Language constrained stabilization of discrete-time switched linear systems: an LMI approach

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Abstract: The goal of this paper is to study sufficient conditions to stabilize an autonomous discrete-time switched system, for which the switching law should belong to a constrained language characterized by a nondeterministic automaton. Based on a decomposition into strongly connected components of the automaton, it is shown that it suffices to consider only a nontrivial strongly connected component. Sufficient conditions are provided as a set of Linear Matrix Inequalities (LMIs) related to the automaton states and associated with a min-switching strategy. Equivalence with the periodic stabilization is investigated. A numerical example is provided to illustrate the main result.

Keywords: Switched systems, Stabilizability, LMI, Language constrained, Automaton.

1. INTRODUCTION

Discrete-time switched systems consist in the combination of a finite set of difference equations and a switching law, that is a function indicating at each time the single difference equation which is active (Liberzon, 2003). This class of systems belongs to the more generic class of hybrid ones. Switched systems are of particular interest in the community of automatic control because they can model complex systems that are encountered in practice. In particular the stability of switched systems is not simple and has generated a large literature (see for some samples (Gurvits, 1995; Daafouz et al., 2002; Sun and Ge, 2011)).

The problem of stabilizability of autonomous switched systems via the switching law is the natural extension of the stability analysis of those systems. Several contributions have been provided by taking several tools or points of view. Without being exhaustive, we can cite the min-switching strategy associated with Lyapunov-Metzler inequalities (Geromel and Colaneri, 2006), the use of joint spectral subradius (Jungers and Mason, 2017), the graph control Lyapunov functions (Lee and Hu, 2016, 2017a) extending the contribution in (Ahmadi et al., 2014), or periodic control Lyapunov functions (Lee and Hu, 2017b) and finally the geometric approach (Fiacchini and Jungers, 2014) that has been augmented and compared with LMIs tools and Lyapunov-Metzler inequalities in (Fiacchini et al., 2016).

A large number of these contributions assumes that the switching law can be chosen at each time among the whole set of modes, which is a very conservative assumption in practice. Actually the switched systems are frequently associated with a collection of constraints: (average) dwell time, maximal duration in a mode, modal constraints prohibiting some transitions, *etc.* An elegant way to avoid a frame dedicated to a specific constraint is to consider language constraints that are generated by a nondeterministic finite state automaton (Bang-Jensen and Gutin, 2008). Although such a framework has been already studied for stability analysis (Mancilla-Aguilar et al., 2005; Wang et al., 2017) or (Athanasopoulos et al., 2017), there are only few contributions on the stabilizability of language constrained discrete-time switched systems. Among them we can cite the extension of the geometric approach (Fiacchini et al., 2018) leading to necessary and sufficient conditions and also the extension of Lyapunov-Metzler inequalities (Jungers et al., 2016) leading to sufficient conditions. Here, we consider a distinct point of view and we extend the sufficient conditions based on LMIs introduced in the unconstrained case in (Fiacchini et al., 2016). The decomposition into strongly connected components is recalled from (Jungers et al., 2016) to check only the stabilizability of a nontrivial strongly connected component.

The paper is organized as follows. Section 2 details the problem formulation. Section 3 presents as preliminaries the tools dedicated to automata and also the main guidelines of the result. Section 4 offers the main contribution in Theorems 9 and 13. It is shown that under LMIs conditions, the system is stabilizable and we exhibit two kinds of min-switching stabilizing policies: one straightforwardly deduced from the LMIs and one which is modified but associated to a exponentially stabilizing control Lyapunov function. Discussions related to retrieving the results for the periodic stabilizability belong to Section 5. A numerical example is provided in Section 6 before concluding remarks in Section 7.

Notation: The notation is standard. $\mathbb{R}^+$ denotes the nonnegative reals. Given $n \in \mathbb{N}$, define $\mathbb{N}_n = \{x \in \mathbb{N} : 1 \leq x \leq n\}$. Given $\alpha \in \mathbb{R}^n$, α_i denote its i-th element. A^T denotes the transpose of matrix A. I_n ($0_{n\times p}$) is the n-order identity matrix ($n\times p$-order null matrix). For a finite set $\mathcal{S}$, $|\mathcal{S}|$ denotes its cardinality. A real square matrix is called *Schur* if its eigenvalues have absolute value less than one.

2. PROBLEM FORMULATION

Let us consider the discrete-time switched linear system

$$x_{k+1} = A_{\sigma(k)}x_k, \tag{1}$$

with $x_k \in \mathbb{R}^n$ the state and $\sigma : \mathbb{N} \to \mathbb{N}_q$ the switching law and finally matrices $A_i \in \mathbb{R}^{n\times n}$ for all $i \in \mathbb{N}_q$. Here we consider that the switching law $\sigma(\cdot)$ is the controlled input of the system (1) and should be an infinite word of the constrained language defined by a nondeterministic finite automaton (Baier and Katoen, 2008).

Definition 1. A *nondeterministic finite automaton* is a tuple $\mathcal{A} = (\mathcal{S}, \Sigma, \delta, \mathcal{S}^0)$ where $\mathcal{S}$ is a finite set of automaton states; Σ is a finite set of labels for the transitions; $\delta : \mathcal{S} \times \Sigma \to 2^{\mathcal{S}}$ is a set-valued transition map, and $\mathcal{S}^0 \subseteq \mathcal{S}$ is a subset of initial states.

In this paper, the labels are the modes of the system (1), leading to $\Sigma = \mathbb{N}_q$ and we denote $p = |\mathcal{S}|$ the number of automaton states and $\mathcal{S} = \{e_i\}_{i\in\mathbb{N}_p}$. In order to introduce Problem 3, let us provide the following definition.

Definition 2. The system (1) is *globally exponentially stabilizable* relatively to the language $L(\mathcal{A})$, if there are $c \geq 0$, $\lambda \in [0, 1)$ and $s \in \mathcal{S}^0$, such that for all $x \in \mathbb{R}^n$, a switching law $\sigma(\cdot)$, function of (x, s) exists as an infinite word of $L(\mathcal{A})$ and verifies $\|x^\sigma_k(x)\| \leq c\lambda^k\|x\|$. Such a switching law is called an *Exponentially Stabilizing and Language Constrained Switching Law* (ESLCSL).

Problem 3. Let us consider the system (1) and the automaton $\mathcal{A}$ defining the language constraints for the switching laws. Determine switching laws $\sigma \in L(\mathcal{A})$ that are ESLCSL.

3. PRELIMINARIES

This section is devoted to several notions and notations related to automata.

Definition 4. A *transition* is a triplet $\tau = (\check{s}, i, \hat{s}) \in \mathcal{S} \times \Sigma \times \mathcal{S}$ such that $\hat{s} \in \delta(\check{s}, i)$. A *path* of $\mathcal{A}$ is a sequence of transitions $p = \{(\check{s}_1, i_1, \hat{s}_1), (\check{s}_2, i_2, \hat{s}_2), \cdots\}$ such that $\check{s}_{k+1} = \hat{s}_k$. The set of finite paths of m transitions is denoted $\mathcal{P}_m(\mathcal{A})$ and the set of infinite paths $\mathcal{P}_\infty(\mathcal{A})$.

For a path $p = \{(\check{s}_1, i_1, \hat{s}_1), \cdots, (\check{s}_m, i_m, \hat{s}_m)\} \in \mathcal{P}_m(\mathcal{A})$, the projection $w(p) = (i_1, i_2, \cdots, i_m) \in \Sigma^m$ is a *word*, admissible of the language $L(\mathcal{A})$ characterized by the automaton $\mathcal{A}$, and we denote $w_j(p) = i_j$ for $j \in \mathbb{N}_m$. In adddition, $\pi(p) = (\check{s}_1, \cdots, \check{s}_m, \hat{s}_m) \in \mathcal{S}^{m+1}$ is the projection of the path over the set of automaton states, and we denote $\pi_j(p) = \check{s}_j$ with $j \in \mathbb{N}_m$ and $\pi_{m+1}(p) = \hat{s}_m$. A path $p \in \mathcal{P}_m(\mathcal{A})$ is a *cycle* of the automaton if $\pi_{m+1}(p) = \pi_1(p)$. We denote $l(p)$ the length of p, that is the number of its transitions. Moreover for a sake of clarity, we will denote $\mathbb{A}_{w(p)} = A_{i_m}A_{i_{m-1}}\cdots A_{i_1}$. For two paths $p_1 \in \mathcal{P}_{m_1}(\mathcal{A})$ and $p_2 \in \mathcal{P}_{m_2}(\mathcal{A})$, if these paths are compatible, in the sense that the last automaton state of p_1 is the first automaton state of p_2, that is $\pi_{l(p_1)+1}(p_1) = \pi_1(p_2)$, then we can define the concatenation of these paths denoted $p_1 \circ p_2 \in \mathcal{P}_{m_1+m_2}(\mathcal{A})$. We have thus $\mathbb{A}_{w(p_1\circ p_2)} = \mathbb{A}_{w(p_2)}\mathbb{A}_{w(p_1)}$. A state $s \in \mathcal{S}$ is said to be *reachable* from a state $\tilde{s} \in \mathcal{S}$ if $s = \tilde{s}$ or if there exists a finite path $p \in \mathcal{P}(\mathcal{A})$ such that $\pi_1(p) = \tilde{s}$ and $\pi_{l(p)}(p) = s$. With some abuse of vocabulary, we will say that a switching law $\sigma : \mathbb{N} \to \Sigma$ belongs to the language $L(\mathcal{A})$ if there exists a path $p \in \mathcal{P}_\infty(\mathcal{A})$ such that $w_k(p) = \sigma(k)$, $\forall k \in \mathbb{N}$.

Problem 3 can be solved by determining a *path control policy* $p_\nu : \mathbb{R}^n \times \mathcal{S}^0 \to \mathcal{P}_\infty(\mathcal{A})$ such that $w(p_\nu(x, s))$ is a ESLCSL. Let us consider now tools related to $\mathcal{A}$.

A finite directed graph or *digraph* can be associated with the automaton $\mathcal{A}$: $\mathcal{G} = (\mathcal{S}, \mathcal{E})$, with the set of vertices $\mathcal{S}$ and the set of edges $\mathcal{E} = \{(s, r) \in \mathcal{S}^2,\ \exists\ell \in \Sigma, r \in \delta(s, \ell)\}$.

Definition 5. Let $(s, r) \in \mathcal{S}^2$. s and r are *strongly connected* if $s = r$ or if there exist a path $p \in \mathcal{P}_m$ such that $\pi_1(p) = s$, $\pi_{m+1}(p) = r$ and a path $\tilde{p} \in \mathcal{P}_{\tilde{m}}$ such that $\pi_1(\tilde{p}) = r$, $\pi_{\tilde{m}+1}(\tilde{p}) = s$. This relation is an equivalence relation on the nodes.

Thanks to this equivalence relation, the set $\mathcal{S}$ may be partitioned into disjoint sets as follows.

Definition 6. Strongly Connected Components (SCCs). Let $\mathcal{G} = (\mathcal{S}, \mathcal{E})$ be a finite digraph and $\mathcal{C} \subset \mathcal{S}$. $\mathcal{C}$ is *strongly connected* if for every pair of vertices $(s, r) \in \mathcal{C}^2$, s and r are strongly connected. $\mathcal{C}$ is a *strongly connected component (SCC)* of the digraph $\mathcal{G}$, if this is a strongly connected set and if no additional vertex of $\mathcal{S}$ can be included in $\mathcal{C}$ without breaking its strongly connectivity. In other words, this is an equivalence class for the relation of strongly connectivity. A SCC $\mathcal{C}$ is called *trivial* if $\mathcal{C} = \{s\}$ and $(s, s) \notin \mathcal{E}$. A SCC $\mathcal{C}$ is called *terminal* if there is no SCC $\mathcal{D} \neq \mathcal{C}$ such that $(s, r) \in \mathcal{E}$ for some $s \in \mathcal{C}$ and $r \in \mathcal{D}$. Finite digraph $\mathcal{G}$ is *cyclic* if and only if $\mathcal{G}$ contains a nontrivial SCC.

By definition, the SCCs are disjoint and their union is equal to $\mathcal{S}$. In the following, we will denote $\mathcal{C}^{\mathcal{G}}_i$ ($i \in \mathbb{N}_d$) the d SCCs of the digraph $\mathcal{G}$ and $d_i = |\mathcal{C}^{\mathcal{G}}_i|$. The digraph of the SCCs (for which the vertices are the SCCs), or *condensation graph* is acyclic by definition of the SCCs. We can thus define a *partial relation order* between the SCCs. We denote $\mathcal{C}_i \succeq \mathcal{C}_j$ if there exists a path between one vertex in $\mathcal{C}_i$ and a vertex in $\mathcal{C}_j$.

Some generic observations for our contribution follow:

- We are interesting into switching laws of infinite length for exponential stabilization, then we can assume without loss of generality that the automaton has only *non-blocking* states ($\forall s \in \mathcal{S}$, $\exists j \in \Sigma$ such that $\delta(s, j) \neq \emptyset$) and also consider that there is no terminal SCC which is trivial.
- Thanks to the partial relation order concerning the SCCs, for any $p \in \mathcal{P}_\infty(\mathcal{A})$, $w(p)$ ultimately enters

and does not exit a nontrivial SCC. See for instance (Jungers et al., 2016, Proposition 2).

Based on these observations, we will focus on nontrivial SCCs of the automaton $\mathcal{A}$. Let us introduce the restriction to a nontrivial SCC of a path and of an exponentially stabilizing control Lyapunov function (ECLF) as follows.

Definition 7. Let $\mathcal{C}$ be a nontrivial SCC induced by the automaton $\mathcal{A}$. We define the set of finite paths of m transitions restricted to $\mathcal{C}$, $\mathcal{P}_m(\mathcal{A},\mathcal{C})$ such that

$$\mathcal{P}_m(\mathcal{A},\mathcal{C}) = \{p \in \mathcal{P}_m(\mathcal{A}),\ \pi_j(p) \in \mathcal{C},\ \forall j \in \mathbb{N}_{m+1}\}. \quad (2)$$

Definition 8. Let us consider a nontrivial SCC $\mathcal{C}$ of the digraph $\mathcal{G}$ induced by the automaton $\mathcal{A}$. A nonnegative continuous function $V : \mathbb{R}^n \times \mathcal{C} \to \mathbb{R}^+$ is an exponentially stabilizing control Lyapunov function (ECLF) of the system (1) in $\mathcal{C}$ if for any $(x, r) \in \mathbb{R}^n \times \mathcal{C}$, we have

(1) $\kappa_1\|x\|^2 \leq V(x,r) \leq \kappa_2\|x\|^2$ for some finite positive constants κ_1 and κ_2;
(2) There exists $p_\nu : \mathbb{R}^n \times \mathcal{C} \to \mathcal{P}_1(\mathcal{A},\mathcal{C})$, such that $\pi_1(p_\nu(x,r)) = r$, and $V(A_{w(p_\nu(x,r))}x, \pi_2(p_\nu(x,r))) - V(x,r) \leq -\kappa_3\|x\|^2$, for a constant $\kappa_3 > 0$.

We will decompose Problem 3 into two parts:

(1) For each nontrivial SCC $\mathcal{C}$ of $\mathcal{A}$, sufficient conditions ensuring the stabilizability of the system (1) over $\mathcal{C}$ will be checked, in order to detect if there is at least a SCC that is stabilzable.
(2) If at least a SCC is stabilizable, then we will check if one of them is reachable by an automaton state belonging to the set of initial automaton set $\mathcal{S}^0$ (see for instance the Dijkstra's algorithm (Dijkstra, 1976)). More precisely, let define the set of initial automaton states that can be chosen to reach a SCC that admits an ECLF

$$\tilde{\mathcal{S}}^0 = \mathcal{S}^0 \bigcap \left(\bigcup_{j\in\mathcal{Q}}\bigcup_{i\in\mathbb{N}_d,\mathcal{C}_i\succeq\mathcal{C}_j}\mathcal{C}_i\right) \quad (3)$$

with $\mathcal{Q} = \{i \in \mathbb{N}_d,\ \mathcal{C}_i$ admits an ECLF$\}$. Finally if $\tilde{\mathcal{S}}^0 \neq \emptyset$, then Problem 3 admits a solution.

The next section is devoted to obtain LMI-based sufficent conditions allowing to exponentially stabilize the system (1) with an admissible switching law of the language $L(\mathcal{C})$ generated only by a nontrivial SCC $\mathcal{C}$ of the automaton $\mathcal{A}$ instead of the language $L(\mathcal{A})$.

4. LMI-BASED SUFFICIENT CONDITIONS FOR STABILIZABILITY OF A SCC

The main result of this paper is given in the following theorem.

Theorem 9. Let be $\mathcal{C}$ a nontrivial SCC of the automaton $\mathcal{A}$, containing $h = |\mathcal{C}|$ automaton states, denoted $\mathcal{C} = \{c_1, \cdots, c_h\}$. For given h positive integers $N_i \in \mathbb{N}$, ($i \in \mathbb{N}_h$); let us define $\overline{\mathcal{P}}_{i,N_i}(\mathcal{A},\mathcal{C})$ as the set of paths starting from the state automaton c_i, admissible to the language $\mathcal{A}$ remaining in the SCC $\mathcal{C}$ of length less than or equal to N_i, that is

$$\overline{\mathcal{P}}_{i,N_i}(\mathcal{A},\mathcal{C}) = \bigcup_{j\in\mathbb{N}_{N_i}} \{p \in \mathcal{P}_j(\mathcal{A},\mathcal{C}),\ \pi_1(p) = c_i\} \quad (4)$$

and $\overline{N}_i$ the number of paths in $\overline{\mathcal{P}}_{i,N_i}(\mathcal{A},\mathcal{C})$. If there exist h vectors $\eta_i \in \mathbb{R}^{\overline{N}_i}$ ($i \in \mathbb{N}_h$), such that $\eta_i \geq 0$ and $\sum_{p\in\overline{\mathcal{P}}_{i,N_i}(\mathcal{A},\mathcal{C})} \eta_{i,p} = 1$ (with the abuse of notation that the components of η_i are indexed by the path p) and finally such that the h Linear Matrix Inequalities (LMI) are satisfied.

$$\sum_{p\in\overline{\mathcal{P}}_{i,N_i}(\mathcal{A},\mathcal{C})} \eta_{i,p}\mathbb{A}_{w(p)}^T\mathbb{A}_{w(p)} < I_n, \quad i \in \mathbb{N}_h, \quad (5)$$

then the system (1) is exponentially stabilizable with a switching law admissible to the language $L(\mathcal{C})$. □

Proof 1. The proof is constructive by designing an infinite path p^* that is restricted to $\mathcal{C}$ and that exponentially stabilizes the system (1). If the inequalities (5) hold, then there exists a scalar $\mu \in [0, 1)$ such that

$$\sum_{p\in\overline{\mathcal{P}}_{i,N_i}(\mathcal{A},\mathcal{C})} \eta_{i,p}\mathbb{A}_{w(p)}^T\mathbb{A}_{w(p)} < \mu^2 I_n, \quad i \in \mathbb{N}_h. \quad (6)$$

The latter inequalities yield that for all $x \in \mathbb{R}^n$,

$$\min_{p\in\overline{\mathcal{P}}_{i,N_i}(\mathcal{A},\mathcal{C})} x^T\mathbb{A}_{w(p)}^T\mathbb{A}_{w(p)}x \leq \mu^2x^Tx,\ i \in \mathbb{N}_h. \quad (7)$$

Now to design p^*, let us define iteratively as follows a sequence of indexes of automaton states $\{i_j\}_{j\in\mathbb{N}} \in (\mathbb{N}_h)^{\mathbb{N}}$ and a sequence of instants $\{k_j\}_{j\in\mathbb{N}} \in \mathbb{N}$, with $k_j < k_{j+1}$, $j \in \mathbb{N}$. The algorithm to build p^* is as follows:

Initialization: Choose $k_0 = 0$, arbitrarily $i_0 \in \mathbb{N}_h$, $x_0 \in \mathbb{R}^n$ and finally $p^* = \emptyset$.

Iteration j: Select $\tilde{p}$ such that

$$\tilde{p} \in \arg\min_{p\in\overline{\mathcal{P}}_{i_j,N_{i_j}}(\mathcal{A},\mathcal{C})} x_{k_j}^T\mathbb{A}_{w(p)}^T\mathbb{A}_{w(p)}x_{k_j}, \quad (8)$$

and define i_{j+1} as the unique value in $\mathbb{N}_h$ such that $c_{i_{j+1}}$ is the last automaton state of $\tilde{p}$:

$$\pi_{l(\tilde{p})+1}(\tilde{p}) = c_{i_{j+1}}, \quad (9)$$

and $k_{j+1} = k_j + l(\tilde{p})$. Build p^*_{k+1} by $p^*_k \circ \tilde{p}$, because the definition of i_{j+1} by equation (9) ensures the compatibility of the paths and allows the concatenation. The state is then given by

$$x_{k_j+z} = A_{w_z(\tilde{p})}x_{k_j+z-1}, \quad z \in \{1, \cdots, k_{j+1} - k_j\}, \quad (10)$$

and in closed form

$$x_{k_{j+1}} = \mathbb{A}_{w(\tilde{p})}x_{k_j}. \quad (11)$$

We have only to prove that this resulting trajectory x_k is exponentially stable. From relation (11) and equation (7), we have $\|x_{k_{j+1}}\| \leq \mu\|x_{k_j}\|$, and $\|x_{k_j}\| \leq \mu^j\|x_0\|$, $j \in \mathbb{N}$.

Let us denote a constant $\Gamma \geq \max(1, \max_{i\in\Sigma}(\|A_i\|))$ and $N = \max_{i\in\mathbb{N}_h}(N_i)$. Then between two instants k_{j+1} and k_j: $\|x_k\| \leq \Gamma^{k-k_j}\|x_{k_j}\|$, $k \in \{k_j, \cdots k_{j+1} - 1\}$. Because $k_{j+1} - k_j \leq N_{i_j} \leq N$, $j \geq k/N - 1$ and we infer $\|x_k\| \leq \mu^{k/N-1}\Gamma^{N-1}\|x_0\|$, $k \in \mathbb{N}$, that is the exponential stability of trajectory x_k. ■

Remark 10. It should be noted that the length of the considered paths in inequalities (5), namely N_i are related to the starting automaton state $c_i \in \mathcal{C}$. The LMIs (5) are *a priori* independent and the value of N_i leading the feasibility of each LMI may be sought independently.

Remark 11. The automaton $\mathcal{A}$ has only labels that are letters belonging to Σ. It is possible to complete the automaton $\mathcal{A}$ by considering in addition labels (and further transitions) that are admissible words of $L(\mathcal{A})$. This principle is shown in Figure 1. This operation on the automaton does not modify the language and make appear labels of length less than or equal to N_i involved in (5).

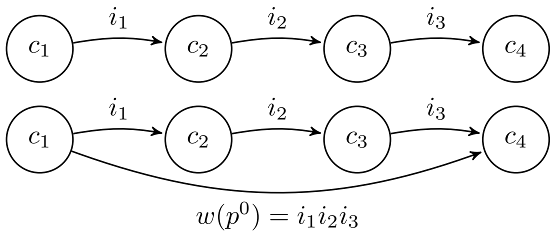

Fig. 1. Example of equivalent paths, with $p^0 = \{(c_1, i_1, c_2), (c_2, i_2, c_3), (c_3, i_3, c_4)\}$.

Theorem 9 shows sufficient conditions to ensure the exponential stabilization of a switched system (1), by providing a switching law solving Problem 3. However neither Euclidean norm nor the function $V : \mathcal{C} \times \mathbb{R}^n \mapsto \mathbb{R}^+$, $V(c_i, x) = \min_{p \in \overline{\mathcal{P}}_{i,N_i}(\mathcal{A},\mathcal{C})} x^T \mathbb{A}_{w(p)}^T \mathbb{A}_{w(p)} x$ are strictly decreasing functions for the resulting trajectory and thus are not associated ECLF. Nevertheless, it is possible to provide an ECLF and a related exponentially stabilizing switching law (in general distinct from the one provided in Theorem 9), based on LMIs (5), at the price of some modifications and by adding the following assumption:

Assumption 12. We assume that the matrices A_i, $i \in \mathbb{N}_q$ are invertible.

The main modification allowing to build an ECLF is based on the following observation. If LMIs (5), involving automaton state dependent lengths of paths, are feasible, they are also feasible by considering a common maximal length $N = \max_{i \in \mathbb{N}_h}(N_i)$, by setting to zero the scalar weights for the extra terms.

Theorem 13. By using the notation of Theorem 9, assume that inequalities (5) and Assumption 12 hold. Let consider $\lambda = \mu^{2/N} \in [0, 1)$, with μ defined in Theorem 9. Let us build iteratively the infinite path p^{**} solving Problem 3:

Initialization: Set $x_0 \in \mathbb{R}^n$. Choose arbitrarily $i_0 \in \mathbb{N}_h$ and $p_0^{**} = \emptyset$.

Iteration at instant $k \in \mathbb{N}$: Select $\hat{p}_k$ such that

$$\hat{p}_k \in \arg \min_{p \in \overline{\mathcal{P}}_{i_k,N}(\mathcal{A},\mathcal{C})} x_k^T \lambda^{-l(p)} \mathbb{A}_{w(p)}^T \mathbb{A}_{w(p)} x_k, \quad (12)$$

that implies in particular that $\pi_1(\hat{p}_k) = c_{i_k}$ and define i_{k+1} as the unique value in $\mathbb{N}_h$ such that

$$\pi_2(\hat{p}_k) = c_{i_{k+1}}, \quad (13)$$

which means that only the first transition in the path $\hat{p}_k$ is applied. Build $p_{k+1}^{**} = p_k^{**} \circ (\pi_1(\hat{p}_k), w_1(\hat{p}_k), \pi_2(\hat{p}_k))$, because the definition of i_{j+1} by equation (13) ensures the compatibility of the paths and allows the concatenation between the path p_k^{**} and the transition $(\pi_1(\hat{p}_k), w_1(\hat{p}_k), \pi_2(\hat{p}_k))$. The path $p^{**} = p_\infty^{**}$ finally verifies $\pi_k(p^{**}) = \pi_1(\hat{p}_k)$ and $\pi_{k+1}(p^{**}) = \pi_2(\hat{p}_k)$. The state is then given by $x_{k+1} = A_{w_1(\hat{p}_k)} x_k$.

We introduce

$$V : \begin{cases} \mathcal{C} \times \mathbb{R}^n \longrightarrow \mathbb{R}^+, \\ (c_i, x) \longmapsto \min_{p \in \overline{\mathcal{P}}_{i,N}(\mathcal{A},\mathcal{C})} x^T \lambda^{-l(p)} \mathbb{A}_{w(p)}^T \mathbb{A}_{w(p)} x \end{cases} \quad (14)$$

which is a Lyapunov function candidate thanks to Assumption 12 and that verifies

$$V(\pi_{k+1}(p^{**}), A_{w_k(p^{**})} x_k) \leq \lambda V(\pi_k(p^{**}), x_k). \quad (15)$$

In other words V defined by (14) is an ECLF for (1). □

Remark 14. It is crucial to mention here that the min-switching strategy given by inclusion (12) differs from the min-switching strategy used in inclusion (8) on two points: the terms are weighted by a scalar depending on the length of each path and also the maximal length of the paths, that is N_i, should be identical ($N_i = N$).

Proof 2. By using μ defined in the proof of Theorem 9, we have $\mu^{-2} = \lambda^{-N} \geq \lambda^{-l(p)}$, for $p \in \overline{\mathcal{P}}_{i,N}(\mathcal{A},\mathcal{C})$, that is

$$\sum_{p \in \overline{\mathcal{P}}_{i,N}(\mathcal{A},\mathcal{C})} \eta_{i,p} \lambda^{-l(p)} \mathbb{A}_{w(p)}^T \mathbb{A}_{w(p)} < I_n. \quad (16)$$

From inequality (16), we infer that the function V defined by equation (14) verifies

$$V(c_i, x) \leq x^T x, \quad \forall i \in \mathbb{N}_h. \quad (17)$$

Then two cases should be considered depending on the length of the path $\hat{p}$ defined in equation (12). If $l(\hat{p}_k) = 1$

$$\begin{aligned} V(\pi_k(p^{**}), x_k) &= V(\pi_1(\hat{p}_k), x_k) \\ &= \lambda^{-1}(x_k^T A_{w_1(\hat{p}_k)}^T) I_n (A_{w_1(\hat{p}_k)} x_k), \end{aligned} \quad (18)$$

and due to inequality (17) and selecting $i = \pi_{k+1}(p^{**}) \in \mathbb{N}_h$ therein, we have

$$V(\pi_k(p^{**}), x_k) \geq \lambda^{-1} V(\pi_{k+1}(p^{**}), A_{w_1(\hat{p}_k)} x_k), \quad (19)$$

that is inequality (15).

If $l(\hat{p}_k) \geq 2$, let decompose the path $\hat{p}_k = \tau_k \circ \hat{\hat{p}}_k$, where the transition τ_k is given by definition by $\tau = (\pi_1(\hat{p}_k), w_1(\hat{p}_k), \pi_2(\hat{p}_k))$ and $\pi_1(\hat{\hat{p}}_k) = \pi_2(\hat{p}_k) = \pi_{k+1}(p^{**})$. Then

$$\begin{aligned} V(\pi_k(p^{**}), x_k) &= V(\pi_1(\hat{p}_k), x_k) \\ &= \lambda^{-l(\hat{p}_k)} x_k^T \mathbb{A}_{w(\hat{p}_k)}^T \mathbb{A}_{w(\hat{p}_k)} x_k, \\ &= \lambda^{-1} x_k^T A_{w_1(\hat{p}_k)}^T (\lambda^{-l(\hat{\hat{p}}_k)} \mathbb{A}_{w(\hat{\hat{p}}_k)}^T \mathbb{A}_{w(\hat{\hat{p}}_k)}) A_{w_1(\hat{p}_k)} x_k, \end{aligned} \quad (20)$$

The path $\hat{\hat{p}}_k$, of length up to $N - 1$, belongs by definition to $\overline{\mathcal{P}}_{\pi_{k+1}(p^{**}),N}(\mathcal{A},\mathcal{C})$ (and not necessarily to $\overline{\mathcal{P}}_{\pi_{k+1}(p^{**}),N_{\pi_{k+1}(p^{**})}}(\mathcal{A},\mathcal{C})$). That yields

$$V(\pi_k(p^{**}), x_k) \geq \lambda^{-1} V(\pi_{k+1}(p^{**}), A_{w_1(\hat{p}_k)} x_k), \quad (21)$$

implying (15). □

5. RELATION WITH PERIODIC STABILIZABILITY

This section is devoted to show the equivalence between the feasibility of our condition and the periodic stabilizability in the case of language constrained switching law, to extend (Fiacchini et al., 2016, Theorem 22). First of all, we need to define the periodic stabilizability in the context of language constrained switched system.

Definition 15. The system (1) is *periodic stabilizable with the language constraint* $L(\mathcal{A})$ if there exist an automaton state $s \in \mathcal{S}^0$ (or *ultimately periodic stabilizable* if there exists a reachable automaton state $s \in \mathcal{S}$ from $\mathcal{S}^0$) and a cyclic path $p_{\text{per}} \in \mathcal{P}_m(\mathcal{A})$ with $\pi_1(p_{\text{per}}) = s$ such that $\mathbb{A}_{w(p_{\text{per}})}$ is Schur.

A cyclic path belongs to a SCC. As in the previous sections, we will consider only a SCC $\mathcal{C}$ of the automaton $\mathcal{A}$ and a periodic stabilizability restricted to this SCC $\mathcal{C}$.

Theorem 16. The system (1) is periodic stabilizable on a SCC $\mathcal{C}$ of $\mathcal{A}$ if and only if there exist $h = |\mathcal{C}|$ natural integers N_i, $(i \in \mathbb{N}_h)$ and vectors η_i in the simplex of dimensions $\overline{N}_i$, such that LMIs (5) are satisfied.

Proof 3. We prove both implications. Let us assume that the system (1) is periodic stabilizable and prove that the LMIs (5) are satisfied. There exists a cyclic path $p_{\text{per}} \in \mathcal{P}_m(\mathcal{A})$ such that $\mathbb{A}_{w(p_{\text{per}})}$ is Schur. Let us denote $c_{i_{\text{per}}} = \pi_1(p_{\text{per}})$. Then there exists a concatenated path $p = \circ^{n_{i_{\text{per}}}} p_{\text{per}} = p_{\text{per}} \circ \cdots \circ p_{\text{per}}$ ($n_{i_{\text{per}}}$ times) such that

$$\mathbb{A}^T_{w(p)}\mathbb{A}_{w(p)} = (\mathbb{A}^{n_1}_{w(p_{\text{per}})})^T \mathbb{A}^{n_1}_{w(p_{\text{per}})} < I_n \qquad (22)$$

for a sufficiently large $n_1 \in \mathbb{N}$. Note that the concatenation is possible thanks to the fact that p_{per} is cyclic. Thus we obtain a particular solution to the LMI related to the automaton state $c_{i_{\text{per}}}$ of $\mathcal{C}$, with $N_i = n_{i_{\text{per}}} l(p_{\text{per}})$. For the LMIs related to the other automaton states of $\mathcal{C}$, we use the connectivity of the automaton states. For all $i \in \mathbb{N}_h / \{i_{\text{per}}\}$, there exist a finite path in $\mathcal{C}$, $p^{i\to i_{\text{per}}}$ with $\pi_1(p^{i\to i_{\text{per}}}) = i$ and $\pi_{l(p^{i\to i_{\text{per}}})+1}(p^{i\to i_{\text{per}}}) = i_{\text{per}}$ and $n_i \in \mathbb{N}$ sufficiently large such that

$$\begin{aligned}&\mathbb{A}^T_{w(p^{i\to i_{\text{per}}} \circ^{n_i} p_{\text{per}})}\mathbb{A}_{w(p^{i\to i_{\text{per}}} \circ^{n_i} p_{\text{per}})} \\ &= (\mathbb{A}^{n_i}_{w(p_{\text{per}})}\mathbb{A}_{w(p^{i\to i_{\text{per}}})})^T \mathbb{A}^{n_i}_{w(p_{\text{per}})}\mathbb{A}_{w(p^{i\to i_{\text{per}}})} < I_n, \qquad (23)\end{aligned}$$

which is a particular solution of the LMIs (5) related to the automaton states c_i, with $N_i = n_i l(p^{i\to i_{\text{per}}})$.

Now let us prove the reciprocity. Assume that LMIs (5) are feasible and consider $\mu \in [0, 1)$ such that LMIs (6), with common maximal length $N = \max_{i\in\mathbb{N}_h}(N_i)$ without loss of generality. Let us start for a given LMIs (6). For each path $p \in \overline{\mathcal{P}}_{i,N}(\mathcal{A}, \mathcal{C})$, we modify the term $\mathbb{A}^T_{w(p)}\mathbb{A}_{w(p)} = \mathbb{A}^T_{w(p)} I_n \mathbb{A}_{w(p)}$ in order to insert one of the LMIs (6) and to concatenate the paths. However to allow the concatenation, we need to select the LMI (6) related to the last automaton state of p, $\pi_{l(p)+1}(p)$, which depends on the considered path p. It results

$$\mathbb{A}^T_{w(p)}\Big(\sum_{\tilde{p}\in\overline{\mathcal{P}}_{\pi_{l(p)+1},N}(\mathcal{A},\mathcal{C})} \eta_{\pi_{l(p)+1},\tilde{p}}\mathbb{A}^T_{w(\tilde{p})}\mathbb{A}_{w(\tilde{p})}\Big)\mathbb{A}_{w(p)} < \mu^2 \mathbb{A}^T_{w(p)}\mathbb{A}_{w(p)}. \qquad (24)$$

By taking the weighted sum over $\overline{\mathcal{P}}_{i,N}(\mathcal{A}, \mathcal{C})$ leads to

$$\sum_{p\in\overline{\mathcal{P}}_{i,N}(\mathcal{A},\mathcal{C})} \sum_{\tilde{p}(p)\in\overline{\mathcal{P}}_{\pi_{l(p)+1},N}(\mathcal{A},\mathcal{C})} \eta_{i,p}\eta_{\pi_{l(p)+1}(p),\tilde{p}(p)} \times \mathbb{A}^T_{w(p\circ\tilde{p}(p))}\mathbb{A}_{w(p\circ\tilde{p}(p))} < \mu^4 I_n. \qquad (25)$$

By noticing that the concatenated paths $p\circ\tilde{p}(p)$ belongs to $\overline{\mathcal{P}}_{i,2N}(\mathcal{A}, \mathcal{C})$ and that the weights $\eta_{i,p}\eta_{\pi_{l(p)+1}(p),\tilde{p}(p)}$ can be reformulated, we can build a new vector $\tilde{\eta}$ in the simplex of dimension $2N$ such that

$$\sum_{p\in\overline{\mathcal{P}}_{i,2N}(\mathcal{A},\mathcal{C})} \tilde{\eta}_{i,p}\mathbb{A}^T_{w(p)}\mathbb{A}_{w(p)} < \mu^4 I_n. \qquad (26)$$

For the paths of $\overline{\mathcal{P}}_{i,2N}(\mathcal{A}, \mathcal{C})$ which cannot be decomposed into such a concatenation $p \circ \tilde{p}$, the associated weight in LMI (26) is set to zero. This is, for instance, always the case for the paths of length one. By following the same way, we obtain for any integer $z \in \mathbb{N}$ that

$$\sum_{p\in\overline{\mathcal{P}}_{i,zN}(\mathcal{A},\mathcal{C})} \hat{\eta}_{i,p}\mathbb{A}^T_{w(p)}\mathbb{A}_{w(p)} < \mu^{2z} I_n. \qquad (27)$$

Now we set z such that $\mu^{2z} < (n^2\Gamma^{2h})^{-1}$, where $\Gamma \geq \max(1, \max_{i\in\Sigma}(\|A_i\|))$. This implies that there exists $p_m \in \overline{\mathcal{P}}_{i,zN}(\mathcal{A}, \mathcal{C})$ such that

$$Tr(\mathbb{A}^T_{w(p_m)}\mathbb{A}_{w(p_m)}) = \min_{p\in\overline{\mathcal{P}}_{i,zN}(\mathcal{A},\mathcal{C})} Tr(\mathbb{A}^T_{w(p)}\mathbb{A}_{w(p)}) < \frac{1}{n\Gamma^{2h}}.$$

The SCC $\mathcal{C}$ containing h automaton states, there exists a path p_{comp} of length less than or equal to h joining $\pi_{l(p_m)+1}(p_m) = \pi_1(p_{\text{comp}})$ to $\pi_1(p_m) = i = \pi_{l(p_{\text{comp}})+1}(p_{\text{comp}})$. Finally it yields

$$\begin{aligned}&Tr(\mathbb{A}^T_{w(p_m\circ p_{\text{comp}})}\mathbb{A}_{w(p_m\circ p_{\text{comp}})}) \\ &\quad = Tr(\mathbb{A}^T_{w(p_m)}\mathbb{A}_{w(p_m)}\mathbb{A}_{w(p_{\text{comp}})}\mathbb{A}^T_{w(p_{\text{comp}})}) \\ &\quad \leq Tr(\mathbb{A}^T_{w(p_m)}\mathbb{A}_{w(p_m)})Tr(\mathbb{A}_{w(p_{\text{comp}})}\mathbb{A}^T_{w(p_{\text{comp}})}) \\ &\quad \leq n\Gamma^{2l(p_{comp})}Tr(\mathbb{A}^T_{w(p_m)}\mathbb{A}_{w(p_m)}) \\ &\quad \leq n\Gamma^{2h}Tr(\mathbb{A}^T_{w(p_m)}\mathbb{A}_{w(p_m)}) < 1, \qquad (28)\end{aligned}$$

implying that $\mathbb{A}_{w(p_m\circ p_{\text{comp}})}$ is Schur with $p_m \circ p_{\text{comp}}$ being a cyclic path. □

6. ILLUSTRATION

Let us consider the switched system (1) with $q = 3$ modes, $x_0 = \begin{pmatrix} 2 & -1 \end{pmatrix}^T$ and

$$[\, A_1 \,|\, A_2 \,|\, A_3 \,] = \left[\begin{array}{cc|cc|cc} 1.3 & 0.8 & 0.6 & 0 & 1.1 & 0 \\ 0 & 0.5 & -0.4 & -1/0.6 & 0 & -0.9 \end{array}\right].$$

The automaton $\mathcal{A}$ defining the constrained language is the one in Figure 2, with all the automaton states as initial ones: $\mathcal{S}^0 = \{e_1, e_2\}$. We select the automaton state e_1 as the initial one in the simulation. By applying Theorem 9, the inequalities (5) are feasible with $N_1 = 3$, and $N_2 = 4$, leading to the following number of terms in the LMIs: $\overline{N}_1 = 48$ and $\overline{N}_2 = 192$.

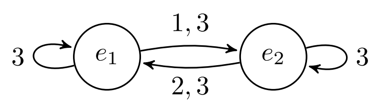

Fig. 2. Automaton related to the numerical example.

The constant mode 3 is admissible, roughly speaking there exists (at least) a path $p \in \mathcal{P}_\infty(\mathcal{A})$ such that $w_k(p) = 3$, but the matrix A_3 is not Schur.

An exponential stabilizing path p^* defined in the proof of Theorem 9 is depicted as follows: $w(p^*)$ and $\pi(p^*)$ in Figure 3. The resulting trajectory of the state of the switched system (1) is given in Figure 4. One can observe that the close loop trajectory converges to the origin as expected.

7. CONCLUSION

The problem of obtaining sufficient conditions to stabilize an autonomous discrete-time switched system with lan-

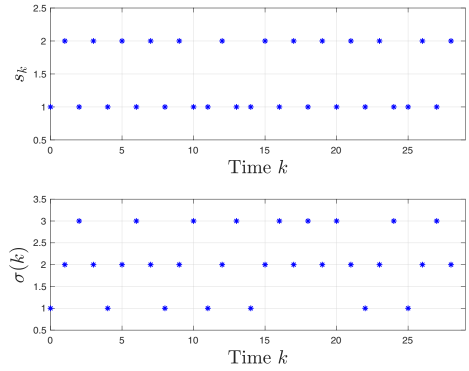

Fig. 3. Trajectory of the automaton state $k \mapsto s_k = \pi_k(p^*)$ (top) and switching law $\sigma(k) = w_k(p^*)$ (bottom) by applying Theorem 9.

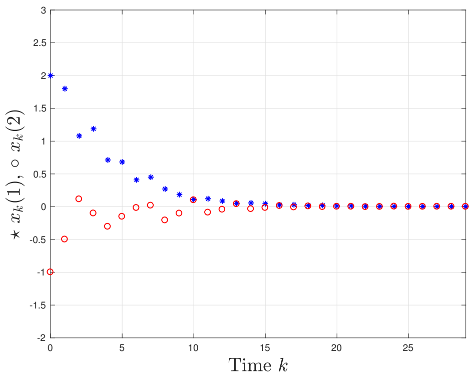

Fig. 4. Trajectory $k \mapsto x_k$ for the resulting switching law.

guage constraints for the switching law has been studied in this paper. The nondeterministic automaton characterizing the language is decomposed into strongly connected components allowing one to focus only on a nontrivial strongly connected component, in addition of the reachability of this component from the initial automaton states. The stabilizability reduced to this strongly connected component is checked with a finite set of Linear Matrix Inequalities related to its automaton states. When these LMIs are feasible, an exponential stabilizing switching law, that belongs to the language constraints, is provided thanks to a min-switching strategy. It is also shown that the feasibility of the LMIs is equivalent to the periodic stabilizability taking into account the language constrained. A numerical example has illustrated the result.

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REFERENCES

Ahmadi, A.A., Jungers, R.M., Parrilo, P.A., and Roozbehani, M. (2014). Joint spectral radius and path-complete graph Lyapunov functions. *SIAM Journal on Control and Optimization*, 52(1), 687–717.

Athanasopoulos, N., Smpoukis, K., and Jungers, R.M. (2017). Invariant sets analysis for constrained switching systems. *IEEE Control Systems Letters*, 1(2), 256–261.

Baier, C. and Katoen, J.P. (2008). *Principles of Model Checking*. The MIT Press.

Bang-Jensen, J. and Gutin, G.Z. (2008). *Digraphs: theory, algorithms and applications*. Springer.

Daafouz, J., Riedinger, P., and Iung, C. (2002). Stability analysis and control synthesis for switched systems: A switched Lyapunov function approach. *IEEE Transactions on Automatic Control*, 47, 1883–1887.

Dijkstra, E. (1976). *A Discipline of Programming*. NJ: Prentice Hall.

Fiacchini, M., Girard, A., and Jungers, M. (2016). On the stabilizability of discrete-time switched linear systems: Novel conditions and comparisons. *IEEE Transactions on Automatic Control*, 61(5), 1181–1193.

Fiacchini, M. and Jungers, M. (2014). Necessary and sufficient condition for stabilizability of discrete-time linear switched systems: A set-theory approach. *Automatica*, 50(1), 75–83.

Fiacchini, M., Jungers, M., and Girard, A. (2018). Stabilization and control Lyapunov functions for language constrained discrete-time switched linear systems. *Automatica, regular paper*, 93, 64–74.

Geromel, J.C. and Colaneri, P. (2006). Stability and stabilization of discrete-time switched systems. *International Journal of Control*, 79(7), 719–728.

Gurvits, L. (1995). Stability of discrete linear inclusion. *Linear Algebra and its Applications*, 231, 47–85.

Jungers, M., Girard, A., and Fiacchini, M. (2016). Language constrained stabilization of discrete-time switched linear systems: a Lyapunov-Metzler inequalities approach. In *Proceedings of 55th IEEE Conference on Decision and Control*, 5539–5544. Las Vegas, USA.

Jungers, R.M. and Mason, P. (2017). On feedback stabilization of linear switched systems via switching signal control. *SIAM J. Control Optim.*, 55(2), 1179–1198.

Lee, D. and Hu, J. (2016). Graph control Lyapunov function for stabilization of discrete-time switched linear systems. In *Proceedings of the 2016 American Control Conference (ACC)*, 697–702.

Lee, D. and Hu, J. (2017a). On the graph control Lyapunov function for stabilization of discrete-time switched linear systems. In *Proceedings of the 2017 American Control Conference (ACC)*, 5159–5164.

Lee, D. and Hu, J. (2017b). Periodic stabilization of discrete-time switched linear systems. *IEEE Transactions on Automatic Control*, 62(7), 3382–3394.

Liberzon, D. (2003). *Switching in Systems and Control*. Birkhäuser, Boston, MA.

Mancilla-Aguilar, J.L., Garcia, R., Sontag, E., and Wang, Y. (2005). Uniform stability properties of switched systems with switchings governed by digraphs. *Nonlinear Analysis*, 63, 472–490.

Sun, Z. and Ge, S.S. (2011). *Stability Theory of Switched Dynamical Systems*. Springer.

Wang, Y., Roohi, N., Dullerud, G.E., and Viswanathan, M. (2017). Stability analysis of switched linear systems defined by regular languages. *IEEE Transactions on Automatic Control*, 62(5), 2568–2575.