# Coherent interaction graphs: a nondeterministic geometry of interaction for MLL


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We introduce the notion of coherent graphs, and show how those can be used to define dynamic semantics for Multiplicative Linear Logic. The models thus obtained are finite with respect to several aspects (finite graphs, finite generation of types) and thus improve in this way previous constructions by Seiller [14, 16]. We also discuss how the added notion of coherence can also be used to introduce non-determinism.


## 1 Introduction

Dynamic semantics of proofs take their origin in Girard's *Geometry of Interaction program* [5], whose aim was to provide a semantic account for the dynamics of cut-elimination. Indeed, while the proofs-as-programs correspondence expresses that $\beta$-reduction of lambda-terms corresponds to cut-elimination, its extension to categorical models (the so-called Curry-Howard-Lambek correspondence) fails as it represent those operations as a simple equality.

In this work, we introduce a new dynamic semantics for Multiplicative Linear Logic (MLL). Inspired from the second author's Interaction Graphs [14] where proofs are interpreted as directed (edge-)weighted graphs, this work was initially motivated by the wish to make the Interaction Graphs models finite. The way to impose finiteness is by endowing graphs with a coherence relation on their edges. While this allows to define models in which all graphs are finite (thus succeeding w.r.t. the initial aim), we will explain how the obtained models are finite in a more essential way: the model's interpretation of types are finitely generated[1].

We will also discuss how the introduction of the coherence relation allows for the representation of non-determinism, and opens the way to several other applications. Some motivations for dealing with non-determinism are the following:

- *implicit complexity*: to characterise a complexity class such as NP with a GoI model, representing non-determinism is desirable;
- *correctness criteria*: the type of a MLL proof structure is a sort of non-deterministic counter-proof, equivalent to the superposition of its switchings;
- *additive connectives*: indeed, a proof of $A \& B$ can be seen as a non-deterministic superposition of a proof of $A$ and a proof of $B$.

[1] It is even stronger, as they are generated by a single object.


Submitted to:
Linearity-TLLA 2018

# 2 The model

We start by exhibiting the definition and basic geometric properties of coherent graphs. A later section then explains briefly how these result can be used to define models of Multiplicative Linear Logic.

## 2.1 Coherent Graphs

We fix a monoid of weights $\Omega$.

**Definition 1.** A *(edge-)coherent graph* is a directed $\Omega$-weighted graph $G = (V, E, s, t, \omega)$ together with a coherence (i.e. symmetric, reflexive) relation $\frown_G$ on its set of edges $E$.

We denote by $\mathrm{Coh}(G)$ the coherence space $(E, \frown_G)$ associated to a coherent graph.

**Definition 2.** Let $G$ and $H$ be coherent graphs. The underlying graph of the plugging $G \,\square\, H$ is defined as in previous work [14], taking the union of vertices and the disjoint union of edges. The graph $G \,\square\, H$ is then endowed with the coherence defined as the disjoint union of $\frown_G$ and $\frown_H$. Note that this gives $\mathrm{Coh}(G \,\square\, H) = \mathrm{Coh}(G) \,\&\, \mathrm{Coh}(H)$.

An alternating path (or cycle) is *coherent* if all its edges are pairwise coherent, i.e. if it is a clique for the associated coherence relation. Two coherent paths (or cycles) $\pi$, $\pi'$ are *mutually coherent* if and only all edges in $\pi$ are coherent with all edges in $\pi'$.

**Definition 3.** Let $G$ and $H$ be coherent graphs. Their *execution* $G :: H$ is the coherent graph defined as:

- the set of vertices is defined as the symmetric difference $V_G \bigtriangleup V_H$;
- edges are coherent alternating paths of source and target in $G \,\square\, H$;
- the weight of an edge $e$ is the product of the weights of the edges in the path $e$ represents;
- the coherence is mutual coherence of paths.

We also want to define a notion of weight of a cycle, analogous to the weight of a path used above. This seems straightforward at first by considering a cycle as a path from a vertex to itself. But the requirement of invariance under change of base point prompts us to consider *conjugacy classes* of weights.

**Definition 4.** We define $x \sim_p y \in \Omega \Leftrightarrow \exists a, b \in \Omega \,/\, x = ab, y = ba$. The *conjugacy relation* $\sim$ over $\Omega$ is the transitive closure of $\sim_p$; it is an equivalence relation, but not necessarily a monoid congruence.

This is one of the standard generalizations of the notion of conjugacy from group theory, see e.g. [10] and references therein. From now on, we will call the equivalence classes for $\sim$ in $\Omega$ *weight classes*.

**Definition 5.** We denote by $\mathrm{Loop}(G, H)$ the set of coherent alternating cycles in $G \,\square\, H$. It is a coherence space (with the mutual coherence of cycles) labeled by weight classes.

**Proposition 6** ("Trefoil property"). *If $V_F \cap V_G \cap V_H = \varnothing$, the set of alternating coherent cycles in $F \,\square\, G \,\square\, H$ is in label-preserving bijection with* $\mathrm{Loop}(F, G) \sqcup \mathrm{Loop}(F :: G, H)$.

*This extends to an isomorphism of coherence spaces if we define $C \in \mathrm{Loop}(F, G)$ to be coherent to $C' \in \mathrm{Loop}(F :: G, H)$ if and only if $C$ is mutually coherent with all the edges of $F :: G$ (i.e. coherent alternating paths in $F \,\square\, G$) appearing in $C'$.*

**Corollary 7** (Adjunction). *If $V_G \sqcup V_H = V_F$, then the following label-preserving isomorphism of coherence spaces holds:* $\mathrm{Loop}(F, G \sqcup H) \cong \mathrm{Loop}(F, G) \sqcup \mathrm{Loop}(F :: G, H)$.

## 2.2 Model of MLL

As in previous interaction graphs models, the tensor product will be represented as the disjoint union of (coherent) graphs. Therefore, the three-terms adjunction obtained in Corollary 7 almost provides the basis for an adequate model of Multiplicative Linear Logic. The term $\mathrm{Loop}(F,G)$ requires us to keep track of cycles that may disappear during the execution. This was done in earlier constructions [14, 16] by considering a measurement of cycles and associating graphs with a scalar – the *wager* – that captured the measurement of disappearing cycles. In this work, we choose to keep the set of cycles itself.

**Definition 8.** A *coherent project* $\mathfrak{a}$ is a pair of a directed $\Omega$-weighted graph $\mathrm{G}(\mathfrak{a}) = (V_{\mathfrak{a}}, E_{\mathfrak{a}}, s_{\mathfrak{a}}, t_{\mathfrak{a}}, \omega_{\mathfrak{a}})$ and a set of cycles $\mathrm{C}(\mathfrak{a})$, together with a coherence relation $\frown_{\mathfrak{a}}$ on the set $E_{\mathfrak{a}} \sqcup \mathrm{C}(\mathfrak{a})$. In particular, $(\mathrm{G}(\mathfrak{a}), \frown_{\mathfrak{a}}\restriction_{\mathrm{G}(\mathfrak{a})})$ is a coherent graph. We denote a coherent project as $\mathrm{G}(\mathfrak{a}) \frown_{\mathfrak{a}} \mathrm{C}(\mathfrak{a})$.

**Definition 9.** Let $\mathfrak{a}$, $\mathfrak{b}$ be coherent projects. We define $\mathrm{Loop}(\mathfrak{a},\mathfrak{b}) = \mathrm{C}(\mathfrak{a}) \sqcup \mathrm{C}(\mathfrak{b}) \sqcup \mathrm{Loop}(\mathrm{G}(\mathfrak{a}),\mathrm{G}(\mathfrak{b}))$.

The execution $\mathfrak{a} :: \mathfrak{b}$ is defined as the project $\mathrm{G}(\mathfrak{a}) :: \mathrm{G}(\mathfrak{b}) \frown_{a::b} \mathrm{Loop}(\mathfrak{a},\mathfrak{b})$, where $\frown_{\mathfrak{a}::\mathfrak{b}}$ is defined from the mutual coherence relation on paths and cycles in the natural way (in particular, its restriction to $\mathrm{G}(\mathfrak{a}) :: \mathrm{G}(\mathfrak{b})$ is $\frown_{\mathrm{G}(\mathfrak{a})::\mathrm{G}(\mathfrak{b})}$). Similarly, $\mathrm{Loop}(\mathfrak{a},\mathfrak{b})$ has a coherence space structure.

When $V_{\mathfrak{a}} \cap V_{\mathfrak{b}} = \varnothing$, $\mathrm{G}(\mathfrak{a}) :: \mathrm{G}(\mathfrak{b})$ is in fact $\mathrm{G}(\mathfrak{a}) \sqcup \mathrm{G}(\mathfrak{b})$. In this case, we write $\mathfrak{a} \otimes \mathfrak{b}$ instead of $\mathfrak{a} :: \mathfrak{b}$.

The next step is to define *orthogonality*, which accounts for negation at the level of projects. Orthogonality is strongly connected to the correctness criteria for proof nets (cf. [14, 12]); intuitively projects represent both proofs and the correctness graphs of the criteria. We will revisit this point in a formal way at the end of section 5.

The definition of orthogonality is dependent on a *pole*, which was in earlier work defined through a measurement of cycles. We here consider the equivalent but somehow simpler definition of a pole as a predicate over possible outcomes of the execution between projects having the same set of vertices.

**Definition 10.** A *pole* is a class of coherence spaces labeled by weight classes, closed under label-preserving isomorphisms of coherence spaces.

The latter condition prevents the pole from inspecting information such as the length of cycles.

**Definition 11.** Given a pole $\perp\!\!\!\perp$, we say that two coherent projects $\mathfrak{a}$ and $\mathfrak{b}$ are orthogonal (w.r.t. $\perp\!\!\!\perp$) when $\mathrm{Loop}(\mathfrak{a},\mathfrak{b}) \in \perp\!\!\!\perp$. This is written $\mathfrak{a} \perp\!\!\!\perp \mathfrak{b}$.

**Proposition 12.** *For any pole $\perp\!\!\!\perp$ and coherent project $\mathfrak{a},\mathfrak{b},\mathfrak{c}$, we have $\mathfrak{a} :: \mathfrak{b} \perp\!\!\!\perp \mathfrak{c} \Leftrightarrow \mathfrak{b} :: \mathfrak{c} \perp\!\!\!\perp \mathfrak{a}$.*

This proposition specialises as the adjunction $\mathfrak{a} :: \mathfrak{b} \perp\!\!\!\perp \mathfrak{c} \Leftrightarrow \mathfrak{b} \otimes \mathfrak{c} \perp\!\!\!\perp \mathfrak{a}$ when $V_{\mathrm{G}(\mathfrak{b})} \cap V_{\mathrm{G}(\mathfrak{c})} = \varnothing$, from which we can define a model. We now fix a pole $\perp\!\!\!\perp$ and construct the corresponding model.

**Definition 13.** A *conduct* is a set $\mathbf{A}$ of coherent projects whose vertex sets are all equal to a common set $V_{\mathbf{A}}$, such that $\mathbf{A} = \mathbf{A}^{\perp\!\!\!\perp\perp\!\!\!\perp}$.

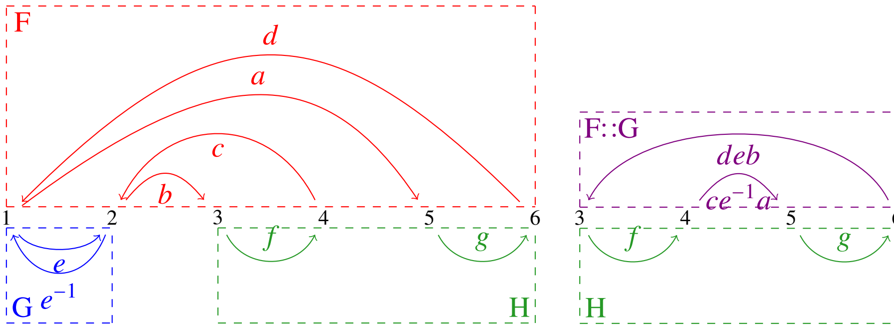
**Definition 14.** Let $\mathbf{A}$ and $\mathbf{B}$ be two conducts such that $V_{\mathbf{A}} \cap V_{\mathbf{B}} = \varnothing$.
We define $\mathbf{A} \parr \mathbf{B} = \{\mathfrak{a} \otimes \mathfrak{b} \mid \mathfrak{a} \in \mathbf{A}, \mathfrak{b} \in \mathbf{B}\}^{\perp\!\!\!\perp}$, $\mathbf{A} \otimes \mathbf{B} = (\mathbf{A} \parr \mathbf{B})^{\perp\!\!\!\perp}$ and $\mathbf{A} \multimap \mathbf{B} = \mathbf{A}^{\perp\!\!\!\perp} \parr \mathbf{B}$.

**Theorem 15.** *For all conducts $\mathbf{A}$, $\mathbf{B}$ with $V_{\mathbf{A}} \cap V_{\mathbf{B}} = \varnothing$, we have $\mathbf{A} \multimap \mathbf{B} = \{\mathfrak{f} \mid \forall \mathfrak{a} \in \mathbf{A}, \mathfrak{f} :: \mathfrak{a} \in \mathbf{B}\}$.*

Strictly speaking, in order to build a denotational model of MLL, we need to work "modulo delocation" to avoid being hindered by the locative assumption $V_{\mathbf{A}} \cap V_{\mathbf{B}} = \varnothing$, as in previous work. If we put aside this technical issue, we have:

**Theorem 16.** *The category of conducts (as objects) and coherent projects (as morphisms) is $\star$-autonomous.*

An important example of pole is $\perp = \{\varnothing\}$. Two projects $\mathfrak{a}$ and $\mathfrak{b}$ are orthogonal for this pole when $C(\mathfrak{a}) = C(\mathfrak{b}) = \mathrm{Loop}(\mathfrak{a},\mathfrak{b}) = \varnothing$; we denote this by $\mathfrak{a} \perp \mathfrak{b}$. This orthogonality corresponds to the *nilpotency* condition of earlier versions of GoI. Note also that the model induced by this pole satisifies the *Mix rule*, that is, $\mathbf{A} \otimes \mathbf{B} \subset \mathbf{A} \parr \mathbf{B}$.




(a) $F \,\square\, (G \cup H)$ (b) $(F ::_{\text{simple}} G) \,\square\, H$

Figure 1: Counter-example of the adjunction for simple paths and cycles

## 3 Simple and elementary paths in coherent graphs

The notion of coherent graphs, together with the constructions leading to models of multiplicative linear logic, was initially considered in Seiller's PhD thesis [15]. The motivation for the extension of graphs with a notion of coherence was to limit the execution. Indeed, the usual execution in Seiller's interaction graphs models leads to infinite graphs, since the execution was defined simply as the set of alternating paths without further restrictions. Therefore, as soon as a path went through a cycle once, similar paths going through the same cycle an arbitrary number of times existed as well, producing an infinite number of edges. To obtain a model in which the execution between two finite graphs is still a finite graph, it is natural to consider restricting our attention to simple paths and cycles.

**Definition 17.** A path (resp. cycle) is *simple* if it does not visit any vertex more than once.

But restricting the execution to simple paths and considering simple cycles breaks the trefoil property, and therefore the adjunction.

**Example 18.** Figure 1 shows how the adjunction fails when one considers only simple paths and cycles. On the left are three graphs, with no simple alternating cycle between them. The simple execution – i.e. the set of alternating *simple* paths between $F$ and $G$ – is shown on top of Figure 1b. One can see that there is now a simple alternating cycle $C$ between this graph and $H$. But the edges $deb$ and $ce^{-1}a$ correspond to paths in $F \,\square\, G$ which both visit the vertices 1 and 2, and so, the cycle corresponding to $C$ in $F \,\square\, (G \cup H)$ (left-hand figure) is not simple, since it visits 1 and 2 twice.

Coherent graphs were therefore a way of making the trefoil property work while restricting the execution to simple paths.

*Remark* 19. To ensure the finiteness of execution, another natural choice would be to restrict to paths and cycles without repeating *edges*, instead of vertices. This may be done in a similar way to what we are about to show here, but using a non-reflexive coherence relation.

**Definition 20.** Let $G$ be a directed graph. The *simple coherence* relation on $E_G$ is defined as follows: two edges are coherent if and only if they do *not* have a common vertex.

**Definition 21.** A coherent graph $G$ is *simple* if the coherence relation $\frown_G$ is included in the simple coherence.

**Proposition 22.** *If $G$ and $H$ are simple, then $G :: H$ is simple.*

Moreover, as shown in the previous section, the trefoil property, hence the adjunction, holds in this setting. One can also check that the model does indeed restrict the paths and cycles to simple ones (and in general excludes some simple paths and cycles, namely those that make the adjunction fail). This shows that we succeeded in obtaining a model with finite (coherent) graphs only.

**Proposition 23.** *Let G and H be two simple coherent graphs on the same set of vertices. If an alternating path or cycle in $G \square H$ is coherent, then it is simple. The converse holds if $\frown_G$ and $\frown_H$ are equal to the simple coherences.*

For instance, the *chordless coherence* defined in section 5 is included in the simple coherence.

## 4 Non-determinism and (wrong) additives

We now argue that coherent interaction graphs should be seen as a *non-deterministic* version of interaction graphs. This raises the following question: why not just represent the set of possibilities as a formal sum of proofs? The advantage of using the coherence relation instead is that it can represent proofs in which small sub-proofs involve non-deterministic choice, without having to duplicate the context. More formally, this corresponds to proofs in a sequent calculus enriched with the SUM rule

$$\frac{\vdash \Gamma \quad \dots \quad \vdash \Gamma}{\vdash \Gamma}$$

In our model, this rule is interpreted as the *incoherent union* (denoted by $+$) of two coherent graphs with the same vertices: in $G+H$, edges coming from $G$ are incoherent with edges coming from $H$.

In [11], which introduces this rule, proof equivalence is extended with associativity and distributivity over arbitrary contexts of SUM [11, Fig. 6]. Applying these rules, one can turn any proof using SUM into a proof with a single SUM rule at the root – and similarly, one may see a coherent interaction graph as the formal sum of its maximal edge-cliques. But this may lead to an exponential blow-up of the size of the proof; thus, one should refrain from doing so eagerly it when studying the complexity of normalization. Indeed, Maurel's motivation for studying the SUM rule in [11] is to characterize the class NP.

This use of a coherence space structure also appears[2] in Girard's *transcendental syntax*. In the deterministic case [7], *constellations* are roughly undirected interaction hypergraphs, whose hyperedges are called *stars*. Equipping constellations with a coherence relation on their stars gives rise to an "analytics of non-determinism" [8]. Implicit complexity is alluded to as a motivation: referring to the kind of non-determinism which arises from a vertex having multiple outgoing edges – which suffices to characterize NL and the bidirectional multihead automata hierarchy [17] – Girard writes: "Since the various alternatives are not correlated, we obtain a sort of Alzheimer non-determinism – the kind at work in LOGSPACE computation. In particular, the seminal NP satisfiablity problem [...] cannot be handled in this way."

That said, these coherent constellations are mainly intended to express the additive connectives of linear logic. Since our extension of interaction graphs with coherence is analogous, one may expect Girard's treatment of additives to be transposable here. This would give:

- for $A \oplus B$, $\mathsf{inl}(G) = G \sqcup \varnothing_B$ and $\mathsf{inr}(H) = \varnothing_A \sqcup H$,
- for $A \& B$, $G \& H = (G \sqcup \varnothing_B) + (\varnothing_A \sqcup H)$: a proof of $A \& B$ consists of a non-deterministic sum of a proof of $A$ and a proof of $B$.

[2] A precursor to this idea is the coherence on monomial proof nets [6, §A.1] for MALL. In this case, the maximal cliques correspond to the *slices* of the proof net.

However, this fails to define a denotational model of MALL. Indeed, consider two linear maps $f : A \multimap C$ and $g : B \multimap C$, their copairing $\langle f, g\rangle : A \oplus B \multimap C$, and some $a : A$. Then, in general, the interpretation of $f(a)$ and that of $\langle f, g\rangle(\mathsf{inl}(a))$ differ: the latter contains any edge of $g$ whose endpoints are both in $C$. This problem is not new; it also occurs with the representation of additives by means of *sliced* interaction graphs (see the discussion in [16, §5.2] for more details and a concrete example, which can be translated to coherent graphs), and it also seems to affect Girard's constellations.

## 5 Principal conducts and the cographic correctness criterion

In this section, we focus on the model induced by the pole $\perp = \{\varnothing\}$.

**Definition 24.** Let $\mathbf{A}$ be a conduct. A *generator* of $A$ is a project $\mathfrak{a} = (A \frown_{\mathfrak{a}} \varnothing) \in \mathbf{A}^{\perp}$ such that $\mathbf{A} = \{\mathfrak{a}\}^{\perp}$.

Every conduct $\mathbf{A}$ has a generator: just take the non-deterministic sum of all graphs in $\mathbf{A}^{\perp}$ whose edges are all pairwise coherent. But this doesn't simplify much the description of $\mathbf{A}$. What we would like is to find generators with a reasonable size, e.g. with a number of edges polynomial in the number of vertices. Coherent interaction graphs allow us to build a model of MLL consisting entirely of conducts with such generators.

**Definition 25.** A conduct $\mathbf{A}$ is *principal* if it admits a generator without parallel edges, i.e. multiple edges with both the same sources and the same targets. It is *bi-principal* if both $\mathbf{A}$ and $\mathbf{A}^{\perp}$ are principal.

**Theorem 26.** *Principal conducts are closed under $\otimes$ and $\parr$, and bi-principal conducts constitute a non-trivial model of MLL.*

The non-triviality comes from the fact that the unique conduct on $\{*\}$ is bi-principal. This conduct does not contain any coherent graph, but this tells us that to get a truly non-empty bi-principal conduct, we can just take any provable formula and interpret every atom by $\{*\}$. The rest of the theorem is a consequence of the following explicit constructions on generators.

**Proposition 27.** *For all $\mathfrak{a} = A \frown_{\mathfrak{a}} \varnothing$ and $\mathfrak{b} = B \frown_{\mathfrak{b}} \varnothing$, we have $\{\mathfrak{a}\}^{\perp} \parr \{\mathfrak{b}\}^{\perp} = \{\mathfrak{a} \otimes \mathfrak{b}\}^{\perp}$ and $\{\mathfrak{a}\}^{\perp} \otimes \{\mathfrak{b}\}^{\perp} = \{\mathfrak{a} \parr \mathfrak{b}\}^{\perp}$, where $\mathfrak{a} \parr \mathfrak{b} = (A \parr B) \frown_{\mathfrak{a} \parr \mathfrak{b}} \varnothing$, $V(A \parr B) = V(A) \sqcup V(B)$, and*

$$(E(A \parr B), \frown_{\mathfrak{a} \parr \mathfrak{b}}) = (E(A), \frown_{\mathfrak{a}}) \oplus (E(B), \frown_{\mathfrak{b}}) \oplus (V(A) \times V(B) \sqcup V(B) \times V(A), \varnothing).$$

*That is, for each $u \in V(A)$ and $v \in V(B)$, we add $(u, v)$ and make it incoherent with all other edges, and similarly from $V(B)$ to $V(A)$.*

Among bi-principal conducts, we have all those built from $\{*\}$ using the multiplicative connectives. Let us take a closer look at what those look like. The following shows that the coherence relation of the canonical generator of such a conduct is entirely determined by its vertices and edges.

**Definition 28.** Let $G$ be a graph. The *chordless coherence* relation is defined as follows: $e, f \in E(G)$ are incoherent if and only if either $e$ and $f$ are incident[3], or some $g \in E(G)$ is incident to both $e$ and $f$.

**Proposition 29.** *Any generator obtained from the one-vertex graph by the constructions of the previous proposition is equipped with the chordless coherence relation.*

Our choice of naming is justified by the following.

[3] That is, $e$ and $f$ have a common endpoint. This does not depend on the directions of $e$ and $f$.

**Proposition 30.** *Let $G$ and $H$ be two graphs equipped with the chordless coherence relation. An alternating cycle in $G \,\square\, H$ is coherent if and only if there is no edge of $G \,\square\, H$ outside the cycle between two vertices of the cycle (such an edge is called a* chord *in graph theory). The same holds for alternating paths in $G \,\square\, H$ with endpoints in $G \,\triangle\, H$.*

Thus, it remains only to understand the set of graphs (without coherence) generated by $\sqcup$ and $⅋$. First, one can remark that those canonical generators are all symmetric directed graphs; it will be more convenient to consider them as *undirected* graphs. In fact, it is common to associate an undirected graph to a classical propositional formula by interpreting $\vee$ as a disjoint union and $\wedge$ as the dual operation (see e.g. [1]). This is exactly what we do with MLL formulae; note that the $⅋$ on conducts is interpreted as a $\sqcup$ on generators, and thus corresponds to the classical conjunction. The graphs obtained this way are *cographs* [2], a well-known class of undirected graphs with many different characterizations.

The cograph associated to a MLL formula has already been studied before: it is used in a correctness criterion for MLL with the Mix rule, first described by Retoré [13], and later rediscovered by Ehrhard [3] who suggests that it must be connected to the Geometry of Interaction. This criterion also underlies Hughes's "combinatorial proofs" for classical propositional logic [9].

**Theorem 31** (Cographic correctness criterion [13, 3])**.** *Let $P$ be a cut-free MLL proof structure with atomic axioms and a single conclusion $\phi$. Let $G$ be the cograph corresponding to $\phi$, and $M$ be the perfect matching on $V(G)$ induced by the axiom links of $P$. Then $P$ is a MLL+Mix proof net if and only if there is no chordless alternating cycle between $G$ and $M$.*

This "no chordless alternating cycle" condition is just orthogonality between coherent interaction graphs: $G \perp M$ where $G$ is equipped with the chordless coherence relation. The "only if" condition can be recovered immediately by interpreting $P$ in our model, sending the atoms of $\phi$ to $\{*\}$ so that $G$ is exactly the canonical generator of the bi-principal conduct corresponding to $\phi$, and $M$ is the element of this conduct corresponding to $P$.

Note that this ties in to the intution of coherent graphs as "sparse" non-deterministic proofs. Geometry of Interaction was born from the observation that in a proof net, the switchings of the lower half (the forest of $\otimes$/$⅋$ links), which are involved in correctness criteria, may be seen as counter-proofs. But this associates to a type exponentially many tests which generate it, forgetting the fact that this set of tests has a concise representation. Here, we translate this lower part of a proof net into a single counter-proof with at most quadratic size, which can be seen as a non-deterministic superposition of switchings.

## 6 Perspectives

The obvious direction for future work is to extend the model to larger fragments of linear logic.

- Additives are already nearly representable, but they remain problematic for reasons similar to other GoI models. In the sliced graph model [16], this is solved by taking the quotient by an observational equivalence. But with the pole $\perp = \{\varnothing\}$, whose relevance we have illustrated, this quotient trivializes the model, so another solution is desirable.
- Exponentials could be added by imitating the infinite constructions of previous GoIs for MELL. However, it would be more interesting to find a way to keep the finiteness properties of our coherent graph model of MLL. The question of obtaining finite generators for exponential conducts is related to the role of "factory tests" (*usine*) in transcendental syntax.
- The ability to take non-deterministic sums in our model also suggests an extension to differential linear logic [4].

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