Control Synthesis for Stochastic Switched Systems using the Tamed Euler Method

Adrien Le Coënt * **Laurent Fribourg** ** **Jonathan Vacher** ***

* *Department of computer science, Aalborg University, Selma Lagerløfs Vej 300, Aalborg, Denmark.*
** *LSV, CNRS, INRIA, ENS Paris-Saclay, 61 Av. du Pdt. Wilson, 94235, Cachan, France.*
*** *Albert Einstein College of Medicine, Block Research Pavilion, Department of Systems and Computational Biology, 1301 Morris Park Avenue, 10461 Bronx, NY, USA.*

Abstract: In this paper, we explain how, under the one-sided Lipschitz (OSL) hypothesis, one can find an error bound for a variant of the Euler-Maruyama approximation method for stochastic switched systems. We then explain how this bound can be used to control stochastic switched switched system in order to stabilize them in a given region. The method is illustrated on several examples of the literature.

Keywords: Stochastic systems, numerical simulation, control system synthesis, switched control systems, nonlinear control systems.

1. INTRODUCTION

Symbolic methods for the verification and control synthesis of hybrid systems (and, particularly, "switched systems") have received significant attention in the past few years. One distinguishes two main classes of symbolic methods for hybrid systems: indirect methods and direct methods (Asarin et al. (2000)).

Indirect methods proceed by constructing a finite abstraction of the original system by discretization of the dense state space $\mathbb{R}^d$ (where d is the dimension of the state space). Among the indirect methods, one of the most successful proceeds by *approximate bisimulation* (Girard et al. (2010)). This method originally designed for deterministic switched systems has been recently extended for *stochastic switched systems* (Zamani et al. (2015, 2014, 2017)). This approach relies on the hypothesis of *incremental stability* of the stochastic switched system (or existence of a common/multiple Lyapunov function).

A direct method proceeds by working directly at the level of the dense state system $\mathbb{R}^d$; it computes "trajectory tubes", which are over-approximations of the set of all the controlled trajectories starting at a given subregion of $\mathbb{R}^d$. In previous work, We have followed such a direct approach (eg, (Fribourg et al. (2014))). The idea is to start with two given hyperrectangles R and S of $\mathbb{R}^d$, (with $R \subseteq S$): one covers R with a finite number of subregions (of the form of balls or sub-rectangles), and finds by exhaustive search, for each subregion, a "control pattern" (i.e., a finite control sequence) such that the trajectories starting from the subregion and controlled by the pattern go back to R while never leaving S. Such a direct method ensures the so-called property of "(R, S)-stability". We have recently applied such a direct method in the deterministic framework, using the Euler approximation scheme for calculating over-approximations of tubes of trajectories (Le Coënt et al. (2017)). We show here how to extend this direct method in order to treat stochastic switched systems. The method is an extension of the deterministic one, but replaces the classical Euler approximation scheme, by a variant of the stochastic Euler-Maruyama scheme (Hutzenthaler et al. (2012)). The correctness of these Euler-based methods does not rely on the hypothesis of incremental stability as in (Zamani et al. (2015, 2017)), but on the hypothesis of 'one-sided Lipschitz (OSL)' condition with constant $\lambda \in \mathbb{R}^d$ (also called 'monotonicity'/'dissipativity', see (von Renesse and Scheutzow (2010))). It can be seen that if a stochastic switched system satisfies an OSL condition with $\lambda < 0$, then the function $V(x, x') = \|x - x'\|^2$ is a *common incremental Lyapunov function* in the sense of (Zamani et al. (2014)), from which it follows that the switched system is incrementally stable, and can be treated by approximate bisimulation. However, Euler-based methods also apply when the system is not incrementally stable, in which case the constant λ is necessarily positive. We thus consider a class of systems different from that of (Zamani et al. (2014)).

The plan of the paper is as follows: In Section 2, we give an explicit upper bound on the mean square error of the tamed Euler method for SDEs under OSL condition. We apply the result in order to ensure properties of stochastic switched systems, such as "(R, S)-stability" (Section 3). We conclude in Section 4.

This work is supported by the LASSO project financed by an ERC adv. grant; and the DiCyPS project funded by Innovation Fund Denmark.

2. BOUNDING THE ERROR OF THE TAMED EULER METHOD

2.1 Assumptions

The symbol $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$.The symbol $\langle\cdot,\cdot\rangle$ denotes the scalar product of two vectors of $\mathbb{R}^d$. Given a point $x \in \mathbb{R}^d$ and a positive real $r > 0$, the ball $B(x,r)$ of centre x and radius r is the set $\{y \in \mathbb{R}^d \mid \|x - y\| \leqslant r\}$.

Let $\tau \in (0,\infty)$ be a fixed real number, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with normal filtration $(\mathcal{F}_t)_{t\in[0,\tau]}$, let $d, m \in \mathbb{N} := \{1,2,\dots\}$ let $W = (W^{(1)},\dots,W^{(m)}) : [0,R]\times\Omega \to \mathbb{R}^m$ be an m-dimensional standard $(W_t)_{t\in[0,\tau]}$-Brownian motion and let $x_0 : \Omega \to \mathbb{R}^d$ be an $\mathcal{F}_0/\mathcal{B}(\mathbb{R}^d)$-measurable mapping with $\mathbb{E}[\|x_0\|^p] < \infty$ for all $p \in [1,\infty)$. Moreover, let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a continuously differentiable function whose derivative grows at most polynomially. Formally, let us suppose the existence of constants $D \in \mathbb{R}_{\geqslant 0}$ and $q \in \mathbb{N}$ such that, for all $x, y \in \mathbb{R}^d$

$$\|f(x) - f(y)\|^2 \leqslant D\|x - y\|^2(1 + \|x\|^q + \|y\|^q) \tag{H1}$$

Let $g = (g_{i,j})_{i\in\{1,\dots,d\},j\in\{1,\dots,m\}} : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be a globally Lipschitz continuous function: there exists $L_g \in \mathbb{R}_{\geqslant 0}$ such that, for all $x, y \in \mathbb{R}^d$

$$\|g(x) - g(y)\| \leqslant L_g\|x - y\| \tag{H2}$$

Finally, let us suppose that f is globally one-sided Lipschitz with constant $\lambda \in \mathbb{R}$:

$$\exists\lambda \in \mathbb{R}\ \forall x, y \in \mathbb{R}^d : \langle f(y) - f(x), y - x\rangle \leqslant \lambda\,\|y - x\|^2 \tag{H3}$$

Then consider the Stochastic Differential Equations (SDE):

$$\mathrm{d}X_t = f(X_t)\mathrm{d}t + g(X_t)\mathrm{d}W_t, \qquad X_0 = x_0 \tag{1}$$

for $t \in [0,\tau]$. The drift coefficient f is the infinitesimal mean of the process X and the diffusion coefficient g is the infinitesimal standard deviation of the process X. Under the above assumptions, the SDE (1) is known to have a unique strong solution. More formally, there exists an adapted stochastic process $X : [0,\tau]\times\Omega \to \mathbb{R}^d$ with continuous sample paths fulfilling

$$X_{t,x_0} = x_0 + \int_0^t f(X_s)\mathrm{d}s + \int_0^t g(X_s)\mathrm{d}W_s$$

for all $t \in [0,\tau]$ $\mathbb{P}$-a.s. (see, e.g., (Oksendal (2002))).

We denote by X_{t,x_0} the solution of Equation (1) at time t from initial condition $X_{0,x_0} = x_0$ $\mathbb{P}$-a.s., in which x_0 is a random variable that is measurable in $\mathcal{F}_0$.

Remark 1. Constants λ, L_g and D can be computed using (constrained) optimization algorithms (see (Le Coënt et al. (2017))).

2.2 Tamed Euler approximation

The standard way to extend the classical Euler method for ordinary differential equations to the SDE (1) is the Euler-Maruyama scheme (Maruyama (1955)). More precisely, given $z : \Omega \to \mathbb{R}^d$ an $\mathcal{F}_0/\mathcal{B}(\mathbb{R}^d)$-measurable mapping with $\mathbb{E}[\|z\|^p] < \infty$ for all $p \in [1,\infty)$, the explicit Euler-Maruyama (EM) method for the SDE (1) is given by the mappings $Y^N_{n,z} : \Omega \to \mathbb{R}^d$, $n \in \{0,1,\dots,N\}$, which satisfy $Y^N_{0,z} = z$ and

$$Y^N_{n+1,z} = Y^N_{n,z} + \frac{\tau}{N}\cdot f(Y^N_{n,z}) + g(Y^N_{n,z})(W_{(n+1)\tau/N} - W_{n\tau/N})$$

for all $n \in \{0,1,\dots,N-1\}$ and all $N \in \mathbb{N}$. See (Maruyama (1955)). Unfortunately, the convergence results for the EM scheme does not hold when the drift function f of the SDE (1) behaves polynomially (and not linearly). For the sake of generality, we will now adopt a refined scheme, which has been proposed recently in order to overcome this difficulty (Hutzenthaler et al. (2012)). Let $\underline{X}^N_{n,z} : \Omega \to \mathbb{R}^d$,

$$\underline{X}^N_{n+1,z} = \underline{X}^N_{n,z} + \frac{\frac{\tau}{N}\cdot f(\underline{X}^N_{n,z})}{1 + \frac{\tau}{N}\cdot\|f(\underline{X}^N_{n,z})\|} + g(\underline{X}^N_{n,z})(W_{\frac{(n+1)\tau}{N}} - W_{\frac{n\tau}{N}}) \tag{2}$$

for all $n \in \{0,1,\dots,N-1\}$ and all $N \in \mathbb{N}$. We refer to the numerical method (2) as a *tamed Euler scheme* (Hutzenthaler et al. (2012)). In this method the drift term $\frac{\tau}{n}\cdot f(\underline{X}^N_{n,z})$ is "tamed" by the factor $1/(1 + \frac{\tau}{N}\cdot\|f(\underline{X}^N_{n,z})\|)$ for $n \in \{0,1,\dots,N-1\}$ and $N \in \mathbb{N}$ in (2).

A time continuous interpolation of the time discrete numerical approximations (2) is also introduced in (Hutzenthaler et al. (2012)) as follows. Let $\tilde{X}^N_z : [0,\tau]\times\Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, be a sequence of stochastic processes given by

$$\tilde{X}^N_{t,z} = \tilde{X}^N_{n,z} + \frac{(t - n\tau/N)\cdot f(\tilde{X}^N_{n,z})}{1 + \tau/N\cdot\|f(\tilde{X}^N_{n,z})\|} + g(\tilde{X}^N_{n,z})(W_t - W_{\frac{n\tau}{N}})$$

for all $t \in [\frac{n\tau}{N}, \frac{(n+1)\tau}{N}]$, $n \in \{0,1\dots,N-1\}$ and all $N \in \mathbb{N}$. Note that $\tilde{X}^N_{t,z} : [0,\tau]\times\Omega \to \mathbb{R}^d$ is an adapted stochastic process with continuous sample paths for every $N \in \mathbb{N}$.

Let us define $\underline{X}^N_{t,z}$ by

$$\underline{X}^N_{t,z} := \underline{X}^N_{n,z} \quad \text{for } t \in [\frac{n\tau}{N}, \frac{(n+1)\tau}{N}).$$

Note that $\tilde{X}^N_{t,z} = \underline{X}^N_{t,z} = \underline{X}^N_{n,z}$ at time $t = \frac{n\tau}{N}$ for $n \in \{0,1,\dots,N\}$.

The following theorem is proven in (Hutzenthaler et al. (2012)):

Theorem 1. (Hutzenthaler et al. (2012)). Let us suppose (H1) (H2) and (H3). Let the setting in this section be fulfilled, and $z : \Omega \to \mathbb{R}^d$ be an $\mathcal{F}_0/\mathcal{B}(\mathbb{R}^d)$-measurable mapping with $\mathbb{E}[\|z\|^p] < \infty$ for all $p \in [1,\infty)$. Then, for all $p \in [1,\infty)$

$$\sup_{N\in\mathbb{N}}\ \sup_{n\in\{0,1,\dots,N\}} \mathbb{E}[\|\underline{X}^N_{n,z}\|^p] < \infty$$

This theorem allows to ensure the strong convergence of the tamed Euler method. Any number N of subsampling steps can thus be used. This number is now left implicit for the sake of simplicity. From Theorem 1, it follows (cf. Lemma 4.3, (Higham et al. (2002))):

Lemma 1. Let us suppose (H1) (H2) and (H3). Let the setting in this section be fulfilled, and $z : \Omega \to \mathbb{R}^d$ be an $\mathcal{F}_0/\mathcal{B}(\mathbb{R}^d)$-measurable mapping with $\mathbb{E}[\|z\|^p] < \infty$ for all $p \in [1,\infty)$. Then, for any even integer $r \geqslant 2$, there exist two constants $E_{r,z}$ and $F_{r,z}$ such that

$$\sup_{0\leqslant t\leqslant\tau} \mathbb{E}\|\underline{X}_{t,z} - \tilde{X}_{t,z}\|^r \leqslant (\Delta_t)^{\frac{r}{2}}(E_{r,z}(\Delta_t)^{\frac{r}{2}} + F_{r,z}d).$$

with $\Delta_t = \tau/N$ and:

$$E_{r,z} = 2^r(\|f(0)\|^r + D2^{\frac{r+1}{2}}$$
$$(1 + \mathbb{E}\sup_{0\leqslant t\leqslant\tau}\|\underline{X}_{t,z}\|^{qr})^{\frac{1}{2}}(\mathbb{E}\sup_{0\leqslant t\leqslant\tau}\|\underline{X}_{t,z}\|^{2r})^{\frac{1}{2}}),$$

$F_{r,z} = 2^r(\|g(0)\|^{2r} + L_g^r \mathbb{E}\sup_{0\leqslant t\leqslant\tau}\|\underline{X}_{t,z}\|^{\frac{r}{2}})$.

Remark 2. Constants $E_{r,z}$ and $F_{r,z}$ are computed using the constants λ and L_g (see Remark 1), and the expected values of $\underline{X}_{t,z}$ at each time $t = 0, \Delta t, 2\Delta t, \ldots, N\Delta t$. These expected values are computed using a Monte Carlo method (by averaging here the value of 10^4 samplings).

2.3 Mean square error bounding

The following Theorem holds for SDE (1). This corresponds to a stochastic version of Theorem 1 of (Le Coënt et al. (2017)), showing that a similar result holds on average, using the tamed Euler method of (Hutzenthaler et al. (2012)). It is an adaptation of Theorem 4.4 in (Higham et al. (2002)).

Theorem 2. Given the SDE system (1) satisfying (H1)-(H2)-(H3). Let $\delta_0 \in \mathbb{R}_{\geqslant 0}$. Suppose that z is a random variable on $\mathbb{R}^d$ such that

$$\mathbb{E}[\|x_0 - z\|^2] \leqslant \delta_0^2.$$

Then, we have, for all $\tau \geqslant 0$:

$$\mathbb{E}[\sup_{0\leqslant t\leqslant\tau}\|X_{t,x_0} - \tilde{X}_{t,z}\|^2] \leqslant \delta_{\tau,\delta_0}^2,$$

with $\delta_{\tau,\delta_0}^2 := \beta(\tau)e^{\gamma\tau}$, where:

$$\gamma = 2(\sqrt{\Delta_t} + 2\lambda + L_g^2 + 128L_g^4), \text{ and}$$

$$\begin{aligned}\beta(\tau) &= 2\delta_0^2 + 2\tau\Delta_t L_g^2(1 + 128L_g^2)(F_{2,z}d + E_{2,z}\Delta_t)\\ &+ 4\tau\sqrt{\Delta_t}D(F_{4,z}d + E_{4,z}\Delta_t^2)^{\frac{1}{2}}\\ &(1 + 4\mathbb{E}\sup_{0\leqslant t\leqslant\tau}\|\underline{X}_{t,z}\|^{2q} + 4\mathbb{E}\sup_{0\leqslant t\leqslant\tau}\|\tilde{X}_{t,z}\|^{2q})^{\frac{1}{2}}.\end{aligned} \tag{3}$$

with $\Delta_t = \tau/N$.

Proof. The proof closely follows the proof of Theorem 4.4 in (Higham et al. (2002)). Let $e_t = X_{t,x_0} - \tilde{X}_{t,z}$. We have, for all $0 \leqslant t \leqslant \tau$:

$$\mathrm{d}e_t = (f(X_{t,x_0}) - f(z))\mathrm{d}t + (g(X_{t,x_0}) - g(z))\mathrm{d}W_t. \tag{4}$$

Then, by using Equation (4) and the integral version of Itô formula applied to function $x \mapsto \|x\|^2$ we obtain

$$\begin{aligned}\|e_t\|^2 &= \|e_0\|^2 + \int_0^t 2\langle e_s,\, f(X_{s,x_0}) - f(\underline{X}_{s,z})\rangle\mathrm{d}s\\ &+ \int_0^t \|g(X_{s,x_0}) - g(\underline{X}_{s,z})\|^2\mathrm{d}s + M(t),\end{aligned} \tag{5}$$

where $e_0 = x_0 - z$, and

$$M(t) = \int_0^t 2\langle e_s,\, g(X_{s,x_0}) - g(\underline{X}_{s,z})\rangle\mathrm{d}W_s.$$

So we have using (H2):

$$\begin{aligned}\|e_t\|^2 &\leqslant \|e_0\|^2 + \int_0^t 2\langle e_s,\, f(X_{s,x_0}) - f(\tilde{X}_{s,z})\rangle\mathrm{d}s\\ &+ L_g^2\int_0^t \|X_{s,x_0} - \underline{X}_{s,z}\|^2\mathrm{d}s\\ &+ \int_0^t 2\langle e_s, f(\tilde{X}_{s,z}) - f(\underline{X}_{s,z})\rangle\mathrm{d}s + M(t).\end{aligned} \tag{6}$$

So we have using (H3) and Young's inequality:

$$\begin{aligned}\|e_t\|^2 &\leqslant \|e_0\|^2 + \int_0^t (2\lambda\|e_s\|^2 + L_g^2\|e_s\|^2)\mathrm{d}s\\ &+ L_g^2\int_0^t \|\underline{X}_{s,z} - \tilde{X}_{s,z}\|^2\mathrm{d}s\\ &+ \int_0^t (\frac{1}{\sqrt{\Delta_t}}\|f(\tilde{X}_{s,z}) - f(\underline{X}_{s,z})\|^2 + \sqrt{\Delta_t}\|e_s\|^2)\mathrm{d}s + M(t).\end{aligned} \tag{7}$$

So we have using (H1), for all $0 \leqslant t \leqslant \tau$:

$$\begin{aligned}\|e_t\|^2 &\leqslant \|e_0\|^2 + (\sqrt{\Delta_t} + 2\lambda + L_g^2)\int_0^t \|e_s\|^2\mathrm{d}s\\ &+ L_g^2\int_0^t \|\underline{X}_{s,z} - \tilde{X}_{s,z}\|^2\mathrm{d}s\\ &+ \frac{D}{\sqrt{\Delta_t}}\int_0^t (1 + \|\underline{X}_{s,z}\|^q + \|\tilde{X}_{s,z}\|^q)\|\underline{X}_{s,z} - \tilde{X}_{s,z}\|^2\mathrm{d}s\\ &+ M(t).\end{aligned} \tag{8}$$

It follows using Lemma 1 for $r = 2$, and Cauchy-Schwarz inequality:

$$\begin{aligned}\mathbb{E}[\sup_{0\leqslant s\leqslant t}\|e_s\|^2] &\leqslant \mathbb{E}\|e_0\|^2 + (\sqrt{\Delta_t} + 2\lambda + L_g^2)\int_0^t \mathbb{E}\|e_s\|^2\mathrm{d}s\\ &+ L_g^2\tau\Delta_t(E_{2,z}\Delta_t + F_{2,z}d)\\ &+ \frac{D}{\sqrt{\Delta_t}}\\ &\int_0^t (\mathbb{E}(1 + \|\underline{X}_{s,z}\|^q + \|\tilde{X}_{s,z}\|^q)^2)^{\frac{1}{2}}(\mathbb{E}\|\underline{X}_{s,z} - \tilde{X}_{s,z}\|^4)^{\frac{1}{2}}\mathrm{d}s\\ &+ m(t),\end{aligned} \tag{9}$$

where

$$m(t) = \mathbb{E}[\sup_{0\leqslant s\leqslant t}\|M(s)\|].$$

Hence, using using Lemma 1 for $r = 4$, and inequality $(a + b)^r \leqslant 2^r(a^r + b^r)$:

$$\begin{aligned}\mathbb{E}[\sup_{0\leqslant s\leqslant t}\|e_s\|^2] &\leqslant \mathbb{E}\|e_0\|^2 + (\sqrt{\Delta_t} + 2\lambda + L_g^2))\int_0^t \mathbb{E}\|e_s\|^2\mathrm{d}s\\ &+ L_g^2\tau\Delta_t(E_{2,z}\Delta_t + F_{2,z}d)\\ &+ 2D\tau\sqrt{\Delta_t}(E_{4,z}\Delta_t^2 + F_{4,z}d)^{\frac{1}{2}}\\ &(1 + 4\mathbb{E}\sup_{0\leqslant t\leqslant\tau}\|\underline{X}_{t,z}\|^{2q} + 4\mathbb{E}\sup_{0\leqslant t\leqslant\tau}\|\tilde{X}_{t,z}\|^{2q})^{\frac{1}{2}} + m(t).\end{aligned} \tag{10}$$

On the other hand, from the Burkholder-Davis-Gundy inequality, we get:

$$m(t) \leqslant 16\mathbb{E}[\int_0^t \|e_s\|^2\|g(X_{s,x_0}) - g(\underline{X}_{s,z})\|^2\mathrm{d}s]^{\frac{1}{2}}$$

Hence, using (H2):

$$m(t) \leqslant 16L_g^2\mathbb{E}[\sup_{0\leqslant s\leqslant t}\|e_s\|^2\int_0^t \|X_{s,x_0} - \underline{X}_{s,z}\|^2\mathrm{d}s]^{\frac{1}{2}}$$

Then, using Young's inequality (for any $\alpha > 0$):

$$m(t) \leqslant 8L_g^2(\alpha\mathbb{E}[\sup_{0\leqslant s\leqslant t}\|e_s\|^2] + \frac{1}{\alpha}\mathbb{E}[\int_0^t \|X_{s,x_0} - \underline{X}_{s,z}\|^2\mathrm{d}s]).$$

Hence, by using Lemma 1 for $r = 2$:

$$m(t) \leqslant 8\alpha L_g^2 \mathbb{E}[\sup_{0\leqslant s\leqslant t} \|e_s\|^2]$$
$$+ \frac{8L_g^2}{\alpha}\int_0^t \mathbb{E}[\sup_{0\leqslant r\leqslant s}\|e_r\|^2]\mathrm{d}s \tag{11}$$
$$+ \frac{8L_g^2}{\alpha}\tau\Delta_t(E_{2,z}\Delta_t + F_{2,z}d).$$

Hence, letting $\alpha = \frac{1}{16L_g^2}$, we have by replacing in (10):

$$\begin{aligned} &\frac{1}{2}\mathbb{E}[\sup_{0\leqslant s\leqslant t}\|e_s\|^2] \leqslant \\ &\delta_0^2 + (\sqrt{\Delta_t} + 2\lambda + L_g^2 + 128L_g^4)\int_0^t \mathbb{E}[\sup_{0\leqslant r\leqslant s}\|e_r\|^2]\mathrm{d}s \\ &+ \tau(L_g^2 + 128L_g^4)\Delta_t(E_{2,z}\Delta_t + F_{2,z}d) \\ &+ \tau 2D\sqrt{\Delta_t}(E_{4,z}\Delta_t^2 + F_{4,z}d)^{\frac{1}{2}} \\ &(1 + 4\mathbb{E}\sup_{0\leqslant t\leqslant\tau}\|\underline{X}_{t,z}\|^{2q} + 4\mathbb{E}\sup_{0\leqslant t\leqslant\tau}\|\tilde{X}_{t,z}\|^{2q})^{\frac{1}{2}}. \end{aligned} \tag{12}$$

It results from Gronwall's inequality:

$$\mathbb{E}[\sup_{0\leqslant t\leqslant\tau}\|e_t\|^2] = \beta(\tau)e^{\gamma\tau},$$

with

$$\begin{aligned} &\gamma = 2(\sqrt{\Delta_t} + 2\lambda + L_g^2 + 128L_g^4), \text{ and} \\ &\beta(\tau) = 2\delta_0^2 + 2\tau(\Delta_t L_g^2(1 + 128L_g^2)(F_{2,z}d + E_{2,z}\Delta_t) \\ &+ 4\tau\sqrt{\Delta_t}D(F_{4,z}d + E_{4,z}\Delta_t^2)^{\frac{1}{2}} \\ &(1 + 4\mathbb{E}\sup_{0\leqslant t\leqslant\tau}\|\underline{X}_{t,z}\|^{2q} + 4\mathbb{E}\sup_{0\leqslant t\leqslant\tau}\|\tilde{X}_{t,z}\|^{2q})^{\frac{1}{2}}. \end{aligned} \tag{13}$$

It follows from Theorem 2 and Jensen's inequality:

Proposition 1. Consider two points x_0 and z of $\mathbb{R}^d$,and a positive real number δ_0. Suppose that $x_0 \in B(z, \delta_0)$ (i.e. $\|x_0 - z\| \leqslant \delta_0$). Then $\mathbb{E}X_{t,x_0} \in B(\tilde{X}_{t,z}, \delta_{t,\delta_0})$ for all $t \in [0, \tau]$.

It also follows from Theorem 2:

Proposition 2. In the setting of Theorem 2, the expression δ_{τ,δ_0} tends to

$$\delta_0\sqrt{2}e^{2\lambda\tau + L_g^2 + 128L_g^4}$$

when Δ_t tends to 0 (i.e., when N tends to ∞).

2.4 Implementation

This method has been implemented in the interpreted language Octave, and the experiments performed on a 2.80 GHz Intel Core i7-4810MQ CPU with 8 GB of memory. The implementation is an adaptation of the program described in (Le Coënt et al. (2017)) for controlling deterministic switched systems, but makes use of the tamed Euler scheme for SDEs (with the error function δ given in Theorem 2) instead of the classical Euler scheme.

Example 1. Consider the following system, corresponding to the example in Section 6.2 of (Zamani et al. (2015)) (cf. (Zamani et al. (2014))) for mode $u = 1$:

$$dx_1 = (-0.25x_1 + x_2 + 0.25)dt + 0.05x_1 dW_t^1$$

$$dx_2 = (-2x_1 - 0.25x_2 - 2)dt + 0.05x_2 dW_t^2$$

The program gives (for $\tau = 1$, $\Delta_t = \tau/10^4$): $q = 0$, $D = 1.36$, $L_g = 0.05$, $\lambda = 0.25$; and for $z = (-4, -3.8)$: $E_{2,z} = 893.3$, $E_{4,z} = 2.14 \cdot 10^5$, $F_{2,z} = 0.002$, $F_{4,z} = 4.9 \cdot 10^{-6}$.

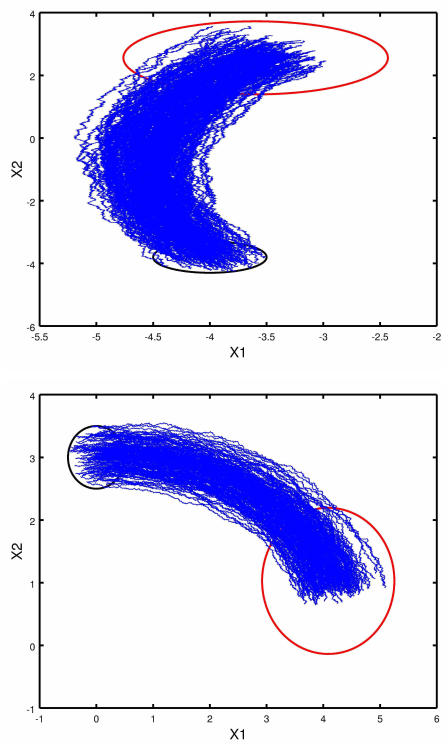

Fig. 1. Simulations of Example 1 with mode $u = 1$ and initial ball $B((-4, 3.8), 0.5)$, and mode $u = 2$ and initial ball $B((0, 3), 0.5)$; $\tau = 1$.

Consider now the system corresponding to the example of (Zamani et al. (2015)) for mode $u = 2$:

$$dx_1 = (-0.25x_1 + 2x_2 - 0.25)dt + 0.05x_1 dW_t^1$$

$$dx_2 = (-x_1 - 0.25x_2 + 1)dt + 0.05x_2 dW_t^2$$

The program gives (for $\tau = 1$, $\Delta_t = \tau/10^4$): $q = 0$, $D = 1.36$, $L_g = 0.05$, $\lambda = 0.25$, and, for $z = (0, 3)$: $E_{2,z} = 543.2$, $E_{4,z} = 7.94 \cdot 10^4$, $F_{2,z} = 0.0442$, $F_{4,z} = 0.00178$.

Both computations take less than 10 s. of CPU time. Simulations of the two systems are given in Figure 1 for mode $u = 1$ and starting point $z = (-4, 3.8)$, and mode $u = 2$ and starting point $z = (0, 3)$. The initial ball ($t = 0$) is depicted in black, the final ball ($t = \tau$) in red, and 200 random sampling trajectories in blue for $t \in [0, \tau]$.

3. SAMPLED STOCHASTIC SWITCHED SYSTEMS

3.1 Stochastic switched system as a finite collection of SDEs

We now consider a finite number of SDEs. Each SDE is referred to as a *mode* j, and the set of modes is referred to as $U = \{1, \ldots, M\}$. We will denote by X_{t,x_0}^j the solution at time t of the system:

$$\begin{aligned} \mathrm{d}x(t) &= f_j(x(t)) + g_j(x(t))\mathrm{d}W_t^j, \\ x(0) &= x_0. \end{aligned} \tag{14}$$

where x_0 is a random variable that is measurable in $\mathcal{F}_0$. Hypotheses (H1-H2-H3), as defined in Section 2, are naturally extended to every mode j of U. Accordingly, constants L_g, λ, F associated to SDE (1) in Section 2, now become L_{g_j}, λ_j, F_j respectively, for each $j \in U$.

Likewise, for each $j \in U$, the nonnegative real $(\delta_{t,\delta_0})^2$ becomes $(\delta^j_{t,\delta_0})^2$ for each mode j; the approximate continuous-time solution of (14) starting from z, is denoted by $\tilde{X}^j_{t,z}$, and the approximate staircase solution by $\underline{X}^j_{t,z}$.

3.2 Control patterns

The control laws that we now consider are "piecewise constant of duration τ" in the sense that, every τ seconds, they select a given mode (see (Zamani et al. (2015))). We call "(control) pattern of length k" a sequence of k modes (i.e., an element of U^k). Each pattern π of the form $j_1 j_2 \cdots j_k$ corresponds to the selection of mode j_1 for time $t \in [0, \tau)$, then mode j_2 for $t \in [\tau, 2\tau)$, and so on, until $t = k\tau$. We assume that the solution of the system is continuous at sampling instants $t = \tau, 2\tau, \ldots$ (which means that there is no "reset" of the system at sampling instants).

Given a stochastic switched system, a pattern π of length k and an initial random variable z, one constructs the "approximate solution controlled by π" by composing together the approximations obtained by successive application of the modes of π. Formally, the "continuous" approximate solution $\tilde{X}^\pi_{t,z}$ is defined at time $t \in [0, k\tau]$ as follows:

- $\tilde{X}^\pi_{t,z} = \tilde{X}^j_{t,z}$ if $\pi = j \in U$, $k = 1$ and $t \in [0, \tau]$, and
- $\tilde{X}^\pi_{(k-1)\tau+t',z} = \tilde{X}^j_{t,z'}$ with $z' = \tilde{X}^{\pi'}_{(k-1)\tau,z}$ if $k \geqslant 2$, $t' \in [0, \tau]$, $\pi = \pi' * j$ for some $j \in U$ and $\pi' \in U^{k-1}$.

The "staircase" approximate solution $\underline{X}^\pi_{t,z}$ is defined analogously. Likewise, given an initial error radius $\delta_0 > 0$ and a pattern π of length $k \geqslant 1$, one defines the error radius δ^π_{t,δ_0} as follows:

- $\delta^\pi_{t,\delta_0} = \delta^j_{t,\delta_0}$ if $\pi = j \in U$, $k = 1$ and $t \in [0, \tau]$, and
- $\delta^\pi_{(k-1)\tau+t',\delta_0} = \delta^j_{t',\delta'}$ with $\delta' = \delta^{\pi'}_{(k-1)\tau,\delta_0}$, if $k \geqslant 2$, $t' \in [0, \tau]$, $\pi = \pi' * j$ for some $j \in U$ and $\pi' \in U^{k-1}$.

3.3 Controlled (R, S)-stability

Given a rectangle $R \subset \mathbb{R}^d$ and a rectangle $S \subset \mathbb{R}^d$ such that $R \subseteq S$, we now extend the problem of "controlled (R, S)-stability", as defined in (Le Coënt et al. (2017)) for deterministic switched systems, to SDEs, as follows:

For any starting point $x_0 \in R$, find a pattern π of length k such that

- $\mathbb{E}X^\pi_{t,x_0} \in R$ for $t = k\tau$
- $\mathbb{E}X^\pi_{t,x_0} \in S$ for all $t = \tau, 2\tau, 3\tau, \ldots$.

It is easy to see that, in order to solve this problem, it suffices to exhibit a finite set of points $z_1, \ldots, z_p$ of S, and a positive real $\delta_0 > 0$ such that:

(1) all the balls $B(z_i, \delta_0)$, $i = 1, \ldots, p$, cover R, and are included into S (i.e. $R \subseteq \bigcup_{i=1}^p B(z_i, \delta_0) \subseteq S$);
(2) for each $i = 1, \ldots, p$, there is a pattern π of length k such that:
    - $B_{i,\pi,t} \subseteq S$ for $t = \tau, 2\tau, \ldots, (k-1)\tau$, and
    - $B_{i,\pi,t} \subseteq R$ for $t = k\tau$.

    where $B_{i,\pi,t} := B(\mathbb{E}\tilde{X}^\pi_{t,z_i}, \delta^\pi_{t,\delta_0})$.

The program mentioned in Section 2.4, has been extended in order to find, by exhaustive enumeration, patterns that make the balls covering R return to R, and such that the intermediate balls (at $t = \tau, 2\tau, \ldots$) belong to S. Details of the algorithm are omitted to conserve space, please refer to (Le Coënt et al. (2016)) for more information on the algorithm. We now give an application of this program.

Example 2. Consider the system (see (Zamani et al. (2014, 2015))):

$$dx_1 = (-0.25x_1 + ux_2 + (-1)^u 0.25)dt + 0.01x_1 dW^1_t$$

$$dx_2 = ((u-3)x_1 - 0.25x_2 + (-1)^u(3-u))dt + 0.01x_2 dW^2_t$$

where $u = 1, 2$.

For $\tau = 0.5$, $\Delta_t = 10^{-4}$, one finds (for all modes $u = 1, 2$):

$q = 0$, $D = 1.36$, $L_g = 0.01$, $\lambda = 0.25$; for $z = (-4, -3.8)$: $E_{2,z} = 893.31$, $E_{4,z} = 2.14 \cdot 10^5$, $F_{2,z} = 0.002$, $F_{4,z} = 4.9 \cdot 10^{-6}$; and for $z = (0, 3)$: $E_{2,z} = 543.22$, $E_{4,z} = 7.94 \cdot 10^4$, $F_{2,z} = 0.0442$, $F_{4,z} = 0.00178$.

Our program shows (R, S)-stability of the system for $R = [-5, 5] \times [-4.4]$ and $S = [-8, 8] \times [-7, 7]$: given a covering of R with balls of radius $\delta_0 = 0.1$, the program finds, by exhaustive search, patterns of length $\leqslant 5$ that make the balls return to R. It takes 6 hours of CPU time. Figure 2 depicts in black the initial balls (at $t = 0$) centered at two corners of R; and for each initial ball, the pattern that sends the ball back to R (at time $t = k\tau$); the intermediate balls (at $t = \tau, 2\tau, \ldots, (k-1)\tau$) are depicted in red, and 200 sampling trajectories drawn in blue.

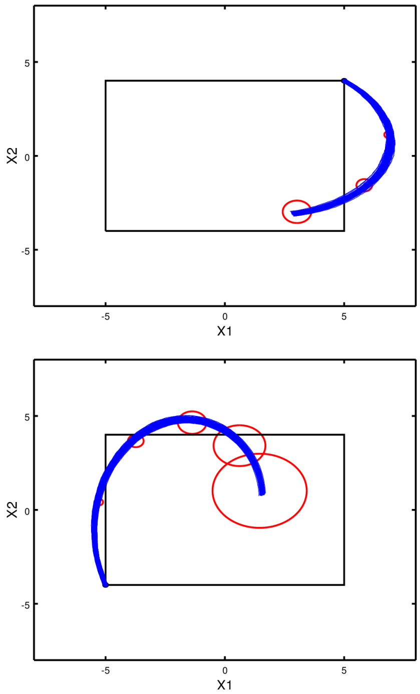

Fig. 2. Simulations of Example 2 from the initial balls $B((5, 4), 0.1)$ and $B((-5, -4), 0.1)$ using patterns $(2 \cdot 2 \cdot 2)$ and $(1 \cdot 1 \cdot 1 \cdot 1 \cdot 1)$ resp., with $\tau = 0.5$.

3.4 Other applications

Our Euler-based method can also be used to control systems in order to achieve *reachability* properties. We sketched out this point in the following example.

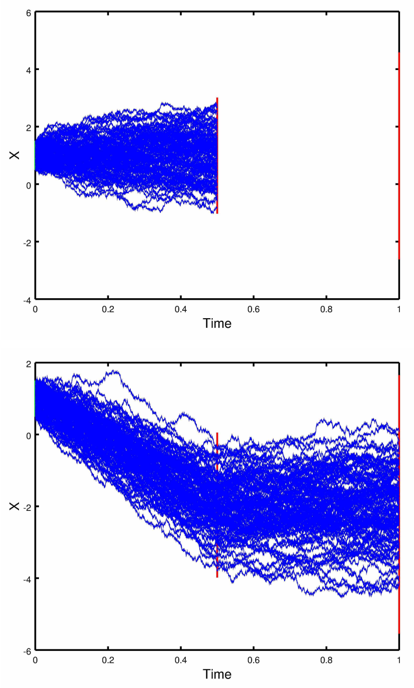

Fig. 3. Simulations of Example 3 without control ($u = 0$) and with control pattern $(-6 \cdot 0)$ from the initial ball $B(1, 0.5)$.

Example 3. (the slit problem)

The problem is adapted from (Morzfeld (2015)). The controlled dynamics is:

$$dX = udt + dW, \qquad X_0 = 1$$

with mode $u \in \{-6, -5, -4, -3, -2, 1, 0, 1, 2, 3, 4, 5, 6\}$. We have (at $t = 0.5$) a slit at $x \in [-1, -4]$. The objective is thus to control the system so that $x(t) \in S = [-1, -4]$ at $t = 0.5$.

One has, for all modes: $q = 0$, $D = 0$, $L_g = 0$, $\lambda = 0$. For $\delta_0 = 0.5$, an initial point $z = 1$ and a sampling time $\tau = 0.5$ with subsampling $\Delta_t = 10^{-3}$, one has for mode $u = -6$: $E_{2,z} = 144$, $E_{4,z} = 20736$, $F_{2,z} = 4$, $F_{4,z} = 16$; and for mode $u = 0$: $E_{2,z} = 0, E_{4,z} = 0, F_{2,z} = 4, F_{4,z} = 16$.

Suppose that all the trajectories start at x_0 with $x_0 \in B(z, \delta_0)$ (i.e., $|x_0 - z| \leqslant 0.5$), with $z = 1$ and $\delta_0 = 0.5$. When there is no control ($u = 0$), at time $t = 0.5$, the expected value of X_{t,x_0} is in $B(z_1, \delta_{t,\delta_0})$ with $z_1 = 1$ and $\delta_{t,\delta_0} = 2$. From Markov's inequality, it follows that the trajectories pass by $S = [-1, -4]$ at $t = 0.5$ with *low probability*: see Figure 3. On the other hand, with control $u = -6$, at time $t = \tau = 0.5$, the expected value of X_{t,x_0} is now in $B(z_1, \delta_{t,\delta_0})$ with $z_1 = -2$ and $\delta_{\tau,\delta_0} = 2$. This explains why the trajectories now pass by $S = [-1, -4]$ at $t = 0.5$ with *high probability.*

4. FINAL REMARKS AND FUTURE WORK

We have explained how to use an Euler-based method in order to control stochastic switched systems. We have focused our work on the property of (R, S)-stability, but it can also be used for achieving reachability properties. In the future, we plan to experiment the method with examples where the drift functions behave polynomially. We would also like to find bounds not only for the expected values of the solutions, but for their variance.

REFERENCES

Asarin, E., Bournez, O., Dang, T., Maler, O., and Pnueli, A. (2000). Effective synthesis of switching controllers for linear systems. *Proceedings of the IEEE*, 88(7), 1011–1025.

Fribourg, L., Kühne, U., and Soulat, R. (2014). Finite controlled invariants for sampled switched systems. *Formal Methods in System Design*, 45(3), 303–329. doi: 10.1007/s10703-014-0211-2.

Girard, A., Pola, G., and Tabuada, P. (2010). Approximately bisimilar symbolic models for incrementally stable switched systems. *IEEE Transactions on Automatic Control*, 55(1), 116–126.

Higham, D.J., Mao, X., and Stuart, A.M. (2002). Strong convergence of Euler-type methods for nonlinear stochastic differential equations. *SIAM Journal on Numerical Analysis*, 40(3), 1041–1063.

Hutzenthaler, M., Jentzen, A., and Kloeden, P.E. (2012). Strong convergence of an explicit numerical method for SDEs with nonglobally lipschitz continuous coefficients. *Annals of Applied Probability*, 22(4), 1611–1641.

Le Coënt, A., Alexandre dit Sandretto, J., Chapoutot, A., and Fribourg, L. (2016). An improved algorithm for the control synthesis of nonlinear sampled switched systems. *Formal Methods in System Design.* doi:10.1007/s10703-017-0305-8.

Le Coënt, A., De Vuyst, F., Chamoin, L., and Fribourg, L. (2017). Control synthesis of nonlinear sampled switched systems using Euler's method. In *Proceedings of 3rd Intl Workshop on Symbolic and Numerical Methods for Reachability Analysis (SNR 2017).*

Maruyama, G. (1955). Continuous Markov processes and stochastic equations. *Rend. Circ. Mat. Palermo*, 2(4), 48–90.

Morzfeld, M. (2015). Implicit sampling for path integral control, Monte Carlo localization, and SLAM. *Journal of Dynamic Systems, Measurement, and Control*, 137(5).

Oksendal, B. (2002). *Stochastic Differential Equations: An Introduction with Applications.* Springer.

von Renesse, M.K. and Scheutzow, M. (2010). Existence and uniqueness of solutions of stochastic functional differential equations. *Random Operators and Stochastic Equations*, 18(3). doi:10.1515/rose.2010.015.

Zamani, M., Abate, A., and Girard, A. (2015). Symbolic models for stochastic switched systems: A discretization and a discretization-free approach. *Automatica*, 55, 183–196. doi:10.1016/j.automatica.2015.03.004.

Zamani, M., Esfahani, P.M., Majumdar, R., Abate, A., and Lygeros, J. (2014). Symbolic control of stochastic systems via approximately bisimilar finite abstractions. *IEEE Trans. Automat. Contr.*, 59(12), 3135–3150. doi: 10.1109/TAC.2014.2351652.

Zamani, M., Tkachev, I., and Abate, A. (2017). Towards scalable synthesis of stochastic control systems. *Discrete Event Dynamic Systems*, 27(2), 341–369. doi: 10.1007/s10626-016-0233-6.