Compositional Synthesis of Finite Abstractions for Continuous-Space Stochastic Control Systems: A Small-Gain Approach ⋆

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Abstract: This paper is concerned with a compositional approach for constructing finite abstractions (a.k.a. finite Markov decision processes) of interconnected discrete-time stochastic control systems. The proposed framework is based on a notion of so-called *stochastic simulation function* enabling us to use an abstract system as a substitution of the original one in the controller design process with guaranteed error bounds. In the first part of the paper, we derive sufficient small-gain type conditions for the compositional quantification of the distance in probability between the interconnection of stochastic control subsystems and that of their (finite or infinite) abstractions. In the second part of the paper, we construct finite abstractions together with their corresponding stochastic simulation functions for the class of linear stochastic control systems. We apply our results to the temperature regulation in a circular building by constructing compositionally a finite abstraction of a network containing 1000 rooms. We use the constructed finite abstractions as substitutes to synthesize policies compositionally regulating the temperature in each room for a bounded time horizon.

Keywords: Interconnected Stochastic Control Systems, Compositionality, Finite Abstractions, Finite Markov Decision Processes, Small-Gain Conditions, Formal Synthesis.

1. INTRODUCTION

Despite being present in many application domains, large-scale interconnected systems are inherently difficult to analyze and control. Here, we will leverage *decomposition* and *abstraction* as two key tools to tackle the aforementioned difficulty, by either breaking the design and analysis object into semi-independent parts or by aggregating states and eliminating unnecessary details. Employing abstractions of subsystems as a replacement is a promising approach in the controller design process. These abstractions allow us to design controllers for them, and then refine the controllers to the ones for the concrete subsystems, while providing quantified errors for the overall interconnected system in this controller synthesis detour. In particular, construction of *finite* abstractions was introduced in recent years as a method to reduce the complexity of controller synthesis problems particularly for enforcing complex logical properties. Finite abstractions are approximate descriptions of the continuous-space control systems in which each discrete state corresponds to a collection of continuous states of the original system. Since the abstractions are finite, algorithmic machineries from computer science are applicable to synthesize controllers enforcing complex properties, e.g. expressed as temporal logic formulae, over concrete systems.

In the past few years, there have been several results on the construction of (in)finite abstractions for stochastic systems. Existing results for *continuous-time* stochastic systems include infinite approximation techniques for jump-diffusion systems (Julius and Pappas (2009)), finite bisimilar abstractions for incrementally stable stochastic switched systems (Zamani et al. (2015)), randomly switched stochastic systems (Zamani and Abate (2014)), and stochastic control systems without discrete dynamics (Zamani et al. (2014)). Recently, compositional construction of infinite abstractions (reduced order models) is discussed in (Zamani et al. (2017)) for jump-diffusion systems using small-gain type conditions.

For *discrete-time* stochastic models with continuous state spaces, construction of finite abstractions is initially proposed in (Abate et al. (2008)) for formal verification and synthesis. The construction algorithms are improved in terms of scalability in (Soudjani and Abate (2013)). Extension of such techniques to infinite horizon properties is proposed in (Tkachev and Abate (2011)) and formal abstraction-based policy synthesis is discussed in (Tkachev et al. (2013)). Recently, compositional construction of finite abstractions is discussed in (Soudjani et al. (2015a)) using dynamic Bayesian networks, and infinite abstractions (reduced order models) in (Lavaei et al. (2017)) and (Lavaei et al. (2018a)) using small-gain type conditions and dissipativity-type properties of subsystems and their abstractions, respectively, both for discrete-time stochastic control systems. Our proposed approach extends the abstraction techniques in (Soudjani et al. (2015a)) from verification to synthesis, by proposing a different quantification of the abstraction error and lever-

⋆ This work was supported in part by the German Research Foundation (DFG) through the grant ZA 873/1-1.

aging *small-gain* type conditions. Although the results in (Lavaei et al. (2017)) deal only with infinite abstractions which might not be amenable to the algorithmic controller synthesis procedures, our proposed approach here considers finite abstractions which are the main tools for automated synthesis of controllers for complex logical properties.

Our main contribution is to provide a compositional approach for the construction of finite abstractions of interconnected discrete-time stochastic control systems. The proposed technique leverages sufficient *small-gain* type conditions to establish the compositionality results. In particular, it relies on a relation between each subsystem and its abstraction characterized by the existence of *stochastic simulation functions*. These types of relations enable us to quantify the error in probability between the interconnection of concrete subsystems and that of their finite abstractions. In addition, we show constructively how to synthesize *finite* abstractions of stabilizable linear stochastic control subsystems. We illustrate the effectiveness of our results by regulating temperatures in a circular building containing 1000 rooms. We leverage the constructed finite abstractions as substitutes to synthesize policies compositionally regulating the temperature in each room for a bounded time horizon. Proofs of statements are omitted due to space limitations.

Related work. Compositional construction of finite abstractions for interconnected discrete-time stochastic control systems is also proposed recently in (Lavaei et al. (2018b)), but using a different compositionality scheme based on *dissipativity theory*. In general, the proposed compositional synthesis approach here is much *less conservative* than the one proposed in (Lavaei et al. (2018b)) (see case study at the end) since the overall approximation error is computed based on the *maximum* of the errors of subsystems instead of their *linear combinations* which is the case in (Lavaei et al. (2018b)).

2. DISCRETE-TIME STOCHASTIC CONTROL SYSTEMS

2.1 Notation

We denote the set of nonnegative integers by $\mathbb{N} := \{0, 1, 2, \dots\}$ and the set of positive integers by $\mathbb{N}_{\geq 1} := \{1, 2, 3, \dots\}$. The symbols $\mathbb{R}$, $\mathbb{R}_{>0}$, and $\mathbb{R}_{\geq 0}$ denote the set of real, positive and nonnegative real numbers, respectively. Given N vectors $x_i \in \mathbb{R}^{n_i}$, $n_i \in \mathbb{N}_{\geq 1}$, and $i \in \{1, \dots, N\}$, we use $x = [x_1; \dots ; x_N]$ to denote the corresponding vector of dimension $\sum_i n_i$. Given a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes the infinity norm of x. Symbols I_n and $\mathbf{1}_n$ denote respectively the identity matrix in $\mathbb{R}^{n\times n}$ and the column vector in $\mathbb{R}^{n\times 1}$ with all elements equal to one. The identity function and composition of functions are denoted by id and symbol $\circ$, respectively. We denote by $\mathsf{diag}(a_1, \dots , a_N)$ a diagonal matrix in $\mathbb{R}^{N\times N}$ with diagonal matrix entries $a_1, \dots , a_N$ starting from the upper left corner. Given functions $f_i : X_i \to Y_i$, for any $i \in \{1, \dots , N\}$, their Cartesian product $\prod_{i=1}^N f_i : \prod_{i=1}^N X_i \to \prod_{i=1}^N Y_i$ is defined as $(\prod_{i=1}^N f_i)(x_1, \dots , x_N) = [f_1(x_1); \dots ; f_N(x_N)]$. For any set A we denote by $A^{\mathbb{N}}$ the Cartesian product of a countable number of copies of A, i.e., $A^{\mathbb{N}} = \prod_{k=0}^{\infty} A$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, is said to be a class $\mathcal{K}$ function if it is continuous, strictly increasing, and $\gamma(0) = 0$. A class $\mathcal{K}$ function $\gamma(\cdot)$ is said to be a class $\mathcal{K}_\infty$ if $\gamma(r) \to \infty$ as $r \to \infty$.

2.2 Discrete-Time Stochastic Control Systems

We consider stochastic control systems in discrete time (dt-SCS) defined by the tuple

$$\Sigma = (X, U, W, \varsigma, f, Y, h), \tag{1}$$

where $X \subset \mathbb{R}^n$ is a Borel space as the state space of the system. We denote by $(X, \mathcal{B}(X))$ the measurable space with $\mathcal{B}(X)$ being the Borel sigma-algebra on the state space. Sets $U \subset \mathbb{R}^m$ and $W \subset \mathbb{R}^p$ are Borel spaces as the *external* and *internal* input spaces of the system. Notation ς denotes a sequence of independent and identically distributed (i.i.d.) random variables from a sample space Ω to the set V_ς,

$$\varsigma := \{\varsigma(k) : \Omega \to V_\varsigma,\ k \in \mathbb{N}\}.$$

The map $f : X \times U \times W \times V_\varsigma \to X$ is a measurable function characterizing the state evolution of the system. Finally, set $Y \subset \mathbb{R}^q$ is a Borel space as the output space of the system. Map $h : X \to Y$ is a measurable function that maps a state $x \in X$ to its output $y = h(x)$.

For given initial state $x(0) \in X$ and input sequences $\nu(\cdot) : \mathbb{N} \to U$ and $w(\cdot) : \mathbb{N} \to W$, evolution of the state of dt-SCS Σ can be written as

$$\Sigma : \begin{cases} x(k + 1) = f(x(k), \nu(k), w(k), \varsigma(k)) \\ y(k) = h(x(k)) \end{cases} \quad k \in \mathbb{N}. \tag{2}$$

Given the dt-SCS in (1), we are interested in *Markov policies* to control the system.

Definition 1. A Markov policy for the dt-SCS Σ in (1) is a sequence $\rho = (\rho_0, \rho_1, \rho_2, \dots)$ of universally measurable stochastic kernels ρ_n (Bertsekas and Shreve (1996)), each defined on the input space U given $X \times W$ and such that for all $(x_n, w_n) \in X \times W$, $\rho_n(U|(x_n, w_n)) = 1$. The class of all such Markov policies is denoted by Π_M.

We associate respectively to U and W the sets $\mathcal{U}$ and $\mathcal{W}$ to be collections of sequences $\{\nu(k) : \Omega \to U,\ k \in \mathbb{N}\}$ and $\{w(k) : \Omega \to W,\ k \in \mathbb{N}\}$, in which $\nu(k)$ and $w(k)$ are independent of $\varsigma(t)$ for any $k, t \in \mathbb{N}$ and $t \geq k$. For any initial state $a \in X$, $\nu(\cdot) \in \mathcal{U}$, and $w(\cdot) \in \mathcal{W}$, the random sequences $x_{a\nu w} : \Omega \times \mathbb{N} \to X$, $y_{a\nu w} : \Omega \times \mathbb{N} \to Y$ satisfying (2) are called respectively the *solution process* and *output trajectory* of Σ under external input ν, internal input w and initial state a.

In this paper we are interested in studying interconnected discrete-time stochastic control systems without internal input that result from the interconnection of dt-SCS having both internal and external inputs. In this case, the interconnected dt-SCS without internal input is indicated by the simplified tuple $(X, U, \varsigma, f, Y, h)$ with $f : X \times U \times V_\varsigma \to X$.

System Σ is called finite if X, U, W are finite sets and infinite otherwise. We discuss construction of finite dt-SCS as abstractions of infinite ones in the following subsection.

2.3 Finite Abstractions of dt-SCS

A dt-SCS defined in (1) can be *equivalently* represented as a general Markov decision process (gMDP). This alternative representation is utilized in (Soudjani et al. (2015a); Haesaert et al. (2017)) to abstract a dt-SCS Σ to a *finite* dt-SCS $\widehat{\Sigma}$. The abstraction algorithm is based on finite partitions of sets $X = \cup_i \mathsf{X}_i$, $U = \cup_i \mathsf{U}_i$, and $W = \cup_i \mathsf{W}_i$, and selection of representative points $\bar{x}_i \in \mathsf{X}_i$, $\bar{\nu}_i \in \mathsf{U}_i$, and $\bar{w}_i \in \mathsf{W}_i$ as abstract states and inputs.

Given a dt-SCS $\Sigma = (X, U, W, \varsigma, f, Y, h)$, its finite abstract dt-SCS $\widehat{\Sigma}$ can be represented as

$$\widehat{\Sigma} = (\hat{X}, \hat{U}, \hat{W}, \varsigma, \hat{f}, \hat{Y}, \hat{h}), \tag{3}$$

where $\hat{X} = \{\bar{x}_i, i = 1, \ldots, n_x\}, \hat{U} = \{\bar{u}_i, i = 1, \ldots, n_u\}$, $\hat{W} = \{\bar{w}_i, i = 1, \ldots, n_w\}$ are the sets of selected representative points. Function $\hat{f} : \hat{X} \times \hat{U} \times \hat{W} \times V_\varsigma \to \hat{X}$ is defined as

$$\hat{f}(\hat{x}, \hat{\nu}, \hat{w}, \varsigma) = \Pi_x(f(\hat{x}, \hat{\nu}, \hat{w}, \varsigma)), \tag{4}$$

where $\Pi_x : X \to \hat{X}$ is the map that assigns to any $x \in X$, the representative point $\hat{x} \in \hat{X}$ of the corresponding partition set containing x. The output map $\hat{h}$ is the same as h with its domain restricted to finite state set $\hat{X}$ and the output set $\hat{Y}$ is just the image of $\hat{X}$ under h. The initial state of $\widehat{\Sigma}$ is also selected according to $\hat{x}_0 := \Pi_x(x_0)$ with x_0 being the initial state of Σ.

Remark 2. Abstraction map Π_x used in (4) satisfies the inequality

$$\|\Pi_x(x) - x\| \leq \delta, \quad \forall x \in X, \tag{5}$$

where δ is the state discretization parameter defined as $\delta := \sup\{\|x - x'\|,\ x, x' \in \mathsf{X}_i,\ i = 1, 2, \ldots, n_x\}$. This inequality will be used in Section 5 for compositional construction of finite dt-SCSs. Let us similarly define the abstraction map $\Pi_w : W \to \hat{W}$ on W that assigns to any $w \in W$ representative point $\hat{w} \in \hat{W}$ of the corresponding partition set containing w. This map also satisfies

$$\|\Pi_w(w) - w\| \leq \mu, \quad \forall w \in W, \tag{6}$$

where μ is the internal input discretization parameter defined similar to δ. We use inequality (6) in Section 4 (with μ indexed as μ_{ji} for the pair of subsystems Σ_j and Σ_i) for the compositional abstractions of interconnected systems.

In the next sections, we provide an approach for the *compositional synthesis* of abstractions for interconnected dt-SCS. We first define the notions of stochastic pseudo-simulation and simulation functions for quantifying the error between two dt-SCS (with both internal and external signals) and two interconnected dt-SCS (without internal signals), respectively. Then we employ dynamical representation of finite $\widehat{\Sigma}$ in (4) to compare interconnections of dt-SCS and those of their finite abstract counterparts based on these new notions. Finally, in the case study section, we synthesize policies for abstract dt-SCSs compositionally and refine them back to the original dt-SCSs while providing quantitative guarantees on the quality of the synthesized policies with respect to the satisfaction of local specifications. The provided guarantee is benchmarked against the approach in (Lavaei et al. (2018b)) in the case study section.

3. STOCHASTIC PSEUDO-SIMULATION AND SIMULATION FUNCTIONS

In this section, we first introduce the notion of stochastic pseudo-simulation function (SPSF) for dt-SCS with both internal and external signals. We then define, as a special case of this definition, the notion of stochastic simulation function (SSF) for dt-SCS without internal signals. Both definitions are used to quantify closeness of two dt-SCS, while the latter is specifically employed for interconnected dt-SCS.

Definition 3. Consider dt-SCS $\Sigma = (X, U, W, \varsigma, f, Y, h)$ and $\widehat{\Sigma} = (\hat{X}, \hat{U}, \hat{W}, \varsigma, \hat{f}, \hat{Y}, \hat{h})$, where $\hat{W} \subseteq W$ and $\hat{Y} \subseteq Y$. A function $V : X \times \hat{X} \to \mathbb{R}_{\geq 0}$ is called a stochastic pseudo-simulation function (SPSF) from $\widehat{\Sigma}$ to Σ if there exist $\alpha \in \mathcal{K}_\infty$, $\kappa \in \mathcal{K}$ with $\kappa < \mathrm{id}$, $\rho_{\mathrm{int}}, \rho_{\mathrm{ext}} \in \mathcal{K}_\infty \cup \{0\}$, and constant $\psi \in \mathbb{R}_{\geq 0}$, such that

$$\alpha(\|h(x) - \hat{h}(\hat{x})\|) \leq V(x, \hat{x}), \quad \forall x \in X, \hat{x} \in \hat{X}, \tag{7}$$

and it holds that for all $\hat{\nu} \in \hat{U}$ there exists $\nu \in U$ such that $\forall \hat{w} \in \hat{W}\ \forall w \in W$,

$$\begin{aligned}&\mathbb{E}\Big[V(f(x, \nu, w, \varsigma), \hat{f}(\hat{x}, \hat{\nu}, \hat{w}, \varsigma)) \,|\, x, \hat{x}, \nu, \hat{\nu}, w, \hat{w}\Big]\\ &\quad \leq \max\Big\{\kappa(V(x, \hat{x})), \rho_{\mathrm{int}}(\|w - \hat{w}\|), \rho_{\mathrm{ext}}(\|\hat{\nu}\|), \psi\Big\}. \end{aligned}\tag{8}$$

We write $\widehat{\Sigma} \preceq_{\mathcal{PS}} \Sigma$ if there exists an SPSF V from $\widehat{\Sigma}$ to Σ, and call the control system $\widehat{\Sigma}$ an abstraction of concrete (original) system Σ. Note that $\widehat{\Sigma}$ may be finite or infinite depending on cardinalities of sets $\hat{X}, \hat{U}, \hat{W}$.

Remark 4. Note that the notion of SPSF in Definition 3 is equivalent to the one defined in (Lavaei et al., 2017, Definition 3.1) in the sense that the existence of one implies that of the other one. Although the upper bound in (8) is in the max form, the one in (Lavaei et al., 2017, inequality (4)) is in the additive form.

Remark 5. Second condition in Definition 3 implicitly implies existence of a function $\nu = \nu_{\hat{\nu}}(x, \hat{x}, \hat{\nu})$ fulfilling inequality (8). This function is called an *interface function* and can be used to refine a synthesized policy $\hat{\nu}$ for $\widehat{\Sigma}$ to a policy ν for Σ.

In the following definition we adapt the notion of SPSF to dt-SCS without internal signals that comprise interconnected dt-SCS.

Definition 6. Consider two dt-SCS $\Sigma = (X, U, \varsigma, f, Y, h)$ and $\widehat{\Sigma} = (\hat{X}, \hat{U}, \varsigma, \hat{f}, \hat{Y}, \hat{h})$ without internal signals, where $\hat{Y} \subseteq Y$. A function $V : X \times \hat{X} \to \mathbb{R}_{\geq 0}$ is called a *stochastic simulation function* (SSF) from $\widehat{\Sigma}$ to Σ if there exists $\alpha \in \mathcal{K}_\infty$ such that

$$\alpha(\|h(x) - \hat{h}(\hat{x})\|) \leq V(x, \hat{x}), \quad \forall x \in X, \hat{x} \in \hat{X}, \tag{9}$$

and it holds that for all $x \in X$, $\hat{x} \in \hat{X}$, $\hat{\nu} \in \hat{U}$, there exists $\nu \in U$ such that

$$\begin{aligned}&\mathbb{E}\Big[V(f(x, \nu, \varsigma), \hat{f}(\hat{x}, \hat{\nu}, \varsigma)) \,|\, x, \hat{x}, \nu, \hat{\nu}\Big]\\ &\quad \leq \max\Big\{\kappa(V(x, \hat{x})), \rho_{\mathrm{ext}}(\|\hat{\nu}\|), \psi\Big\}, \end{aligned}\tag{10}$$

for some $\kappa \in \mathcal{K}$ with $\kappa < \mathrm{id}$, $\rho_{\mathrm{ext}} \in \mathcal{K}_\infty \cup \{0\}$, and $\psi \in \mathbb{R}_{\geq 0}$.

We write $\widehat{\Sigma} \preceq \Sigma$ if there exists an SSF V from $\widehat{\Sigma}$ to Σ, and call $\widehat{\Sigma}$ an abstraction of Σ.

The next theorem shows how SSF can be used to compare output trajectories of two dt-SCS (without internal signals) in a probabilistic sense. This theorem is borrowed from (Lavaei et al., 2017, Theorem 3.3), and holds for the setting here since our max form of SSF implies the additive form of SSF used there.

Theorem 7. Let $\Sigma = (X, U, \varsigma, f, Y, h)$ and $\widehat{\Sigma} = (\hat{X}, \hat{U}, \varsigma, \hat{f}, , \hat{Y}, \hat{h})$ be two dt-SCS without internal signals, where $\hat{Y} \subseteq$

Y. Suppose V is an SSF from $\widehat{\Sigma}$ to Σ, and there exists a constant $0 < \widehat{\kappa} < 1$ such that the function $\kappa \in \mathcal{K}$ in (10) satisfies $\kappa(r) \geq \widehat{\kappa} r$, $\forall r \in \mathbb{R}_{\geq 0}$. For any external input trajectory $\hat{\nu}(\cdot) \in \hat{\mathcal{U}}$ that preserves Markov property for the closed-loop $\widehat{\Sigma}$, and for any random variables a and $\hat{a}$ as the initial states of the two dt-SCS, there exists an input trajectory $\nu(\cdot) \in \mathcal{U}$ of Σ through the interface function associated with V such that the following inequality holds

$$\mathbb{P}\left\{\sup_{0\leq k\leq T_d} \|y_{a\nu}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\| \geq \varepsilon \,|\, [a;\hat{a}]\right\} \tag{11}$$

$$\leq \begin{cases} 1-(1-\dfrac{V(a,\hat{a})}{\alpha(\varepsilon)})(1-\dfrac{\widehat{\psi}}{\alpha(\varepsilon)})^{T_d} & \text{if } \alpha(\varepsilon)\geq\dfrac{\widehat{\psi}}{\widehat{\kappa}}, \\ (\dfrac{V(a,\hat{a})}{\alpha(\varepsilon)})(1-\widehat{\kappa})^{T_d}+(\dfrac{\widehat{\psi}}{\widehat{\kappa}\alpha(\varepsilon)})(1-(1-\widehat{\kappa})^{T_d}) & \text{if } \alpha(\varepsilon)<\dfrac{\widehat{\psi}}{\widehat{\kappa}}, \end{cases}$$

where the constant $\widehat{\psi} \geq 0$ satisfies $\widehat{\psi} \geq \rho_{\text{ext}}(\|\hat{\nu}\|_\infty) + \psi$.

4. COMPOSITIONAL ABSTRACTIONS FOR INTERCONNECTED SYSTEMS

In this section we analyze networks of stochastic control subsystems and show how to construct their abstractions together with a simulation function based on abstractions and SPSF functions of their subsystems.

4.1 Concrete Interconnected Stochastic Control Systems

Let us consider a collection of concrete stochastic control subsystems

$$\Sigma_i = (X_i, U_i, W_i, \varsigma_i, f_i, Y_i, h_i), \qquad i \in \{1,\ldots,N\}, \tag{12}$$

where their internal inputs and outputs are partitioned as

$$\begin{aligned} w_i &= [w_{i1};\ldots;w_{i(i-1)};w_{i(i+1)};\ldots;w_{iN}], \\ y_i &= [y_{i1};\ldots;y_{iN}], \end{aligned} \tag{13}$$

and their output spaces and functions are of the form

$$Y_i = \prod_{j=1}^{N} Y_{ij}, \quad h_i(x_i) = [h_{i1}(x_i);\ldots;h_{iN}(x_i)]. \tag{14}$$

We interpret the outputs y_{ii} as *external* ones, whereas the outputs y_{ij} with $i \neq j$ are *internal* ones which are used to interconnect these stochastic control subsystems. For the interconnection, we assume that w_{ij} is equal to y_{ji} if there is a connection from Σ_j to Σ_i, otherwise we put the connecting output function identically zero, i.e. $h_{ji} \equiv 0$. Now we are ready to define the *concrete interconnected* stochastic control systems and that of their *abstract interconnection*.

Definition 8. Consider $N \in \mathbb{N}_{\geq 1}$ concrete stochastic control subsystems $\Sigma_i = (X_i, U_i, W_i, \varsigma_i, f_i, Y_i, h_i)$, $i \in \{1,\ldots,N\}$, with the input-output configuration as in (13) and (14). The interconnection of Σ_i for any $i \in \{1,\ldots,N\}$, is the concrete interconnected stochastic control system $\Sigma = (X, U, \varsigma, f, Y, h)$, denoted by $\mathcal{I}(\Sigma_1,\ldots,\Sigma_N)$, such that $X := \prod_{i=1}^N X_i$, $U := \prod_{i=1}^N U_i$, function $f := \prod_{i=1}^N f_i$, $Y := \prod_{i=1}^N Y_{ii}$, and function $h = \prod_{i=1}^N h_{ii}$, subject to the following constraint:

$$\forall i,j \in \{1,\ldots,N\}, i \neq j: \quad w_{ji} = y_{ij}, \quad Y_{ij} \subseteq W_{ji}.$$

4.2 Compositional Abstractions of Interconnected Systems

Suppose we are given N concrete stochastic control subsystems (12) together with their corresponding abstractions

$$\widehat{\Sigma}_i = (\hat{X}_i, \hat{U}_i, \hat{W}_i, \varsigma_i, \hat{f}_i, \hat{Y}_i, \hat{h}_i)$$

where $\hat{W}_i \subseteq W_i$ and $\hat{Y}_i \subseteq Y_i$, with SPSF V_i from $\widehat{\Sigma}_i$ to Σ_i with the associated comparison functions and constants denoted by α_i, κ_i, $\rho_{\text{int}i}$, $\rho_{\text{ext}i}$, and ψ_i. In order to provide one of the main results of the paper, we define a notion of interconnection for abstract stochastic control subsystems.

Definition 9. Consider $N \in \mathbb{N}_{\geq 1}$ abstract stochastic control subsystems $\widehat{\Sigma}_i = (\hat{X}_i, \hat{U}_i, \hat{W}_i, \varsigma_i, \hat{f}_i, \hat{Y}_i, \hat{h}_i)$, $i \in \{1,\ldots,N\}$, with the input-output configuration similar to (13) and (14). The interconnection of $\widehat{\Sigma}_i$ for any $i \in \{1,\ldots,N\}$, is the abstract interconnected stochastic control system $\widehat{\Sigma} = (\hat{X}, \hat{U}, \varsigma, \hat{f}, \hat{Y}, \hat{h})$, denoted by $\widehat{\mathcal{I}}(\widehat{\Sigma}_1,\ldots,\widehat{\Sigma}_N)$, such that $\hat{X} := \prod_{i=1}^N \hat{X}_i$, $\hat{U} := \prod_{i=1}^N \hat{U}_i$, function $\hat{f} := \prod_{i=1}^N \hat{f}_i$, $\hat{Y} := \prod_{i=1}^N \hat{Y}_{ii}$, and function $\hat{h} = \prod_{i=1}^N \hat{h}_{ii}$, subject to the following constraint:

$$\forall i,j \in \{1,\ldots,N\}, i \neq j: \hat{w}_{ji} = \Pi_{w_{ji}}(\hat{y}_{ij}), \Pi_{w_{ji}}(\hat{Y}_{ij}) \subseteq \hat{W}_{ji}.$$

Now we raise the following *small-gain assumption* that is essential for the main compositionality result of the paper.

Assumption 1. Assume that $\mathcal{K}_\infty$ functions κ_{ij} defined as

$$\kappa_{ij}(r) := \begin{cases} \kappa_i(r) & \text{if } i = j \\ 2\rho_{\text{int}i}(2(\alpha_j^{-1}(r))) & \text{if } i \neq j, \end{cases}$$

satisfy

$$\kappa_{i_1 i_2} \circ \kappa_{i_2 i_3} \circ \ldots \circ \kappa_{i_{r-1} i_r} \circ \kappa_{i_r i_1} < \text{id} \tag{15}$$

for all sequences $(i_1,\ldots,i_r) \in \{1,\ldots,N\}^r$ and $r \in \{1,\ldots,N\}$. Note that small-gain condition (15) implies (Rüffer, 2010, Theorem 5.5) the existence of $\mathcal{K}_\infty$ functions $\sigma_i > 0$, satisfying

$$\max_{i,j}\left\{\sigma_i^{-1} \circ \kappa_{ij} \circ \sigma_j\right\} < \text{id}, \quad i,j = \{1,\ldots,N\}. \tag{16}$$

In the next theorem, we leverage small-gain Assumption 1 to quantify the error between the interconnection of stochastic control subsystems and that of their abstractions in a compositional way.

Theorem 10. Consider the interconnected dt-SCS $\Sigma = \mathcal{I}(\Sigma_1,\ldots,\Sigma_N)$ induced by $N \in \mathbb{N}_{\geq 1}$ stochastic control subsystems Σ_i. Suppose that each Σ_i and its abstraction $\widehat{\Sigma}_i$ admit a SPSF V_i. If Assumption 1 holds, then function $V(x,\hat{x})$ defined as

$$V(x,\hat{x}) := \max_i\{\sigma_i^{-1}(V_i(x_i,\hat{x}_i))\}, \tag{17}$$

for σ_i as in (16), is an SSF function from $\widehat{\Sigma} = \widehat{\mathcal{I}}(\widehat{\Sigma}_1,\ldots,\widehat{\Sigma}_N)$ to Σ provided that $\max_i \sigma_i^{-1}$ is concave.

5. CONSTRUCTION OF FINITE ABSTRACTIONS

In this section, we consider Σ as an infinite dt-SCS and $\widehat{\Sigma}$ as its *finite* abstraction constructed as in Section 2.3. We impose conditions on the infinite dt-SCS Σ enabling us to find SPSF from its finite abstraction $\widehat{\Sigma}$ to Σ.

Here, we focus on the class of linear dt-SCS and of stochastic pseudo-simulation functions that are square root of quadratic functions. First, we formally define a linear dt-SCS Σ. Afterwards, we construct its finite abstraction $\widehat{\Sigma}$ as in (3), and then provide conditions under which a candidate V is an SPSF from $\widehat{\Sigma}$ to Σ.

Dynamics of a linear dt-SCS are given by

$$\Sigma: \begin{cases} x(k+1) = Ax(k) + B\nu(k) + Dw(k) + \bar{N}\varsigma(k), \\ y(k) = Cx(k), \end{cases} \tag{18}$$

where the additive noise $\varsigma(k)$ is a sequence of independent random vectors with multivariate standard normal distributions. We use the tuple

$$\Sigma = (A, B, C, D, \bar{N}),$$

to refer to a linear dt-SCS of the form (18). Consider the following function

$$V(x, \hat{x}) = ((x - \hat{x})^T \tilde{M}(x - \hat{x}))^{\frac{1}{2}}, \tag{19}$$

where $\tilde{M}$ is a positive-definite matrix of appropriate dimension. In order to show that V in (19) is an SPSF from $\widehat{\Sigma}$ to Σ, we require the following assumption on Σ.

Assumption 11. Let $\Sigma = (A, B, C, D, \bar{N})$. Assume that there exist matrices $\tilde{M} \succ 0$, and K of appropriate dimensions such that the matrix inequality

$$(1 + 2\pi)(A + BK)^T \tilde{M}(A + BK) \preceq \widehat{\kappa}\tilde{M}, \tag{20}$$

holds for some constant $0 < \widehat{\kappa} < 1$ and $\pi > 0$.

Note that condition (20) is nothing more than pair (A, B) being stabilizable. Now, we have the main result of this section.

Theorem 12. Assume system $\Sigma = (A, B, C, D, \bar{N})$ satisfies Assumption 11. Let $\widehat{\Sigma}$ be its finite abstraction as in Subsection 2.3 with state discretization parameter δ. Then function V defined in (19) is an SPSF from $\widehat{\Sigma}$ to Σ.

6. CASE STUDY

To demonstrate the effectiveness of our approach, we apply it to the temperature regulation in a circular building by constructing compositionally a finite abstraction of a network containing 1000 rooms.

Consider a network of $n \geq 3$ rooms each equipped with a heater and connected circularly. The model of this case study is adapted from (Meyer et al. (2017)) by including stochasticity in the model as additive noise. The evolution of temperatures T can be described by the interconnected linear dt-SCS

$$\Sigma : \begin{cases} T(k+1) = \bar{A}T(k) + \gamma T_h \nu(k) + \beta T_E + \varsigma(k), \\ y(k) = T(k), \end{cases}$$

where $\bar{A}$ is a matrix with diagonal elements $\bar{a}_{ii} = (1 - 2\eta - \beta - \gamma\nu_i(k))$, $i \in \{1, \ldots, n\}$, off-diagonal elements $\bar{a}_{i,i+1} = \bar{a}_{i+1,i} = \bar{a}_{1,n} = \bar{a}_{n,1} = \eta$, $i \in \{1, \ldots, n-1\}$, and all other elements are identically zero. Parameters η, β, and γ are conduction factors respectively between the rooms $i \pm 1$ and the room i, between the external environment and the room i, and between the heater and the room i. Moreover, $T(k) = [T_1(k); \ldots; T_n(k)]$, $\nu(k) = [\nu_1(k); \ldots; \nu_n(k)]$, $\varsigma(k) = [\varsigma_1(k); \ldots; \varsigma_n(k)]$, $T_E = [T_{e1}; \ldots; T_{en}]$, where $T_i(k)$ and $\nu_i(k)$ are taking values in sets $[19, 21]$ and $[0, 0.6]$, respectively, for all $i \in \{1, \ldots, n\}$. Outside temperatures are the same for all rooms: $T_{ei} = -1\,^\circ C$, for all $i \in \{1, \ldots, n\}$, and the heater temperature $T_h = 50\,^\circ C$. Let us consider the individual rooms as Σ_i described as

$$\Sigma_i : \begin{cases} T_i(k+1) = A_i T_i(k) + \gamma T_h \nu_i(k) + D_i w_i(k) + \beta T_{ei} + \varsigma_i(k), \\ y_i(k) = T_i(k), \end{cases}$$

where $A_i = \bar{a}_{ii}$, $i \in \{1, \ldots, n\}$. One can readily verify that $\Sigma = \mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$ where $D_i = [\eta; \eta]^T$, and $w_i(k) = [y_{i-1}(k); y_{i+1}(k)]$ (with $y_0 = y_n$ and $y_{n+1} = y_1$). One can also verify that condition (20) is satisfied with $\tilde{M}_i = 1$, $K_i = 0$, $\pi_i = 1$, $\widehat{\kappa}_i = 0.48$ $\forall i \in \{1, \ldots, n\}$, and $\eta = 0.1, \beta = 0.4, \gamma = 0.5$. For the sake of comparison with (Lavaei et al. (2018b)), we fix here SPSF as $V_i(x_i, \hat{x}_i) = (x_i - \hat{x}_i)^T \tilde{M}_i (x_i - \hat{x}_i)$. Then function $V_i(T_i, \hat{T}_i) = (T_i - \hat{T}_i)^2$ is an SPSF from $\widehat{\Sigma}_i$ to Σ_i satisfying condition (7) with $\alpha_i(s) = s^2$ and condition (8) with $\kappa_i(s) = 0.99s$, $\rho_{\text{int}i}(s) = 0.91s^2$, $\rho_{\text{ext}i}(s) = 0$, $\forall s \in \mathbb{R}_{\geq 0}$, and $\psi_i = 7.6\,\delta^2$.

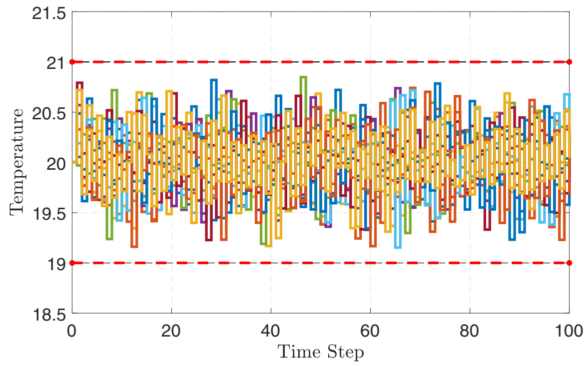

Fig. 1. Closed loop state trajectories of a representative room with different noise realizations in a network of 1000 rooms.

Now we check small-gain condition (15) that is required for the compositionality result. By taking $\sigma_i(s) = s$, $\forall i \in \{1, \ldots, n\}$, condition (15) and as a result condition (16) are always satisfied without any restriction on the number of rooms. Hence, $V(T, \hat{T}) = \max_i (T_i - \hat{T}_i)^2$ is an SSF from $\widehat{\Sigma}$ to Σ satisfying conditions (9) and (10) with $\alpha(s) = s^2$, $\kappa(s) = 0.99\, s$, $\rho_{\text{ext}}(s) = 0$, and $\psi = 7.6\,\delta^2$.

For the simulations, we fix $n = 1000$ and set the state discretization parameter $\delta = 0.005$. The initial states of the interconnected systems Σ and $\widehat{\Sigma}$ are $20\mathbf{1}_{1000}$. Using Theorem 7, we guarantee that the distance between outputs of Σ and $\widehat{\Sigma}$ will not exceed $\varepsilon = 0.5$ during the time horizon $T_d = 100$ with probability at least 98%, i.e.

$$\mathbb{P}(\|y_{a\nu}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\| \leq 0.5,\ \forall k \in [0, 100]) \geq 0.98\,. \tag{21}$$

Note that for the construction of finite abstractions, we have selected the center of partition sets as representative points. This choice has further tightened the above inequality. Moreover, to have a fair comparison with the compositional technique proposed in (Lavaei et al. (2018b)), we assume $\hat{Y}_{ij} = \hat{W}_{ji}$, i.e. μ_{ji}=0 $\forall i, j \in \{1, \ldots, N\}, i \neq j$.

Let us now synthesize a controller for Σ via the abstraction $\widehat{\Sigma}$ such that the controller maintains the temperature of any room in the comfort zone $[19, 21]$. We design a local controller for the abstract subsystem $\widehat{\Sigma}_i$, and then refine it to subsystem Σ_i using interface function. We employ the tool FAUST^2 (Soudjani et al. (2015b)) to synthesize controllers for Σ_i by taking the external input discretization parameter as 0.04 and standard deviation of noise as 0.21, $\forall i \in \{1, \ldots, n\}$. Closed-loop state trajectories of a representative room with different noise realizations are illustrated in Figure 1 with only 10 trajectories. Our simulations show that two out of 100 trajectories violates the specification, which is in accordance with the theoretical guarantee (21).

We now compare the guarantee provided in this paper by that of (Lavaei et al. (2018b)). Our result is based on a

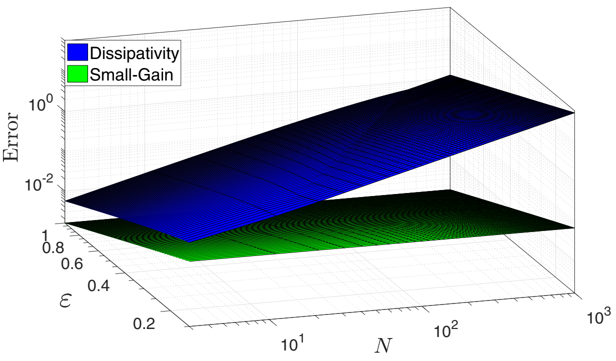

Fig. 2. Temperature control: Comparison of error bound in (11) provided by our approach based on *small-gain* condition with that of (Lavaei et al. (2018b)) based on *dissipativity property*. Plots are in logarithmic scale for a fixed $\delta = 0.005$, and $T_d = 100$.

small-gain approach while the one proposed in (Lavaei et al. (2018b)) uses *dissipativity-type conditions* on subsystems in the network. The comparison is shown in Figure 2 in logarithmic scale, in which we have fixed $\delta = 0.005$ and plotted the error (the upper bound of the probability in (11)) as a function of the number of subsystems N and confidence bound ε (cf. (11)). As seen, our new approach outperforms dramatically the one proposed in (Lavaei et al. (2018b)) since ψ in (11) is independent of the size of the network, and is computed based on the *maximum* of ψ_i of subsystems instead of being a *linear combination* of them. Hence, by increasing the number of subsystems, our error does not change whereas the error computed by the dissipativity approach in (Lavaei et al. (2018b)) will increase.

7. DISCUSSION

In this paper, we provided a compositional approach for the construction of *finite* Markov decision processes of interconnected discrete-time stochastic control systems. First, we introduced new notions of stochastic pseudo-simulation and simulation functions in order to quantify the distance in a probability setting between original stochastic control subsystems and their finite abstractions and their interconnections, respectively. Furthermore, we provided a compositional scheme on the construction of finite Markov decision processes of interconnected discrete-time stochastic control systems using *small-gain* type reasoning. Then, we proposed an approach to construct finite Markov decision processes together with their corresponding stochastic pseudo-simulation functions for a class of discrete-time linear stochastic control systems. Finally, we demonstrated the effectiveness of our proposed results in comparison with the existing ones in (Lavaei et al. (2018b)) based on dissipativity theory.

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