Interacting Particle System-based Estimation of Reach Probability for a Generalized Stochastic Hybrid System

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Abstract: This paper studies estimation of reach probability for a generalized stochastic hybrid system (GSHS). For diffusion processes a well-developed approach in reach probability estimation is to introduce a suitable factorization of the reach probability and then to estimate these factors through simulation of an Interacting Particle System (IPS). The theory of this IPS approach has been extended to arbitrary strong Markov processes, which includes GSHS executions. Because Monte Carlo simulation of GSHS particles involves sampling of Brownian motion as well as sampling of random discontinuities, the practical elaboration of the IPS approach for GSHS is not straightforward. The aim of this paper is to elaborate the IPS approach for GSHS by using complementary Monte Carlo sampling techniques. For a simple GSHS example, it is shown that and why the specific technique selected for sampling discontinuities can have a major influence on the effectiveness of IPS in reach probability estimation.

Keywords: Interacting Particles, Factorization, Rare event, Reach Probability, Stochastic Hybrid System

1. INTRODUCTION

This paper addresses the safety verification of unsafe subsets in the state space of a continuous-time generalized stochastic hybrid system (GSHS). GSHS involves discrete-valued and continuous-valued state components that dynamically interact (e.g. Bujorianu and Lygeros, 2006). Such safety verification problem can also be formulated as a stochastic reachability estimation problem (e.g. Prandini and Hu, 2007; Abate et al., 2009). Reach probability estimation is well studied in control systems domain and in safety domain. In the control domain the focus is on developing an (approximate) abstraction of the system for which it can be shown that the reach probability problem is sufficiently similar (Alur et al., 2000; Julius and Pappas, 2009). Approximate abstractions typically make use of a finite partition of the state space (e.g. Prandini and Hu, 2007; Abate et al., 2011; Di Benedetto et al., 2015).

In the safety domain, reach probability is evaluated by using a finite partition method or by using Monte Carlo (MC) simulation. For realistic applications the latter requires support from analytical methods to accelerate the simulation. Literature on such acceleration distinguishes two main approaches: importance sampling and multi-level splitting. Importance sampling draws samples from a reference stochastic system model in combination with an analytical compensation for sampling from the reference model instead of the intended model. Bucklew (2004) gives an overview of importance sampling and analytical compensation mechanisms. For complex models analytical compensation mechanisms typically fall short and multi-level splitting is the preferred approach (e.g. Botev and Kroese, 2008; L'Ecuyer et al., 2009; Rubinstein, 2010; Morio and Balesdent, 2016).

The basic idea of multi-level splitting is to enclose the target set, i.e. the set for which the reach probability has to be estimated, by a series of strictly increasingly subsets. This allows to split a simulated particle realization of the process considered into multiple copies each time such particle hits one of the enclosing subsets. This multi-level setting allows to express the small reach probability of the inner level set as a product of larger reach probabilities for the sequence of enclosing subsets (see e.g. Glasserman et al, 1999). Cérou et al. (2005, 2006) embedded this multi-level factorization in the Feynman-Kac factorization equation for strong Markov processes (Del Moral, 2004). This Feynman-Kac setting subsequently supported the evaluation of the reach probability through sequential Monte Carlo simulation in the form of an Interacting Particle System (IPS), including proof of convergence (Cérou et al., 2006). Cérou and Guyader (2007) extends the IPS algorithm to adaptively selecting the sequence of subsets to be used for a scalar diffusion. Guyader et al. (2011) further develop the multi-level splitting approach in estimating small reach probabilities by a diffusion process given a quantile. Morio and Balesdent (2016) show the effectiveness of IPS in rare event estimation for simple diffusion examples in aerospace.

Blom et al. (2006, 2007a) applied IPS to rare event estimation for a GSHS model of an advanced air traffic scenario. The hybrid state space of this model is very large, i.e. involving 490 discrete states and a 28-dimensional Euclidean state space. In order to prevent particle depletion or impoverishment a very large number of particles had to be used. In an attempt of improving the quality of the set of particles, Blom et al. (2007b, 2009) develops and applies hybrid extensions of IPS for air traffic. Complementary, Prandini et al. (2011) investigates the integration of air traffic complexity modelling with IPS.

MC simulation of GSHS involves interlacing the simulation of solutions of diffusion equations with the simulation of sudden events and jumps triggered by boundary hittings and in-homogeneous Poisson point processes. For the MC sampling of in-homogeneous Poisson points, there are two approaches (e.g. Glasserman, 2004): thinning of samples from a homogeneous Poisson process, and Bernoulli sampling. In the above mentioned IPS applications to GSHS, a form of Poisson thinning has been used. However, the expectation is that Bernoulli sampling may have a significant advantage. The aim of the present paper is to explicitly address the details of incorporating both MC sampling approaches in an IPS algorithm for an arbitrary GSHS, and to evaluate the effects on IPS performance for a simple rare event estimation example. In doing so, the paper identifies Bernoulli sampling as the fundamentally better choice in IPS based reach probability estimation for a GSHS.

The paper is organized as follows. Section 2 outlines the definition and execution of GSHS. Section 3 reviews IPS theory and algorithmic steps for an arbitrary GSHS. Section 4 incorporates sampling techniques in the basic IPS steps for an arbitrary GSHS. Section 5 applies these sampling techniques in IPS based reach probability estimation for a simple GSHS example. Section 6 draws conclusions.

2. GSHS DEFINITION

Throughout this and the following sections, all stochastic processes are defined on a complete stochastic basis $(\Omega,\mathcal{F},\mathbb{F},\mathrm{P},\mathcal{T})$ with $(\Omega,\mathcal{F},\mathrm{P})$ a complete probability space and $\mathbb{F}$ an increasing sequence of sub-σ-algebra's on the time line $\mathcal{T}=\mathbb{R}_+$, i.e., $\mathbb{F}\triangleq\{\mathcal{J},(\mathcal{F}_t,t\in\mathbb{R}_+),\mathcal{F}\}$, $\mathcal{J}$ containing all P-null sets of $\mathcal{F}$ and $\mathcal{J}\subset\mathcal{F}_s\subset\mathcal{F}_t\subset\mathcal{F}$ for every $s<t$.

(Bujorianu and Lygeros, 2006) formalized the concept of GSHS or general stochastic hybrid automata as follows:

Definition 1 (GSHS). A GSHS is a collection $(\Theta,d,X,f,g,Init,\lambda,R)$ where

- Θ is a countable set of discrete-valued variables;
- $d:\Theta\to\mathbb{N}$ is a map giving the dimensions of the continuous state spaces;
- $X:\Theta\to\mathbb{R}^{d(\cdot)}$ maps each $\theta\in\Theta$ into an open subset X^θ of $\mathbb{R}^{d(\theta)}$;
- $f:\Xi\to\mathbb{R}^{d(\cdot)}$ is a vector field, where $\Xi\triangleq\bigcup_{\theta\in\Theta}\{\theta\}\times X^\theta$;
- $g:\Xi\to\mathbb{R}^{d(\cdot)\times m_{\dim}}$ is a $X^{(\cdot)}$-valued matrix, $m_{\dim}\in\mathbb{N}$;
- Init : $\beta(\Xi)\to[0,1]$ an initial probability measure on Ξ ;
- $\lambda:\bar{\Xi}\to\mathbb{R}^+$ is a transition rate function;
- $R:\bar{\Xi}\times\beta(\bar{\Xi})\to[0,1]$ is a transition measure.

Definition 2 (GSHS Execution). A stochastic process $\{\theta_t,x_t\}$ is called a GSHS execution if there exists a sequence of stopping times $s_0=0<s_1<s_2<\ldots$ such that:

- (θ_0,x_0) is a Ξ-valued random variable satisfying the probability measure *Init*;
- For $t\in\left[s_{j-1},s_j\right)$, $j\geq 1$, $\{\theta_t,x_t\}$ is a solution of the SDE:
$$\begin{aligned} d\theta_t &= 0 \\ dx_t &= f(\theta_t,x_t)dt+g(\theta_t,x_t)dW_t \end{aligned} \tag{1}$$
with W_t m-dimensional standard Brownian motion;
- s_j is the minimum of the following two stopping times: i) first hitting time $>s_{j-1}$ of the boundary of $X^{\theta_{s_{j-1}}}$ by the process $\{x_t\}$; and ii) first moment $>s_{j-1}$ of a transition event to happen at rate $\lambda(\theta_t,x_t)$.
- At stopping time s_j the novel hybrid state $\{\theta_{s_j},x_{s_j}\}$ satisfies the conditional probability measure $p_{\theta_{s_j},x_{s_j}|\theta_{s_j-},x_{s_j-}}(A\,|\,\theta,x)=R((\theta,x),A)$ for any $A\in\beta(\Xi)$.

In order to assure that a GSHS execution has a solution the following assumptions are adopted:

A1 (non-Zeno property): $E\{s_j-s_{j-1}\}>0$, P -a.s.

A2: For each $(\theta_0,x_0)\in\Xi$, equation (1) has a unique solution on a finite time interval $[0,T]$.

A3 λ is measurable and finite valued.

A4 $Init(\Xi)=1$, and $R((\theta,x),\Xi)=1$ for each $(\theta,x)\in\bar{\Xi}$.

Bujorianu and Lygeros (2006) show that the stochastic process $\{\theta_t,x_t\}$ generated by execution of a GSHS satisfies the strong Markov property.

3. IPS BASED REACH PROBABILITY ESTIMATION

3.1. GSHS reach probability

The problem is to estimate the probability γ that $\{\theta_t,x_t\}$ reaches a closed subset $D\subset\Xi$ within finite period $[0,T]$, i.e.

$$\gamma=P(\tau<T) \tag{2}$$

with τ the first hitting time of D by $\{\theta_t,x_t\}$:

$$\tau=\inf\{t>0,(\theta_t,x_t)\in D\} \tag{3}$$

Cérou et al. (2006) developed the IPS theory and algorithmic steps for estimating reach probability for a strong Markov process on a general Polish state space. Thanks to the strong Markov property of the process $\{\theta_t,x_t\}$ defined by the execution of the GSHS in section 2, the IPS approach applies to the estimation of GSHS reach probability.

3.2. Multi-level factorization of reach probability

The underlying principle to factorization of the reach probability $\gamma = P(\tau < T)$ is to introduce a nested sequence of closed subsets D_k of Ξ, such that $D = D_m \subset D_{m-1} \subset \cdots \subset D_1 \subset \Xi$. Let τ_k be the first moment in time that $\{x_t, \theta_t\}$ reaches D_k, i.e.

$$\tau_k = \inf\{t > 0;\ (x_t, \theta_t) \in D_k\} \tag{4}$$

Next, we define $\{0,1\}$-valued random variables $\{\chi_k, k = 1,..,m\}$ as follows:

$$\begin{aligned}\chi_k &= 1, \text{ if } \tau_k < T \text{ or } k = 0\\ &= 0, \text{ else}\end{aligned} \tag{5}$$

By using this χ_k definition, the factorization becomes:

$$\gamma = \prod_{k=1}^{m} \gamma_k \tag{6}$$

with $\gamma_k \triangleq \mathrm{P}(\chi_k = 1 | \chi_{k-1} = 1) = \mathrm{P}(\tau_k < T | \tau_{k-1} < T)$.

3.3. Recursive estimation of the multi-level factors

By using the strong Markov property of $\{\theta_t, x_t\}$, we develop a recursive estimation of γ using the factorization in (6). First we define $\Xi' \triangleq \mathbb{R} \times \Xi$, $\xi_k \triangleq (\tau_k, \theta_{\tau_k}, x_{\tau_k})$, $Q_k \triangleq (0,T) \times D_k$, for $k = 1,\dots,m$, and the following conditional probability measure $\pi_k(B)$ for an arbitrary Borel set B of Ξ':

$$\pi_k(B) \triangleq P(\xi_k \in B \mid \xi_k \in Q_k)$$

Cerou et al. (2006) shows that π_k is a solution of the following recursion of transformations:

$$\pi_{k-1}(\cdot) \xrightarrow{\text{I. mutation}} p_k(\cdot) \xrightarrow{\text{III. selection}} \pi_k(\cdot)$$
$$\downarrow II.\ \text{conditioning}$$
$$\gamma_k$$

where $p_k(B)$ is the conditional probability measure of $\xi_k \in B$ given $\xi_{k-1} \in Q_{k-1}$ i.e.

$$p_k(B) \triangleq P(\xi_k \in B \mid \xi_{k-1} \in Q_{k-1})$$

Because $\{\theta_t, x_t\}$ is a strong Markov process, $\{\xi_k\}$ is a Markov sequence. Hence the mutation transformation (I) satisfies a Chapman-Kolmogorov equation prediction for ξ_k:

$$p_k(B) = \int_{E'} p_{\xi_k|\xi_{k-1}}(B \mid \xi)\pi_{k-1}(d\xi) \text{ for all } B \in \beta(\Xi') \tag{7}$$

For the conditioning transformation (II) this means:

$$\gamma_k = \mathrm{P}(\tau_k < T \mid \tau_{k-1} < T) = \int_{E'} 1_{\{\xi \in Q_k\}} p_k(d\xi). \tag{8}$$

Hence, selection transformation (III) satisfies:

$$\pi_k(B) = \frac{\int_B 1_{\{\xi \in Q_k\}} p_k(d\xi)}{\int_{E'} 1_{\{\xi' \in Q_k\}} p_k(d\xi')} = [\int_B 1_{\{\xi \in Q_k\}} p_k(d\xi)] / \gamma_k. \tag{9}$$

With this, the γ_k terms in (6) are characterized as solutions of a recursive sequence of mutation equation (7), conditioning equation (8) and selection equation (9).

3.4. IPS algorithmic steps for a GSHS

Following Cérou et al. (2006), equations (6)-(9) yield the IPS algorithmic steps for the numerical estimation of γ:

$$\bar{\pi}_{k-1}(\cdot) \xrightarrow{\text{I. mutation}} \bar{p}_k(\cdot) \xrightarrow{\text{III. selection}} \tilde{\pi}_k(\cdot) \xrightarrow{\text{IV. splitting}} \bar{\pi}_k(\cdot)$$
$$\downarrow II.\ \text{conditioning}$$
$$\bar{\gamma}_k$$

A set of N_P particles is used to form empirical density approximations $\bar{\gamma}_k$, $\bar{p}_k$ and $\bar{\pi}_k$ of γ_k, p_k and π_k respectively. By increasing the number N_P of particles in a set, the errors in these approximations will decrease. When simulating particles from Q_{k-1} to Q_k, a fraction $\bar{\gamma}_k$ of the simulated particle trajectories only will reach Q_k within the time period $[0,T]$ considered; these particles form $\tilde{\pi}_k$. In order to start the next IPS cycle with N_P particles, randomly selected particles from $\tilde{\pi}_k$ are copied (also called splitting) and added to $\bar{\pi}_k$. The IPS cycle stops if $\tilde{\pi}_k$ has zero particles. The resulting IPS algorithmic steps are given in Table 1.

Table 1. Algorithm 0; IPS steps for GSHS

0. Initiation: Generate N_P particles $\xi_0^i \sim \pi_0$, $i = 1,..,N_P$, i.e. $\bar{\pi}_0(\cdot) = \sum_{i=1}^{N_p} \frac{1}{N_p} \delta_{\{\xi_0^i\}}(\cdot)$, with Dirac δ. Set $k = 1$.

I. Mutation: $\bar{p}_k(.) = \sum_{i=1}^{N_p} \frac{1}{N_p} \delta_{\{\bar{\xi}_k^i\}}(\cdot)$, where $\bar{\xi}_k^i$ is obtained through simulation of a GSHS execution starting at ξ_{k-1}^i.

II. Conditioning: $\bar{\gamma}_k = \sum_{i=1}^{N_p} \frac{1}{N_p} 1(\bar{\xi}_k^i \in Q_k)$.

III. Selection: $\tilde{\pi}_k(.) = \sum_{i=1}^{N_p} \frac{1}{N_p} 1(\bar{\xi}_k^i \in Q_k) \delta_{\{\bar{\xi}_k^i\}}(\cdot) / \bar{\gamma}_k$.

IV. Splitting: $\bar{\pi}_k(.) = \sum_{i=1}^{N_p} \frac{1}{N_p} \delta_{\{\xi_k^i\}}(.)$, with $\xi_k^i \sim \tilde{\pi}_k(.)$.

If $k < m$, then repeat IPS steps I-IV for $k := k + 1$.

V. $\bar{\gamma} = \prod_{k=1}^{m} \bar{\gamma}_k$

Remark: By using the Feynman-Kac framework of Del Moral (2004), Cérou et al. (2006) proofs that $\bar{\gamma}$ forms an unbiased γ estimate and also derives bounds for $(\bar{\gamma} - \gamma)$.

Next we address the details of Monte Carlo simulation from particle state ξ_{k-1}^i to particle state $\bar{\xi}_k^i$ in mutation step I.

4. GSHS SIMULATION OF THE IPS MUTATION STEP

The IPS mutation step in Table 1 involves simulation from particle state ξ_{k-1}^i to particle state $\bar{\xi}_k^i$. This involves sampling of an in-homogeneous Poisson process and managing hittings of boundaries of Ξ and Q_k. First we address the simulation without taking boundary hittings into account. Subsequently we extend this for boundary hittings.

4.1. Thinning of Poisson process points

A well-known approach in Monte Carlo sampling of an in-homogeneous Poisson process is based on thinning of time points sampled from a homogeneous Poisson process on $[0,T]\times\left[0,\bar{\lambda}\right]$, with $\bar{\lambda} \geq \sup_{(\theta,x)\in\Xi} \lambda(\theta,x)$. The resulting Poisson points happen at unit density on $[0,T]\times\left[0,\bar{\lambda}\right]$. The thinning consists of rejecting all points that lie above the graph of $\lambda(\theta_t,x_t)$. The remaining points, i.e. those at or below the graph of $\lambda(\theta_t,x_t)$, are projected onto the *time*-axis $[0,T]$. The resulting execution of the GSHS, starting from ξ_{k-1}^i, on the interval $[\tau_{k-1},T]$ is described in Table 2.

Table 2. GSHS execution algorithm 1; inputs ξ_{k-1}^i and $\bar{\lambda}$

1. Set $t=\tau_{k-1}$.
2. Generate $u\sim \mathrm{U}(0,1)$.
3. $\Delta_t := -(\ln u)/\bar{\lambda}$.
4. Execute GSHS for $\lambda=0$ from t until $t+\Delta_t$; this yields $(\theta_{t+\Delta_t-},x_{t+\Delta_t-})$.
5. Generate $v\sim \mathrm{U}(0,1)$.
6. If $\lambda\left(\theta_{t+\Delta_t-},x_{t+\Delta_t-}\right)\geq v\bar{\lambda}$, then generate $(\theta_{t+\Delta_t},x_{t+\Delta_t})\sim R((\theta_{t+\Delta_t-},x_{t+\Delta_t-}),(.,.))$.
7. If $t<T$, then set $t:=t+\Delta_t$ and repeat from step 2.

4.2. Bernoulli sampling

An alternative to Monte Carlo sampling of an in-homogeneous Poisson process is Bernoulli sampling at each small time step $\vartriangle$ (Glassermann, 2004, pp. 137-142). The probability that no Poisson point of rate $\bar{\lambda}$ occurs on an interval $(t,t+\vartriangle)$ is $1-\bar{\lambda}\vartriangle$. To simulate the event of at least one transition event to happen, during the time step $\vartriangle$ a sample Δ_t is generated from an exponential density with mean duration $\bar{\lambda}^{-1}$. If $\Delta_t>\vartriangle$ then this sample is rejected. Otherwise at moment $t+\Delta_t$ a uniform sample v is taken from $(0,\bar{\lambda})$. If $\lambda(\theta_{t+\Delta_t},x_{t+\Delta_t})\geq v$ then a transition applies at moment $t+\Delta_t$. In using this approach of Bernoulli sampling, the execution of a GSHS, starting from ξ_{k-1}^i, on the interval $[\tau_{k-1},T]$ is described in Table 3.

Table 3. GSHS execution algorithm 2; inputs ξ_{k-1}^i and $\bar{\lambda}$

1. Set $t=\tau_{k-1}$
2. Generate $u\sim \mathrm{U}(0,1)$.
3. $\Delta_t := \min\{\vartriangle,-(\ln u)/\bar{\lambda}\}$.
4. Execute GSHS for $\lambda=0$ from t to $t+\Delta_t$; this yields $(\theta_{t+\Delta_t-},x_{t+\Delta_t-})$.
5. If $\Delta_t\geq\vartriangle$ then set $t:=t+\vartriangle$, and repeat from step 2.
6. Generate $v\sim \mathrm{U}(0,1)$.
7. If $\lambda\left(\theta_{t+\Delta_t-},x_{t+\Delta_t-}\right)\geq v\bar{\lambda}$, then generate $(\theta_{t+\Delta_t},x_{t+\Delta_t})\sim R((\theta_{t+\Delta_t-},x_{t+\Delta_t-}),(.,.))$.
8. If $t<T$, then set $t:=t+\Delta_t$ and repeat from step 2.

In contrast to algorithm 1, algorithm 2 uses small time steps only. As a consequence, algorithm 2 needs a much larger number of independent uniform samples than algorithm 1. Therefore algorithm 1 is often preferred.

4.3. Execution of GSHS for $\lambda=0$

In step 4 of algorithms 1 and 2, GSHS is executed on interval $(t,t+\Delta_t)$ for $\lambda=0$. If no boundary hitting event occurs, then this can be accomplished by applying Euler-Maruyama integration of eq. (1) along small time steps $\vartriangle$, i.e. using:

$$\begin{aligned} \theta_{t+\vartriangle} &= \theta_t \\ x_{t+\vartriangle} &= f(\theta_t,x_t)\vartriangle+g(\theta_t,x_t)(W_{t+\vartriangle}-W_t) \end{aligned} \tag{10}$$

However, if during any time step $\vartriangle$ one of the boundaries of Ξ or Q_k is passed, then additional MC simulation steps are needed. First of all it is needed to simulate a hitting time τ' of the applicable boundary by the simulated $\{\theta_t,x_t\}$. For this, Glasserman (2004, p. 367) proposes an interpolation of the solution of equation (1) on the $\vartriangle$ interval considered, by simulating a Brownian bridge between the already simulated Brownian motion points W_t and $W_{t+\vartriangle}$. This yields moment τ' and hybrid state $(\theta_{\tau'},x_{\tau'})$ at which this Brownian bridge hits Ξ or Q_k for the first time.

In case of hitting Ξ, then also a jump in $\{\theta_t,x_t\}$ has to be simulated according to transition measure R, i.e. $(\theta_{\tau'},x_{\tau'})\sim R((\theta_{\tau'-},x_{\tau'-}),(.,.))$. Subsequently, the Euler-Maruyama integration has to be completed on the remaining part of the $\vartriangle$ interval, i.e. on $(\tau',t+\vartriangle)$.

In case of hitting Q_k, then both algorithms 1 and 2 should stop. The new particle $\bar{\xi}_k^i$ should include the latest state:

$$(\tau_k,\theta_{\tau_k}^i,x_{\tau_k}^i)=(\tau',\theta_{\tau'},x_{\tau'}) \tag{11}$$

and the period Δ_k^i

$$\Delta_k^i=(t+\Delta_t-\tau') \tag{12}$$

during which GSHS execution remains to be simulated. This means that step 1 in algorithms 1 and 2 has to be replaced by:

1. Set $t=\tau_{k-1}$, $\Delta_t=\Delta_{k-1}^i$; goto step 4 iff $\Delta_t>0$.

5. RARE EVENT SIMULATION OF GSHS EXAMPLE

5.1. Hypothetical car example

Our example does not involve Brownian motion. A car driver in dense fog is heading to a wall at position d_{wall}. If the car is at distance d_{fog} from the wall, then the driver sees the wall for the first time. Then it takes the driver a random reaction delay to start braking; with a density $p_{delay}(s)$. During the reaction delay the velocity of the car does not change; after the reaction delay the car decelerates at constant value $a_{\min}$. We apply IPS to estimate the probability $p_{hit} = \gamma$ that the car hits the wall. Table 4 gives analytically obtained p_{hit} results for various mean reaction delay values μ, $p_{delay}(t) = \mu^{-1} e^{-t/\mu}$, $d_{fog} = 5400m$, $v_0 = 216 \, {}^{km}\!/\!_{h} = 60 \, {}^{m}\!/\!_{s}$, $a_{\min} = -1 \, {}^{m}\!/\!_{s^2}$.

Table 4 Analytical p_{hit} results for different values of μ

μ	p_{hit}
10	2.47875×10^{-3}
5	6.144212×10^{-6}
3.33	1.522998×10^{-8}
2.5	3.775135×10^{-11}
2	9.3576×10^{-14}

5.2. GSHS model

For this example, the discrete set of the GSHS is:

$$\Theta = \{-1, 0, 1, delay, hit\} \tag{13}$$

with -1 decelerating mode, 0 uniform mode, 1 accelerating mode, *delay* a reaction delay mode, and *hit* if the wall has been hit. A transition diagram representing the switchings between these four modes is given in Figure 1.

The continuous state components are $x_t = Col(s_t, y_t, v_t)$, where s_t is the amount of delay passed since the driver could see the wall for the first time, y_t is the position of the car at time t, and v_t is the velocity at time t, Hence, the dimension of the continuous state space is $d(.) = 3$. The subsets X^θ are defined as follows:

$$\begin{aligned} X^0 &= \mathbb{R} \times (-\infty, d_{wall} - d_{fog}) \times \mathbb{R} \\ X^1 &= \mathbb{R} \times (-\infty, d_{wall} - d_{fog}) \times (0, v_{\max}) \\ X^{-1} &= \mathbb{R} \times (-\infty, d_{wall}) \times (0, \infty) \\ X^{delay} &= \mathbb{R} \times (-\infty, d_{wall}) \times \mathbb{R} \\ X^{hit} &= \mathbb{R}^3 \end{aligned} \tag{14}$$

Between switching moment of $\{\theta_t\}$, x_t evolves as follows:

$$\begin{aligned} ds_t &= dt \\ dy_t &= v_t dt \\ dv_t &= \theta_t(\theta_t - 1) a_{\min} / 2 + \theta_t(\theta_t + 1) a_{\max} / 2 \end{aligned} \tag{15}$$

where $a_{\min}$ is the deceleration value and $a_{\max}$ is the acceleration value. The initial measure *Init* generates $\theta_0 = 0$, $s_0 = 0$, $y_0 < d_{wall} - d_{fog}$, $0 < v_0 < v_{\max}$.

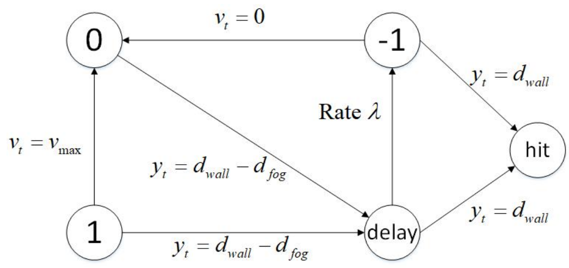

Figure 1. State transition diagram for the car example.

The instantaneous transition rate $\lambda(\theta_t, (s_t, y_t, v_t))$ satisfies:

$$\lambda(\theta, (s, y, v)) = \chi(\theta = delay) p_{delay}(s) / \int_s^\infty p_{delay}(s') ds' \tag{16}$$

The transition measure $R((\theta, (s, y, v)), (.,.))$ satisfies:

$R((1, (s, y, v)), \{0\} \times \{0, y, v\}) = 1$ iff $v = v_{\max}$

$R((-1, (s, y, v)), \{0\} \times \{0, y, v\}) = 1$ iff $v = 0$

$R((0, (s, y, v)), \{delay\} \times \{0, y, v\}) = 1$ iff $y = d_{wall} - d_{fog}$

$R((1, (s, y, v)), \{delay\} \times \{0, y, v\}) = 1$ iff $y = d_{wall} - d_{fog}$

$R((delay, (s, y, v)), \{-1\} \times \{0, y, v\}) = 1$, iff λ generates a point,

$R((delay, (s, y, v)), \{hit\} \times \{0, y, 0\}) = 1$, iff $y = d_{wall}$

$R((-1, (s, y, v)), \{hit\} \times \{0, y, 0\}) = 1$, iff $y = d_{wall}$.

5.3. IPS results

For the application of IPS we adopt m equidistant levels for car passing levels between $d_{wall} - d_{fog}$ and d_{wall}, i.e. $D_k = \Theta \times \mathbb{R} \times (-\infty, d_{wall} - (\frac{m-k}{m}) d_{fog}) \times \mathbb{R}$. Running IPS with $N_p = 10000$ yields the estimated probability $\bar{p}_{hit} = \bar{\gamma}$ results in Table 5 and Table 6 for exponential sampling and Bernoulli sampling respectively.

Table 5 shows that IPS using exponential sampling is not able to estimate $\bar{p}_{hit}$ for μ values of 5 s or smaller.

Table 5. IPS simulation results under exponential sampling for different values of μ, and $\Delta = 0.01s$, $m = 10$ and $N_p = 10000$

μ	$\bar{p}_{hit}$	$\lvert \bar{p}_{hit} - p_{hit} \rvert$	Simulation time (s)
10	2.1×10^{-3}	3.79×10^{-4}	5
5	/	/	/
3.33	/	/	/
2.5	/	/	/
2	/	/	/

Table 6 shows that under Bernoulli sampling the results are far better. For a μ value of 10s the estimation error is more than a factor 10 smaller than under exponential sampling, at the cost of a relative small increase of simulation time. Moreover, in contrast to exponential sampling, for lower μ values Bernoulli sampling based IPS continues to work well. This demonstrates that the factorization of eq. (6) keeps on working well with Bernoulli sampling over a wide range.

Table 6. IPS simulation results under Bernoulli sampling for different values of μ, and $\Delta = 0.01s$, $m = 10$ and $N_p = 10000$

μ	$\overline{p}_{hit}$	$\lvert \overline{p}_{hit} - p_{hit} \rvert$	Simulation time (s)
10	2.5×10^{-3}	2.13×10^{-5}	6.10
5	6.07×10^{-6}	7.40×10^{-8}	5.01
3.33	1.46×10^{-8}	5.89×10^{-10}	4.13
2.5	3.30×10^{-11}	4.80×10^{-12}	3.47
2	9.48×10^{-14}	1.21×10^{-15}	2.96

6. CONCLUSION

The simulation results obtained show that for the hypothetical car example the use of Bernoulli sampling in IPS, instead of exponential sampling, leads to a dramatic improvement in reach probability estimation. As there was no Brownian motion, during the length Δ_t of GSHS execution in algorithms 1 and 2, the prediction from (θ_t, x_t) to $(\theta_{t+\Delta_t}, x_{t+\Delta_t})$ is deterministic. This also means that during the interval $(t, t+\Delta_t)$ one or more of the equidistant levels may be passed. Each time this happens, the practical effectiveness of the factorization in (6) is reduced. Moreover, such particle also has a larger chance to be copied in replacement of a particle that does not reach one of these passing levels. The latter reduces the variability in the set of particles to be used in the next IPS iteration. The chance of both effects can be reduced by shortening the length of Δ_t. The latter is accomplished by the Bernoulli sampling in algorithm 2, but not by the Poisson thinning of algorithm 1.

Taken into account the generality of the above explanation, it is reasonable to expect that the use of Bernoulli sampling, instead of Poisson thinning, in IPS based reach probability estimation will lead to similar dramatic improvements for other GSHS applications that involve random delays but no or insufficient Brownian motion.

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