \mathcal{S}_{\mathcal{Y}}\}) \\
&= \mathrm{lower}(\cup_{\lambda \in \mathcal{L}} \{A(\lambda)y + b(\lambda) \in \mathcal{V} : y \in \mathcal{S}_{\mathcal{Y}}\}) \\
&\supseteq \mathrm{lower}(\cup_{\lambda \in \mathcal{L}} \{A(\lambda)y + b(\lambda) \in \mathcal{V} : y \in \mathcal{S}_{\mathcal{Y}} \cap \mathcal{V}\})
\end{aligned}
$$

which is equal to $\mathcal{F}(\Sigma^\lambda, \mathcal{S})$. The proof of the second part of the statement follows similar steps as in Proposition 1(i).

(ii) Since $\mathcal{V}$ is invariant for the lifted system (13), we have

$$
\begin{aligned}
&\mathrm{lower}(\mathcal{B}(\Sigma^\lambda_{\mathcal{Y}}, \mathcal{S}_{\mathcal{Y}})) \\
&= \mathrm{lower}(\cap_{\lambda \in \mathcal{L}} \{y \in \mathcal{V} : A(\lambda)y + b(\lambda) \in \mathcal{S}_{\mathcal{Y}}\}) \\
&= \cap_{\lambda \in \mathcal{L}} \{x \in \mathbb{R}^n : C(\lambda)x + e(\lambda) \in \mathcal{S}\} = \mathcal{B}(\Sigma^\lambda, \mathcal{Y}). \blacksquare
\end{aligned}
$$

The matrix $A(\lambda)$ and vector $b(\lambda)$ from (5) consist of blocks that are functions of the monomials of λ, up to a maximum degree d. Consequently, it is very difficult to have a computationally efficient expression of the forward and backward reachability maps for the lifted system, even when $\mathcal{Y}$ is polytopic. For this reason, we propose to compute instead reachability operations by overapproximating the lifted set $\mathcal{L}^{[d!]}$. One can use general techniques for approximating semialgebraic sets, see, e.g., Cerone et al. (2012) for overapproximations with polyhedral sets and Dabbene et al. (2017) for the general case. However, as the specific setting involves intersections of the Veronese surface with polyhedra, we propose an overapproximation scheme that exploits this particular set structure. To this purpose, let $\mathcal{Q}$ denote the set of polynomials $\{l_{i,j}(\cdot)\}$ that define $\mathcal{L}$ (12), i.e., $\mathcal{Q} = \{l_{i,j}(\cdot) : i = 1, .., p_j, j = 1, ..., d_\lambda\}$ and $\mathcal{I}_i$ denote the set of pairs of vectors

$$
\mathcal{I}_i := \left\{(\gamma, \delta) : l_{\gamma_j, \delta_j}(\cdot) \in \mathcal{Q}, \sum_{j=1}^{n_k} \delta_j = i\right\}. \tag{14}
$$

Each pair of the elements of these vectors, e.g., (γ_k, δ_k), $\gamma_k \in \mathbb{R}^{n_k}$, $\delta_k \in \mathbb{R}^{n_k}$, $1 \leq k \leq d$, corresponds to a polynomial $l_{\gamma_i, \delta_i}(\lambda)$ such that the product $l_{\gamma_1, \delta_1}(\lambda) l_{\gamma_2, \delta_2}(\lambda) \cdots l_{\gamma_{n_k}, \delta_{n_k}}(\lambda)$ is a polynomial of degree i.

We denote with $\mathcal{L}_i$, $i = 1, ..., d_\lambda$, the semialgebraic sets defined by the products of the polynomial inequalities in $\mathcal{L}$ of degree i

$$
\mathcal{L}_i := \left\{\lambda \in \mathbb{R}^{n_\lambda} : (\gamma, \delta) \in \mathcal{I}_i, \prod_{j=1}^{n_k} l_{(\gamma_j, \delta_j)}(\lambda) \geq 0\right\}. \tag{15}
$$

We let $\mathcal{V}_\lambda$ denote the Veronese surface on the lifted space induced by d_λ, i.e., $\mathcal{V}_\lambda := \left\{z \in \mathbb{R}^{N_{d_\lambda}} : \left(\exists \lambda \in \mathbb{R}^{n_\lambda} : z = \lambda^{[d_\lambda!]}\right)\right\}$ where $N_{d_\lambda} = \binom{n_\lambda + d_\lambda}{d_\lambda} - 1$. We organize a few observations below.

Proposition 3. Consider the set $\mathcal{L}$ (12), the sets $\mathcal{L}_i$ (15), $i = 1, ..., d_\lambda$ and the corresponding lifted sets $\mathcal{Y}_{\mathcal{L}}$ and $\mathcal{Y}_{\mathcal{L}_i}$ in the lifted space $\mathbb{R}^{\binom{n_\lambda + d_\lambda}{d_\lambda} - 1}$. Let $\overline{\mathcal{L}} = \bigcap_{i \in \{1,...,d_\lambda\}} \mathcal{L}_i$ and $\overline{\mathcal{Y}}_{\mathcal{L}} = \bigcap_{i \in \{1,...,d_\lambda\}} \mathcal{Y}_{\mathcal{L}_i}$. The following hold.

[2] $\mathcal{V}$ is backwards invariant with respect to (10).

(i) $\overline{\mathcal{L}} = \mathcal{L}$.
(ii) $\overline{\mathcal{Y}}_{\mathcal{L}} \subset \mathcal{Y}_{\mathcal{L}}$.
(iii) $\mathcal{Y}_{\mathcal{L}} \cap \mathcal{V}_\lambda = \overline{\mathcal{Y}}_{\mathcal{L}} \cap \mathcal{V}_\lambda$.

Proof **(i)** By construction, it holds that $\mathcal{L} \subseteq \bigcap_{i\in\{1,\dots,d_\lambda\}} \mathcal{L}_i$ since $\lambda \in \mathcal{L}$ implies $l_{i,j}(\lambda) \geq 0$ which in turn implies that any combination of products $l_{i,j}(\lambda)$ remains nonnegative. Moreover, $\bigcap_{i\in\{1,\dots,d_\lambda\}} \mathcal{L}_i \subseteq \mathcal{L}$ since for each $j \in \{1,\dots,d_\lambda\}$ the inequalities $l_{i,j}(\lambda) \geq 0$ hold true in the set $\mathcal{L}_j$. Thus, $\overline{\mathcal{L}} = \mathcal{L}$. Item **(ii)** follows by construction of the lifted set (Definition 5), while **(iii)** follows directly from (i) and (ii). ■

In the case $\overline{\mathcal{Y}}_{\mathcal{L}}$ is not bounded, one can add bounding hyperplanes to its description and define a compact set $\overline{\mathcal{Y}}'_{\mathcal{L}}$ with the property $\overline{\mathcal{Y}}'_{\mathcal{L}} \cap \mathcal{V}_\lambda = \overline{\mathcal{Y}}_{\mathcal{L}} \cap \mathcal{V}_\lambda$. For more information see (Athanasopoulos and Jungers, 2016a, Appendix).

Let $\overline{\Sigma}^\lambda_{\mathcal{Y}}$ denote the system in the lifted space $\mathbb{R}^{\binom{n+d}{d}-1}$

$$y^+ = A(\lambda)y + b(\lambda), \tag{16}$$

where $\lambda \in \overline{\mathcal{Y}}_{\mathcal{L}}$. The corollary result below is a consequence of Proposition 3 and Proposition 2.

Corollary 1. Consider the system Σ^λ (11), a set $\mathcal{S} \subset \mathbb{R}^n$, an integer $d > 1$ and the corresponding lifted set $\mathcal{S}_{\mathcal{Y}} \subset \mathbb{R}^N$, $N = \binom{n+d}{d} - 1$ and system $\overline{\Sigma}^\lambda_{\mathcal{Y}}$. The following hold.

(i) $\mathcal{F}(\Sigma^\lambda, \mathcal{S}) \subseteq \mathrm{lower}(\mathcal{F}(\overline{\Sigma}^\lambda_{\mathcal{Y}}, \mathcal{S}_y))$.
(ii) $\mathcal{B}(\Sigma^{\mathrm{sd}}, \mathcal{S}) \supseteq \mathrm{lower}(\mathcal{B}(\overline{\Sigma}^\lambda_{\mathcal{Y}}, \mathcal{S}_y))$.

5. SYSTEMS WITH INPUTS

We consider the simplest form of (1) that includes input signals, namely linear systems with the update rule

$$x^+ = Cx + Du, \tag{17}$$

with $x \in \mathcal{X} \subset \mathbb{R}^n$, $u \in \mathcal{U} \subset \mathbb{R}^m$, where $\mathcal{X}$ is a semialgebraic set. We consider $\mathcal{U}$ to be polyhedral for simplicity. Specifically, by setting $e = Du$ in (4), one can see that the corresponding lifted matrix and vector (5) are polynomial functions of the inputs. The lifted system is expressed by

$$y^+ = A(u)y + (Bu)^{[d!]}, \tag{18}$$

with the elements of $A(u)$ being functions of monomials of the input vector u of at most degree d. Starting from a controlled invariant polyhedral set $\mathcal{Y}$ in the lifted space, we identify an input vector for which there corresponds at least one vector that can be transfered to $\mathcal{Y}$ in one step by following the lifted system dynamics. For linear systems, see, e.g., Athanasopoulos et al. (2014a), this allows to add at each step the convex hull of the added vector with the controlled invariant set, thus leading to an enlarged polyhedral set. While this is not the case for the lifted system (18), we can exploit the fact that for a fixed input (18) becomes affine. Consequently, we can add backward reachable sets to $\mathcal{Y}$ which are unions of polytopes. The proposed approach is shown in algorithmic fashion in Algorithm 1.

A few remarks are in order: The are several ways to compute an initial controlled invariant set $\mathcal{Y}_0$ for the lifted system (18). For example, one can first find a stabilizing linear controller and a corresponding admissible controlled invariant set for the linear system (17). Consequently, $\mathcal{Y}_0$ can be computed as the maximal invariant set of the closed-loop lifted system (which remains linear).

Algorithm 1 Inputs: System (17), (18), $\mathcal{X}, \mathcal{U}, k_1 > 1, k_2 > 1$.
Outputs: Enlarged controlled invariant set $\mathcal{Z}$.

1: Compute lifted system (18), lifted set $\mathcal{Y}$, an initial controlled invariant set $\mathcal{Y}_0$, set $i \leftarrow 0$, $c_1 \leftarrow 0$
2: **while** $i < k_1$ **and** $c_1 = 0$
3: Solve: $\max_{y,u} \mathrm{dist}(y, \mathcal{Y}_i)$
s.t. **(i)** $u \in \mathcal{U}$,
(ii) $y \in \mathcal{Y}$,
(iii) $A(u)y + (Bu)^{[d!]} \in \mathcal{Y}_i$
4: $\Sigma_{y^\star} : y^+ = A(u^\star)y + (Bu^\star)^{[d!]}$
5: $j \leftarrow 0$, $\mathcal{S}_0 \leftarrow \mathcal{Y}_i$, $c_2 \leftarrow 0$
6: **while** $j < k_2$ **and** $c_2 = 0$
7: $\mathcal{S}_{j+1} \leftarrow \mathcal{B}(\Sigma_{y^\star}, \mathcal{S}_j) \cap \mathcal{Y}$
8: **if** $\mathcal{S}_{j+1} \subseteq \cup_j \mathcal{S}_j$
9: $c_2 \leftarrow 1$
10: **end**
11: $j \leftarrow j + 1$
12: **end**
13: $\mathcal{Y}_{i+1} = \mathcal{Y}_i \cup_j \mathcal{S}_j$
14: **if** $\mathcal{Y}_{i+1} = \mathcal{Y}_i$
15: $c_1 \leftarrow 1$
16: **end**
17: $i \leftarrow i + 1$
18: **end**
19: $\mathcal{Z} \leftarrow \mathrm{lower}(\mathcal{Y}_i)$

At the beginning of each iteration (Line 2), an optimization problem (Line 3) is solved that retrieves a vector $y^\star$ that can be transferred in one step to $\mathcal{Y}_i$ with the admissible input $u^\star$. We note that constraints (i), (ii) can be translated to a set of linear inequalities and condition (iii) is equivalent to the satisfaction of polynomial inequalities. Consequently, the optimization problem can be solved using Sum of Squares programming, by formulating it first as a feasibility problem, see e.g., Prajna et al. (2002), Papachristodoulou and Prajna (2005). The objective function in Line 3, namely the distance function with respect to the set $\mathcal{Y}_i$ is a nonlinear function of y. However, since $\mathcal{Y}_i$ is a polytope (or a union of polytopes) we can solve instead a set of optimization problems with linear objective functions. Each such problem aims to maximize the distance of the vector y from a facet of the polytope $\mathcal{Y}_i$(or from a facet of a polytope included in the description of $\mathcal{Y}_i$), see, e.g., Athanasopoulos et al. (2014a).

By fixing the input to be equal to $u^\star$, we retrieve in Line 4 an affine system in the lifted space. At each iteration of the loop in Lines $6 - 11$ the set of vectors that can be brought to the set $\mathcal{Y}_i$ in at most j iterations with the input $u^\star$ is calculated, while all such states are added to $\mathcal{Y}_i$ in Line 13. We underline that since $\mathcal{S}_j$ are unions of polytopes, $\mathcal{Y}_i$ is a union of polytopes as well.

6. CONCLUSIONS

We studied the reachability analysis for general linear dynamics, including uncertainties and inputs, when the involved sets are semialgebraic. We lifted the system in a higher dimensional state space induced by the Veronese embedding and associated the reachability mappings of the lifted system with the ones of the original. Consequently, we established the relationship between the minimal and maximal invariant sets of the two systems as well. By formulating simple equivalence results, we showed that in some cases we can perform exact reachability, while in others we can utilize in a straightforward manner approximations that exploit the structure of the lifted nonlinear system. The proposed approach can be potentially followed in a variety of control problems, for example in path planning, obstacle avoidance and sliding mode control.

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