# On approximate predictability of metric systems ⋆

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**Abstract:** In this paper we introduce and characterize the notion of approximate predictability for the general class of metric systems, which are a powerful modeling framework to deal with complex heterogeneous systems such as hybrid systems. Approximate predictability corresponds to the possibility of predicting the occurrence of specific states belonging to a particular subset of interest, in advance with respect to their occurrence, on the basis of observations corrupted by measurement errors. We establish a relation between approximate predictability of a given metric system and approximate predictability of a metric system that approximately simulates the given one. This relation allows checking approximate predictability of a system with an infinite number of states, provided that one is able to construct a metric system with a finite number of states and inputs, approximating the original one in the sense of approximate simulation. The analysis of approximate predictability of Piecewise Affine (PWA) systems is carried out as an application of the proposed approach.

*Keywords:* hybrid systems, piecewise affine systems, symbolic models, approximate simulation, approximate predictability, approximate diagnosability.

## 1. INTRODUCTION

The study of safety issues in modern control systems is presently one of the most significant challenges, see e.g. (Cárdenas et al. (2008)). In this regard, it is fundamental to be able to understand if the system's behavior enters a given subset of the state space on the basis of the observations. This particular subset of states, which in the following will be called critical set, may represent faulty states, unsafe operations or, more generally, any subset of states which is of particular interest from the system's behavior point of view; a state belonging to the critical set will be called critical state (or also faulty state or fault). The safety problem can thus be addressed in two ways, that is either by detecting the occurrence of states belonging to the critical set within a finite time interval (diagnosability property), or by predicting in a deterministic way the occurrence of specific states belonging to the critical set, in advance with respect to their occurrence (predictability property). In this paper we focus on the latter.

Diagnosability has been extensively studied for: *i)* finite state systems, see e.g. (Lin (1994)), (Sampath et al. (1995)), and (De Santis and Di Benedetto (2017)); *ii)* continuous systems, see e.g (Benosman (2010)) and (Gao et al. (2015)); *iii)* hybrid systems, see e.g. (Narasimhan and Biswas (2007)), (Bayoudh and Travé-Massuyès (2014)), (De Santis and Di Benedetto (2016)), to name a few. In the recent paper (Pola et al. (2018)) a novel notion of approximate diagnosability has been introduced for metric systems, making it possible to deal with measurements affected by errors. Metric systems are a powerful modeling framework to deal with complex heterogeneous systems such as hybrid systems, characterized by the interaction of continuous dynamics, modeling physical processes, and discrete dynamics, modeling computational and communication components, see e.g. (Tabuada (2009)).

In safety critical applications, predicting the future occurrence of particular states of interest is of paramount importance. Indeed, this allows *pro-actively* performing operations on the system to enhance its reliability, optimizing performance or ensuring safety by avoiding abnormal behaviors. Motivated by this need, predictability has been studied for: *i)* discrete event systems, see e.g. (Genc and Lafortune (2009)), (Zaytoon and Lafortune (2013)) and references therein, (Takai and Kumar (2017)); *ii)* continuous systems, see e.g. (Mosterman and Biswas (1997)). In this paper we introduce the notion of approximate predictability for metric systems. In particular, given an accuracy $\rho \geq 0$ and a set of faulty states $\mathcal{F}$, the notion of approximate predictability corresponds to the possibility of distinguishing, from the observations collected up to a certain time instant $T > 0$, state runs that will reach for the first time the set of faulty states $\mathcal{F}$ within a finite time interval $\Delta > 0$ (i.e., before $T + \Delta$), from both state runs that will not reach the set $\mathcal{B}_\rho(\mathcal{F})$, obtained by expanding $\mathcal{F}$ with a factor $\rho$, and state runs that already reached the set of faulty states $\mathcal{F}$ at a certain time instant $t < T$.

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The over approximation of the set $\mathcal{F}$ allows taking into account uncertainties due to measurement errors. When the accuracy $\rho$ is equal to zero, approximate predictability translates to metric systems the notion of (exact) predictability investigated in (Fiore et al. (2018)) for Finite State Machines (FSMs).

The main result of this paper is the relation between approximate predictability of a given metric system and approximate predictability of a metric system that approximately simulates the given one. Thanks to this relation, it is possible to check approximate predictability of a system $\Sigma$ with an infinite number of states, e.g. a nonlinear system or a hybrid system, provided that one is able to construct a metric system that is symbolic (i.e. with a finite number of states and inputs), and that approximates $\Sigma$ in the sense of approximate simulation. The construction of symbolic models for continuous or hybrid control systems can be achieved under the conditions and by means of the numerous results existing in the literature on this topic, see e.g. (Tabuada (2009)) and references therein. To demonstrate this important aspect of the main result, we apply it to study approximate predictability of discrete-time Piecewise Affine (PWA) systems.

The paper is organized as follows. In Section 2 we introduce notation and preliminary definitions. In Section 3 we define the approximate predictability property. In Section 4 we derive the relation between approximate simulation and approximate predictability. This relation is applied in Section 5 to the approximate predictability of PWA systems. Section 6 offers some concluding remarks and the Appendix provides algorithms to check approximate predictability of symbolic metric systems.

## 2. NOTATION AND PRELIMINARY DEFINITIONS

The symbols $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ denote the set of nonnegative integer, real, positive real, and nonnegative real numbers, respectively. The symbol $0_n$ denotes the origin in $\mathbb{R}^n$. Given $a, b \in \mathbb{Z}$, we denote $[a; b] = [a, b] \cap \mathbb{Z}$. For a finite set $X$, the symbol $\mathrm{card}(X)$ denotes the cardinality of $X$. Given a pair of sets $X$ and $Y$ and a relation $\mathcal{R} \subseteq X \times Y$, the symbol $\mathcal{R}^{-1}$ denotes the inverse relation of $\mathcal{R}$, i.e. $\mathcal{R}^{-1} = \{(y, x) \in Y \times X : (x, y) \in \mathcal{R}\}$. Given $X' \subseteq X$ and $Y' \subseteq Y$, we denote $\mathcal{R}(X') = \{y \in Y | \exists x \in X' \text{ s.t. } (x, y) \in \mathcal{R}\}$ and $\mathcal{R}^{-1}(Y') = \{x \in X | \exists y \in Y' \text{ s.t. } (x, y) \in \mathcal{R}\}$. Given a function $f : X \to Y$ and $X' \subseteq X$ the symbol $f(X')$ denotes the image of $X'$ through $f$, i.e. $f(X') = \{y \in Y | \exists x \in X' \text{ s.t. } y = f(x)\}$. Given a set $X$, a set $Y \subseteq X \times X$ is said to be symmetric if $(y, y') \in Y$ implies $(y', y) \in Y$. The minimal symmetric set $Y \subseteq X \times X$ containing a set $Z \subseteq X \times X$ is a symmetric set such that $Z \subseteq Y \subseteq Y'$ for any symmetric set $Y' \subseteq X \times X$ containing $Z$. Given $\theta \in \mathbb{R}^+$ and $x \in \mathbb{R}^n$, we define $\mathcal{B}^n_{[-\theta,\theta[}(x) = \{y \in \mathbb{R}^n | y_i \in [-\theta + x_i, \theta + x_i[, i \in [1; n]\}$, where $x_i$ and $y_i$ denote the $i$–th component of vectors $x$ and $y$, respectively. Note that for any $\theta \in \mathbb{R}^+$, the collection of sets $\mathcal{B}^n_{[-\theta,\theta[}(x)$ is a partition of $\mathbb{R}^n$. A polyhedron $P \subseteq \mathbb{R}^n$ is a set obtained by the intersection of a finite number of (open or closed) half–spaces. A polytope is a bounded polyhedron. Given a set $X$, a function $\mathbf{d} : X \times X \to \mathbb{R}_0^+ \cup \{\infty\}$ is a quasi–pseudo–metric for $X$ if:

(i) for any $x \in X$, $\mathbf{d}(x, x) = 0$ and

(ii) for any $x, y, z \in X$, $\mathbf{d}(x, y) \leq \mathbf{d}(x, z) + \mathbf{d}(z, y)$.

If condition (i) is replaced by:

(i') $\mathbf{d}(x, y) = 0$ if and only if $x = y$,

then $\mathbf{d}$ is said to be a quasi–metric for $X$. If function $\mathbf{d}$ enjoys properties (i), (ii) and property:

(iii) for any $x, y \in X$, $\mathbf{d}(x, y) = \mathbf{d}(y, x)$,

then $\mathbf{d}$ is said a pseudo–metric for $X$. If function $\mathbf{d}$ enjoys properties (i'), (ii) and (iii), it is said a metric for $X$. When function $\mathbf{d}$ is a (quasi) (pseudo) metric for $X$, the pair $(X, \mathbf{d})$ is said a (quasi) (pseudo) metric space. From (Reilly et al. (1982)), given a quasi–pseudo–metric space $(X, \mathbf{d})$, a sequence $\{x_i\}_{i \in \mathbb{N}_0}$ over $X$ is left (resp. right) $\mathbf{d}$–convergent to $x^* \in X$, denoted $\lim\limits_{\leftarrow} x_i = x^*$ (resp. $\lim\limits_{\rightarrow} x_i = x^*$), if for any $\varepsilon \in \mathbb{R}^+$ there exists $N \in \mathbb{N}_0$ such that $\mathbf{d}(x_i, x^*) \leq \varepsilon$ (resp. $\mathbf{d}(x^*, x_i) \leq \varepsilon$) for any $i \geq N$. Given $X \subseteq \mathbb{R}^n$ we denote by $\mathbf{d}_h$ the Hausdorff pseudo–metric induced by the infinity norm $\|\cdot\|$ on $2^X$; we recall that for any $X_1, X_2 \subseteq X$, $\mathbf{d}_h(X_1, X_2) := \max\{\mathbf{d}_h(X_1, X_2), \mathbf{d}_h(X_2, X_1)\}$, where $\mathbf{d}_h(X_1, X_2) = \sup_{x_1 \in X_1} \inf_{x_2 \in X_2} \|x_1 - x_2\|$ is the Hausdorff quasi–pseudo–metric.

## 3. METRIC SYSTEMS AND APPROXIMATE PREDICTABILITY

In this paper we consider the class of metric systems, defined as follows.

*Definition 1.* (Tabuada (2009)) A system $S$ is a tuple

$$S = (X, X_0, U, \longrightarrow, Y, H), \tag{1}$$

where:

- $X$ is the set of states,
- $X_0 \subseteq X$ is the set of initial states,
- $U$ is the set of inputs,
- $\longrightarrow \subseteq X \times U \times X$ is the transition relation,
- $Y$ is the set of outputs,
- $H : X \to Y$ is the output function.

We follow standard practice and denote a transition $(x, u, x') \in \longrightarrow$ of $S$ by $x \xrightarrow{u} x'$. The evolution of a system is captured by the notions of state and output runs. Given a sequence of transitions of $S$

$$x(0) \xrightarrow{u(0)} x(1) \xrightarrow{u(1)} \ldots \xrightarrow{u(l-1)} x(l) \tag{2}$$

with $x(0) \in X_0$, the sequences:

$$x(\cdot) : x(0)\, x(1) \ldots x(l), \tag{3}$$

$$y(\cdot) : H(x(0))\, H(x(1)) \ldots H(x(l)), \tag{4}$$

are called a *state run* and an *output run* of $S$, respectively. State $x(l)$ is called the *ending state* of the state run in (3). The *accessible part* of a system $S$, denoted $\mathrm{Ac}(S)$, is the collection of states of $S$ that are reached by state runs of $S$. System $S$ in (1) is said to be *symbolic* if $\mathrm{Ac}(S)$ and $U$ are finite sets, and *metric* if $X$ is equipped with a metric. Metric systems are general enough to capture heterogeneous dynamics arising for example in cyber-physical systems, see e.g. (Tabuada (2009)). Throughout the paper we assume that the inputs $u$ of metric system $S$ are not available; this assumption corresponds to the point of view of an external observer that cannot have access to the inputs of the system $S$. We now introduce the notion of approximate predictability.

*Definition 2.* Consider $S = (X, X_0, U, \longrightarrow, Y, H)$ a metric system with metric $\mathbf{d}$, and denote by $\mathcal{B}_\rho(x)$ the closed ball induced by metric $\mathbf{d}$ centered at $x \in X$, and with radius $\rho \in \mathbb{R}_0^+$, i.e.:

$$\mathcal{B}_\rho(x) = \{x' \in X | \mathbf{d}(x, x') \leq \rho\}. \tag{5}$$

Given $X' \subseteq X$, define:

$$\mathcal{B}_\rho(X') = \bigcup_{x' \in X'} \mathcal{B}_\rho(x'). \tag{6}$$

Consider a set $\mathcal{F} \subseteq X$ of faulty states of $S$ with $\mathcal{F} \cap X_0 = \emptyset$. Given a desired accuracy $\rho \in \mathbb{R}_0^+$, system $S$ is $(\rho, \mathcal{F})-$predictable if there exists $\Delta \in \mathbb{N}$, such that for any finite state run $x^f(\cdot)$ of $S$ for which the ending state $x^f(t_f) \in \mathcal{F}$, there exists $T \in [t_0, t_f)$ such that the following properties hold:

(i) for any state run $x^s$, with $y^s|_{[t_0,T]} = y^f|_{[t_0,T]}$, $x^s|_{[t_0,T]}$ does not contain states in $\mathcal{F}$;
(ii) for any state run $x^s$ such that $x^s \neq x^f$, with $y^s|_{[t_0,T]} = y^f|_{[t_0,T]}$, and for any infinite state run $x^c(\cdot)$ such that $x^c|_{[t_0,T]} = x^s|_{[t_0,T]}$, $x^c(T + \delta) \in \mathcal{B}_\rho(\mathcal{F})$, for some $\delta \leq \Delta$.

Approximate predictability corresponds to the possibility of distinguishing, from the observations collected in a certain time interval $[t_0; T]$, state runs that will reach for the first time the set of faulty states $\mathcal{F}$ in at most $\Delta > 0$ time steps (i.e., within $T+\Delta$) from both state runs that will not reach the set $\mathcal{B}_\rho(\mathcal{F})$ and state runs that already reached $\mathcal{F}$ at a previous time instant $t < T$. The over approximation of the set $\mathcal{F}$ allows taking into account uncertainties due to measurement errors. The definition above extends to metric systems the notion of (exact) predictability given in (Fiore et al. (2018)) for FSMs [1]. In particular, when $\rho = 0$, the definition above coincides with the one given in (Fiore et al. (2018)). Checking approximate predictability of *symbolic* metric systems is a decidable problem with polynomial computational complexity, as discussed in the Appendix, where we provide an extension of algorithms described in (Fiore et al. (2018)) from exact to approximate predictability.

## 4. MAIN RESULT

In this section we establish the relation between approximate predictability and approximate simulation.
We start by recalling the following definition.

*Definition 3.* Consider a pair of metric systems $S_i = (X_i, X_{0,i}, U_i, \underset{i}{\longrightarrow}, Y_i, H_i)$, $i = 1, 2$, with $X_1$ and $X_2$ subsets of some metric set $X$ equipped with metric $\mathbf{d}$, and let $\varepsilon \in \mathbb{R}_0^+$ be a given accuracy. Consider a relation $\mathcal{R} \subseteq X_1 \times X_2$ satisfying the following conditions:

(i) $\forall x_1 \in X_{0,1}\ \exists x_2 \in X_{0,2}$ such that $(x_1, x_2) \in \mathcal{R}$;
(ii) $\mathbf{d}(x_1, x_2) \leq \varepsilon$, $\forall (x_1, x_2) \in \mathcal{R}$;
(iii) $H_1(x_1) = H_2(x_2)$, $\forall (x_1, x_2) \in \mathcal{R}$.

Relation $\mathcal{R}$ is an *$\varepsilon$-approximate simulation relation* from $S_1$ to $S_2$ if it enjoys conditions (i)–(iii) and the following one:

(iv) $\forall (x_1, x_2) \in \mathcal{R}$ if $x_1 \xrightarrow[1]{u_1} x_1'$ then there exists $x_2 \xrightarrow[2]{u_2} x_2'$ with $(x_1', x_2') \in \mathcal{R}$.

System $S_1$ is $\varepsilon$-simulated by $S_2$, denoted $S_1 \preceq_\varepsilon S_2$, if there exists an $\varepsilon$-approximate simulation relation from $S_1$ to $S_2$. Relation $\mathcal{R}$ is an *$\varepsilon$-approximate bisimulation relation* between $S_1$ and $S_2$ if $\mathcal{R}$ is an $\varepsilon$-approximate simulation relation from $S_1$ to $S_2$, and $\mathcal{R}^{-1}$ is an $\varepsilon$-approximate simulation relation from $S_2$ to $S_1$. Systems $S_1$ and $S_2$ are $\varepsilon$-bisimilar if there exists an $\varepsilon$-approximate bisimulation relation between $S_1$ and $S_2$.

*Remark 1.* The definition above extends the classical definition of bisimulation equivalence of (Milner (1989); Park (1981)) for concurrent processes, to the class of metric systems in the sense of Definition 1; when condition (ii) is removed, it becomes an adaptation to systems of the definition given in (Milner (1989); Park (1981)) for concurrent processes. It slightly differs from the one given in (Girard and Pappas (2007)) where it is assumed that sets $Y_1 = Y_2$ are metric spaces with metric $\mathbf{d}$, and conditions (ii) and (iii) are replaced by $\mathbf{d}(H_1(x_1), H_2(x_2)) \leq \varepsilon, \forall (x_1, x_2) \in \mathcal{R}$.

We can now present the main result of this paper.

*Theorem 1.* Consider a pair of metric systems $S_i = (X_i, X_{0,i}, U_i, \underset{i}{\longrightarrow}, Y_i, H_i)$, $i = 1, 2$, with $X_1$ and $X_2$ subsets of some metric set $X$ equipped with metric $\mathbf{d}$ and suppose that $S_1 \preceq_\varepsilon S_2$. Consider a set $\mathcal{F}_1 \subseteq X_1$ of faulty states for $S_1$ and define the set $\mathcal{F}_2 = \mathcal{B}_\varepsilon(\mathcal{F}_1) \cap X_2$ of faulty states for system $S_2$. If $S_2$ is $(\rho_2, \mathcal{F}_2)-$predictable for some accuracy $\rho_2 \in \mathbb{R}^+$, then $S_1$ is $(\rho_1, \mathcal{F}_1)-$predictable for all $\rho_1 \geq \rho_2 + 2\varepsilon$.

We point out that since $S_1 \preceq_\varepsilon S_2$, in view of conditions (i) and (iv) of Definition 3 we get $\mathcal{R}(\mathcal{F}_1) \neq \emptyset$, where $\mathcal{R}$ is an $\varepsilon$–approximate simulation relation from $S_1$ to $S_2$, and since by condition (ii) of Definition 3 we have $\mathcal{R}(\mathcal{F}_1) \subseteq \mathcal{B}_\varepsilon(\mathcal{F}_1)$, implying $\mathcal{R}(\mathcal{F}_1) \cap X_2 \subseteq \mathcal{B}_\varepsilon(\mathcal{F}_1) \cap X_2$, the set $\mathcal{F}_2$ is nonempty.

Thanks to Theorem 1, it is possible to check approximate predictability of a metric system $S_1$ on the basis of approximate predictability of a metric system $S_2$ such that $S_1 \preceq_\varepsilon S_2$. Therefore, when $S_2$ has fewer states than $S_1$, Theorem 1 can reduce computational complexity in checking approximate predictability of $S_1$. In particular, provided that one is able to construct a symbolic metric system approximating a continuous or hybrid control system $\Sigma$ (with an infinite number of states) in the sense of approximate simulation, Theorem 1 allows leveraging the results reported in the Appendix to check approximate predictability of $\Sigma$.

We recall that the literature on the construction of symbolic models approximating continuous or hybrid control systems is very broad, see e.g. (Tabuada (2009)) and the references therein. Works available in the literature that fit precisely the framework of this paper are (Pola et al. (2016, 2008)) proposing symbolic models for incrementally stable nonlinear systems, (Zamani et al. (2012)) for possibly unstable nonlinear systems, (Girard et al. (2010)) for incrementally stable switched systems, and (Pola and Di Benedetto (2014)) for PWA systems. In the next section

[1] FSMs considered in (Fiore et al. (2018)) coincide with symbolic systems considered in this paper.

the analysis of approximate predictability of PWA systems is carried out as an application of Theorem 1.

## 5. APPROXIMATE PREDICTABILITY OF PIECEWISE AFFINE SYSTEMS

In this section we investigate approximate predictability for the class of discrete–time Piecewise Affine (PWA) systems described by the tuple

$$\Sigma = (\mathbb{R}^n, \mathcal{U}, \{\Sigma_1, \Sigma_2, ..., \Sigma_N\}) \tag{7}$$

where

- $\mathbb{R}^n$ is the state space,
- $\mathcal{U} \subseteq \mathbb{R}^m$ is the set of control inputs,
- $\Sigma_i$, $i = 1, \cdots, N$ is a constrained affine control system defined by:

$$\begin{cases} x_i(t+1) = A_i x_i(t) + B_i u_i(t) + f_i, \\ x_i(t) \in X_i, u_i(t) \in \mathcal{U}, \end{cases} \tag{8}$$

  where $f_i$, $i = 1, \cdots, N$ is a constant vector.

We suppose that the sets $X_i \subseteq \mathbb{R}^n$ are polyhedral, with interior, and that their collection is a partition of $\mathbb{R}^n$; moreover we suppose that the set $\mathcal{U}$ is polyhedral. We denote by $\mathbf{x}(t, x_0, \mathbf{u})$ the state reached by $\Sigma$ at time $t \in \mathbb{N}$ starting from an initial state $x_0 \in \mathbb{R}^n$ with control input $\mathbf{u} : \mathbb{N} \to \mathcal{U}$. Since $\{X_i\}_{i\in[1;N]}$ is a partition of $\mathbb{R}^n$ the PWA system $\Sigma$ is deterministic. We are interested in the evolution of PWA systems within bounded subsets of the state space $\mathbb{R}^n$. Let $\mathcal{X}$ be a polytopic subset of $\mathbb{R}^n$ representing the region of the state space of $\Sigma$ we are interested in. Define:

$$\mathcal{X}_i = X_i \cap \mathcal{X},\ i \in [1; N] \tag{9}$$

and denote by $\mathcal{P}(\mathcal{X})$ the set of polytopic subsets of $\mathcal{X}$.
We now equip the PWA system with an output function. Let $\mathcal{Y}$ be a finite set of outputs and consider an output function $h : \mathcal{P}(\mathcal{X}) \to \mathcal{Y}$. Function $h$ naturally induces an output $y$ on a state $x$ of $\Sigma$ by defining $y = \overline{h}(x)$ where $\overline{h}(x) = h(X_i)$ if $x \in X_i$. In the sequel we refer to the PWA $\Sigma$ equipped with the output function $h$, and we will work with the set $\mathcal{S}(\mathcal{P}(\mathcal{X}), \mathbf{d}_h)$ of pseudo–metric systems with state pseudo–metric space $(\mathcal{P}(\mathcal{X}), \mathbf{d}_h)$. The notion of approximate simulation relations induces certain metrics on $\mathcal{S}(\mathcal{P}(\mathcal{X}), \mathbf{d}_h)$:

*Definition 4.* (Girard and Pappas (2007)) Consider two pseudo–metric systems $S_1, S_2 \in \mathcal{S}(\mathcal{P}(\mathcal{X}), \mathbf{d}_h)$. The simulation metric $\mathbf{d}_\mathrm{s}$ from $S_1$ to $S_2$ is defined by:

$$\mathbf{d}_\mathrm{s}(S_1, S_2) = \inf\{\varepsilon \in \mathbb{R}_0^+ | S_1 \preceq_\varepsilon S_2\}. \tag{10}$$

*Theorem 2.* (Girard and Pappas (2007))
The pair $(\mathcal{S}(\mathcal{P}(\mathcal{X}), \mathbf{d}_h), \mathbf{d}_\mathrm{s})$ is a quasi–pseudo–metric space[2].

The expressive power of the notion of systems as in Definition 1 is general enough to describe the evolution of PWA systems within the bounded region $\mathcal{X}$ of the state space $\mathbb{R}^n$.

*Definition 5.* Given the PWA system $\Sigma$ and the polytopic subset $\mathcal{X}$ of $\mathbb{R}^n$ define the pseudo–metric system:

$$\mathbb{S}(\Sigma) = (\mathbb{X}, \mathbb{X}_0, \mathbb{U}, \longrightarrow, \mathbb{Y}, \mathbb{H}) \tag{11}$$

where:

- $\mathbb{X} = \mathbb{X}_0 = \mathcal{X}$, equipped with $\mathbf{d}_h$,
- $\mathbb{U} = \mathcal{U}$,
- $x \xrightarrow{u} x'$, if $x \in \mathcal{X}_i$ and $x' = A_i x + B_i u + f_i$,
- $\mathbb{Y} = \mathcal{Y}$,
- $\mathbb{H}(x) = \overline{h}(x)$, for any $x \in X$.

System $\mathbb{S}(\Sigma)$ preserves important properties of $\Sigma$, such as reachability and determinism. Also, since $\mathbf{d}_h(\{x\}, \{y\}) = \|x - y\|$, metric properties of $\Sigma$ are naturally transferred to $\mathbb{S}(\Sigma)$ and vice versa. Although system $\mathbb{S}(\Sigma)$ correctly describes $\Sigma$ within the bounded set $\mathcal{X}$, it is not symbolic because $\mathbb{X}$ and $\mathbb{U}$ are not finite sets. For this reason, in the sequel we introduce a sequence of symbolic models $\mathbb{A}_M(\Sigma)$ that approximate the PWA system $\Sigma$. To this purpose we first need to introduce two operators.

*Definition 6.* Given a PWA system $\Sigma$, the bisimulation operator is the map:

$$\mathrm{Bisim} : 2^{\mathcal{X}} \to 2^{\mathcal{X}} \tag{12}$$

that associates to any $Z_1, Z_2, ..., Z_L \subseteq \mathcal{X}$ the collection $\mathrm{Bisim}(\{Z_1, Z_2, ..., Z_L\})$ of sets $\{x \in Z_j | \exists u \in \mathcal{U} \text{ s.t. } A_i x + B_i u + f_i \in Z_{j'}, x \in \mathcal{X}_i\}$ $(j, j' \in [1; L])$.

We can now introduce the splitting operator. We recall that the diameter $\mathrm{Diam}(X)$ of a set $X \subseteq \mathbb{R}^n$ is defined by:

$$\mathrm{Diam}(X) = \sup_{x,y\in X} \|x - y\|. \tag{13}$$

*Definition 7.* Consider a finite collection of polytopes $\mathbb{P} = \{P_1, P_2, ..., P_N\} \subset \mathcal{P}(\mathcal{X})$. A splitting policy with contraction rate $\lambda \in ]0, 1[$ for $\mathbb{P}$ is a map

$$\Phi_\lambda : \mathbb{P} \to 2^{\mathcal{P}(\mathcal{X})} \tag{14}$$

enjoying the following properties:

(i) the cardinality of $\Phi_\lambda(P_i)$ is finite,
(ii) $\Phi_\lambda(P_i)$ is a partition of $P_i$,
(iii) $\mathrm{Diam}(P_i^j) \leq \lambda \mathrm{Diam}(P_i)$ for all $P_i^j \in \Phi_\lambda(P_i)$.

In the sequel, $\mathrm{Split}_\lambda$ denotes a splitting policy with contraction rate $\lambda$ and we abuse notation by writing $\mathrm{Split}_\lambda(\{P_1, P_2, ..., P_N\})$ instead of $\bigcup_{i\in[1;N]} \mathrm{Split}_\lambda(P_i)$.

We now have all the ingredients to introduce a sequence of abstractions $\mathbb{A}_M(\Sigma)$ approximating the PWA system $\Sigma$. Consider the following recursive equations:

$$\begin{cases} \mathbf{X}_0 = \{\mathcal{X}_1, \mathcal{X}_2, ..., \mathcal{X}_N\}, \\ \mathbf{X}_{M+1} = \mathrm{Split}_\lambda(\mathrm{Bisim}(\mathbf{X}_M)), M \in \mathbb{N}. \end{cases} \tag{15}$$

At each order $M \in \mathbb{N}$, the set $\mathbf{X}_M$ naturally induces a system that is formalized as follows.

*Definition 8.* Given the set $\mathbf{X}_M$ define the pseudo–metric system

$$\mathbb{A}_M(\Sigma) = (\mathbb{X}_M, \mathbb{U}_M, \underset{M}{\longrightarrow}, \mathbb{Y}_M, \mathbb{H}_M) \tag{16}$$

where:

- $\mathbb{X}_M = \mathbf{X}_M \cup \{\mathcal{X} \backslash \bigcup_{X\in\mathbf{X}_M} X\}$, equipped with the pseudo–metric $\mathbf{d}_h$; a state in $\mathbb{X}_M$ is denoted by $X_M^j$,
- $X_M^j \xrightarrow[M]{V} X_M^{j'}$ if there exist $x \in X_M^j$ and $u \in \mathcal{U}$ such that $A_i x + B_i u + f_i \in X_M^{j'}$, and $V = \{u \in \mathcal{U} | \exists x \in X_M^j \text{ s.t. } A_i x + B_i u + f_i \in X_M^{j'}\}$, where index $i$ is such that $X_M^j \subseteq \mathcal{X}_i$,

[2] In (Girard and Pappas (2007)) quasi–pseudo–metric spaces are termed directed pseudo–metric spaces.

- $\mathbb{U}_M$ is the collection of all sets $V \subseteq \mathcal{U}$ for which $X_M^j \xrightarrow[M]{V} X_M^{j'}$,
- $\mathbb{Y}_M = \mathcal{Y}$,
- $\mathbb{H}_M(X_M^j) = y$ if $X_M^j \subseteq \mathcal{X}_i$ and $h(\mathcal{X}_i) = y$.

By construction, system $\mathbb{A}_M(\Sigma)$ is symbolic. Symbolic system $\mathbb{A}_{M+1}(\Sigma)$ can be viewed as a refinement of $\mathbb{A}_M(\Sigma)$. The following result holds as a direct consequence of the definition of operator Bisim.

*Proposition 1.* If $\mathbf{X}_M = \text{Bisim}(\mathbf{X}_M)$ then $\mathbb{A}_M(\Sigma)$ and $\mathbb{S}(\Sigma)$ are exactly bisimilar.

We point out that, in general, even if an exact bisimulation of a given PWA system $\Sigma$ exists, there is no guarantee that it can be found by the recursive equations in (15); this is because in general $\text{Split}_\lambda$ does not satisfy the reachability properties of $\Sigma$. On the other hand, as we shall show in the sequel, the splitting operator is a key element to prove the convergence properties of the sequence $\mathbb{A}_M(\Sigma)$.

We now proceed with a step further by providing our approximation scheme and a quantification of its accuracy. Define:

$$\text{Gran}(\mathbb{A}_M(\Sigma)) = \max_{X_M^j \in \mathbb{X}_M} \text{Diam}(X_M^j). \tag{17}$$

Function Gran provides a measure of the "granularity" of the symbolic system $\mathbb{A}_M(\Sigma)$ (i.e. how fine is the covering of the set $\mathcal{X}$).

The following result provides an upper bound for the distance between the PWA system $\Sigma$ and the abstraction $\mathbb{A}_M(\Sigma)$.

*Theorem 3.* $\mathbf{d}_\text{s}(\mathbb{S}(\Sigma), \mathbb{A}_M(\Sigma)) \leq \text{Gran}(\mathbb{A}_M(\Sigma))$.

The rest of this section is devoted to the analysis of the convergence of the sequence $\{\mathbb{A}_M(\Sigma)\}_{M\in\mathbb{N}}$ to $\mathbb{S}(\Sigma)$.

*Lemma 1.* $\text{Gran}(\mathbb{A}_{M+1}(\Sigma)) \leq \lambda \text{Gran}(\mathbb{A}_M(\Sigma))$.

*Theorem 4.* $\mathbb{S}(\Sigma) = \underset{\leftarrow}{\lim} \mathbb{A}_M(\Sigma)$.

Proposition 1, Theorem 3, Lemma 1 and Theorem 4 are straightforward generalizations of Proposition 1, Theorem 2, Lemma 1 and Theorem 3 in (Pola and Di Benedetto (2014)), respectively, from PWA systems to PWA systems equipped with outputs.

We can now state the following results that follow as a direct application of Theorem 1.

*Theorem 5.* Given a critical set $\mathcal{F} \subset \mathbb{R}^n$ and a desired accuracy $\varepsilon \in \mathbb{R}^+$, if there exists $M \in \mathbb{N}$ such that $\mathbb{A}_M(\Sigma)$ is $(k\gamma, \mathcal{F}_\varepsilon)-$predictable for some $k \in \mathbb{N}$, with $\gamma = \text{Gran}(\mathbb{A}_M(\Sigma))$ and $\mathcal{F}_\varepsilon = \mathcal{B}_\varepsilon(\mathcal{F}) \cap \mathbb{X}_M$, then $\Sigma$ is $(\rho, \mathcal{F})-$predictable, for any $\rho \geq k\gamma + 2\varepsilon$.

*Corollary 1.* Given a critical set $\mathcal{F} \subset \mathbb{R}^n$ and a desired accuracy $\varepsilon \in \mathbb{R}^+$, if there exists $M \in \mathbb{N}$ such that $\mathbb{A}_M(\Sigma)$ is $(k\gamma, \mathcal{F}_\varepsilon)-$predictable for some $k \in \mathbb{N}$, with $\gamma = \text{Gran}(\mathbb{A}_M(\Sigma))$ and $\mathcal{F}_\varepsilon = \mathcal{B}_\varepsilon(\mathcal{F}) \cap \mathbb{X}_M$, then $\mathbb{A}_i(\Sigma)$ is $(k\gamma, \mathcal{F}_\varepsilon^i)-$predictable, with $\mathcal{F}_\varepsilon^i = \mathcal{B}_\varepsilon(\mathcal{F}) \cap \mathbb{X}_i$, for any $i \geq M$, $i \in \mathbb{N}$.

Given the critical set $\mathcal{F} \subset \mathbb{R}^n$ and a desired accuracy $\varepsilon \in \mathbb{R}^+$, Theorem 5 allows checking approximate predictability of a PWA system $\Sigma$ by recursion, that is, by constructing the abstraction $\mathbb{A}_1(\Sigma)$ and checking its approximate predictability with respect to $\mathcal{F}_\varepsilon^1 = \mathcal{B}_\varepsilon(\mathcal{F}) \cap \mathbb{X}_1$. If $\mathbb{A}_1(\Sigma)$ is not predictable, then the refinement $\mathbb{A}_2(\Sigma)$ is derived, and so on, until $M \in \mathbb{N}$ is found such that $\mathbb{A}_M(\Sigma)$ is predictable. If this is not found, nothing can be inferred about the approximate predictability of the PWA system $\Sigma$. The sufficient condition provided in Theorem 5 is a direct consequence of the existence of sequence of symbolic models that approximate the PWA system in the sense of approximate simulation.

## 6. CONCLUSIONS

In this paper we introduced and characterized the notion of approximate predictability for metric systems and we described how to check this property for symbolic metric systems. Furthermore, we established a relation between approximate predictability and approximate simulation. This relation allows checking approximate predictability of a system $\Sigma$ with an infinite number of states for which a symbolic metric system (i.e., with a finite number of states and inputs) approximating $\Sigma$ in the sense of approximate simulation can be constructed. This result was applied to the analysis of approximate predictability of piecewise affine systems.

## REFERENCES


Bayoudh, M. and Travé-Massuyès, L. (2014). Diagnosability analysis of hybrid systems cast in a discrete-event framework. *Discrete Event Dynamic Systems*, 24(3), 309–338.

Benosman, M. (2010). A survey of some recent results on nonlinear fault tolerant control. *Math. Probl. Eng.*, 2010.

Cárdenas, A.A., Amin, S., and Sastry, S. (2008). Secure control: Towards survivable Cyber-Physical Systems. In *2008 The 28th International Conference on Distributed Computing Systems Workshops*, 495–500.

De Santis, E. and Di Benedetto, M.D. (2016). Observability of hybrid dynamical systems. *Foundations and Trends in Systems and Control*, 3(4), 363–540.

De Santis, E. and Di Benedetto, M.D. (2017). Observability and diagnosability of finite state systems: a unifying framework. *Automatica*, 81, 115–122.

Fiore, G., De Santis, E., and Di Benedetto, M.D. (2018). Predictability for Finite State Machines: a set-membership approach. In *2018 14th International Workshop on Discrete Event Systems (WODES)*.

Gao, Z., Cecati, C., and Ding, S. (2015). A survey of fault diagnosis and fault–tolerant techniques–part i: Fault diagnosis with model-based and signal-based approaches. *IEEE Transactions on Industrial Electronics*, 62, 3757–3767.

Genc, S. and Lafortune, S. (2009). Predictability of event occurrences in partially-observed discrete-event systems. *Automatica*, 45(2), 301 – 311.

Girard, A. and Pappas, G. (2007). Approximation metrics for discrete and continuous systems. *IEEE Transactions on Automatic Control*, 52(5), 782–798.

Girard, A., Pola, G., and Tabuada, P. (2010). Approximately bisimilar symbolic models for incrementally stable switched systems. *IEEE Transactions of Automatic Control*, 55(1), 116–126.

Lin, F. (1994). Diagnosability of discrete event systems and its applications. *Discrete Event Dynamic Systems*, 4(2), 197–212.

Milner, R. (1989). *Communication and Concurrency.* Prentice Hall.

Mosterman, P.J. and Biswas, G. (1997). Monitoring, prediction, and fault isolation in dynamic physical systems. In *AAAI/IAAI*, 100–105.

Narasimhan, S. and Biswas, G. (2007). Model-based diagnosis of hybrid systems. *IEEE Transactions on Systems, Man, and Cybernetics - Part A: Systems and Humans*, 37(3), 348–361.

Park, D. (1981). Concurrency and automata on infinite sequences. volume 104 of *Lecture Notes in Computer Science*, 167–183.

Pola, G. and Di Benedetto, M.D. (2014). Symbolic models and control of discrete–time piecewise affine systems: An approximate simulation approach. *IEEE Transactions of Automatic Control*, 59(1), 175–180.

Pola, G., Girard, A., and Tabuada, P. (2008). Approximately bisimilar symbolic models for nonlinear control systems. *Automatica*, 44, 2508–2516.

Pola, G., Pepe, P., and Di Benedetto, M.D. (2016). Symbolic models for networks of control systems. *IEEE Transactions on Automatic Control*, 61(11), 3663–3668.

Pola, G., De Santis, E., and Di Benedetto, M.D. (2018). Approximate diagnosis of metric systems. *IEEE Control Systems Letters*, 2(1), 115–120.

Reilly, I., Subrahmanyam, P., and Vamanamurthy, M. (1982). Cauchy sequences in quasi–pseudo–metric spaces. *Monatshefte für Mathematik*, 93(2), 127–140.

Sampath, M., Sengupta, R., Lafortune, S., Sinnamohideen, K., and Teneketzis, D. (1995). Diagnosability of discrete-event systems. *IEEE Transactions on Automatic Control*, 40(9), 1555–1575.

Tabuada, P. (2009). *Verification and Control of Hybrid Systems: A Symbolic Approach.* Springer.

Takai, S. and Kumar, R. (2017). A generalized inference-based prognosis framework for discrete event systems. *IFAC-PapersOnLine*, 50(1), 6819 – 6824. 20th IFAC World Congress.

Zamani, M., Mazo, M., Pola, G., and Tabuada, P. (2012). Symbolic models for nonlinear control systems without stability assumptions. *IEEE Transactions of Automatic Control*, 57(7), 1804–1809.

Zaytoon, J. and Lafortune, S. (2013). Overview of fault diagnosis methods for discrete event systems. *Annual Reviews in Control*, 37(2), 308 – 320.

## 7. APPENDIX

In this section we show how to check approximate predictability of metric *symbolic* systems by extending the algorithm given in (Fiore et al. (2018)) for exact predictability. Given a symbolic metric system $S = (X, X_0, U, \longrightarrow, Y, H)$, let $\mathrm{Post}(x) = \{x' \in X \mid \exists\, x \xrightarrow{u} x'\}$, $\mathrm{Pre}(x') = \{x \in X \mid \exists\, x \xrightarrow{u} x'\}$. For a set $X' \subseteq X$, let $\mathrm{Post}(X') = \bigcup_{x \in X'} \mathrm{Post}(x)$, $\mathrm{Pre}(X') = \bigcup_{x \in X'} \mathrm{Pre}(x)$. Given $S$ and a set of faulty states $\mathcal{F} \subseteq X$, define the symmetric sets:

$$\Pi = \{(x, x') \in X \times X : H(x) = H(x')\} \tag{18}$$

and

$$\Theta = \{(x, x') \in X \times X : x = x'\} \subseteq \Pi. \tag{19}$$

We recall, from e.g. (De Santis and Di Benedetto (2017)), that two state runs of system $S$ are indistinguishable if their corresponding output runs coincide. Let $\mathcal{I}^*$ be the set of all pairs $(x, x') \in \Pi$ reachable from $X_0$ with two indistinguishable state runs. Set $\mathcal{I}^*$ can be computed by using the following recursion:

$$\begin{aligned} \mathcal{I}_1 &= (X_0 \times X_0) \cap \Pi, \\ \mathcal{I}_{k+1} &= \{(x, x') \in \Pi : (\mathrm{Pre}(x) \times \mathrm{Pre}(x')) \cap \mathcal{I}_k \neq \emptyset\} \cup \mathcal{I}_k, \\ k &\in \mathbb{N}. \end{aligned} \tag{20}$$

*Lemma 2.* Consider recursion (20). Then:
*i*) The least fixed point of recursion (20) exists, is unique and is equal to $\mathcal{I}^*$.
*ii*) Recursion (20) reaches the fixed point $\mathcal{I}^*$ in at most $\mathrm{card}(X)^2$ steps.

Given a set $\mathcal{F} \subset X$, $R^{-1}(\mathcal{F})$ is the set of states $x \in X$ from which it is possible to reach the set $\mathcal{F}$ after one step, but not before, that is:

$$R^{-1}(\mathcal{F}) = \{x \in X, x \in \overline{\mathcal{F}} : \mathrm{Post}(x) \cap \mathcal{F} \neq \emptyset\}. \tag{21}$$

Let $F_1(\mathcal{F}) \subset R^{-1}(\mathcal{F})$ be the set of states from which it is possible to reach the set $\mathcal{F}$ in finite time, that is:

$$F_1(\mathcal{F}) = \{x \in X : \forall x(\cdot), x(0) = q, x(i) \in \mathcal{F}, i \in [2, +\infty)\}. \tag{22}$$

Let $F(\mathcal{F})$ be the complement of the maximal set of states starting from which there exists an arbitrarily long state run belonging to $\overline{\mathcal{F}}$, then:

$$F_1(\mathcal{F}) = R^{-1}(\mathcal{F}) \cap F(\mathcal{F}). \tag{23}$$

Given system $S$, define the symbolic metric system $\widetilde{S} = (X, X_0, U, \xrightarrow[\sim]{}, Y, H)$ where $x \xrightarrow[\sim]{u} x'$ if and only if $x \xrightarrow{u} x'$ and $x \notin \mathcal{F}$.

*Assumption 1.* Sets $F_1(\mathcal{F})$ and $R^{-1}(\mathcal{F})$ computed for the symbolic metric system $\widetilde{S}$ are such that $F_1(\mathcal{F}) \neq \emptyset$ and $R^{-1}(\mathcal{F}) = F_1(\mathcal{F})$.

Given system $\widetilde{S}$, let $\widetilde{\mathcal{I}}^*$ be the set of pairs of states reachable from $X_0$ with two indistinguishable state runs, and we also define the following set:

$$\hat{F}_1(\mathcal{F}) = \{x \in F_1(\mathcal{F}) : \mathrm{Pre}(x) \cap \overline{F_1(\mathcal{F})} \neq \emptyset\}. \tag{24}$$

We can state the following:

*Theorem 6.* System $S$ is $(\rho, \mathcal{F})-$predictable if and only if for system $\widetilde{S}$ the following holds:

$$\mathcal{I}^* \cap \left(\hat{F}_1(\mathcal{F}) \times D_1(\mathcal{F})\right) = \emptyset \tag{25}$$

where:

$$D_1(\mathcal{F}) = \overline{F(\mathcal{B}_\rho(\mathcal{F}))} \cup \mathcal{F}. \tag{26}$$

The proof of the above result is a direct consequence of Proposition 8 in (Fiore et al. (2018)).

*Remark 2.* Space and time complexities in computing $\mathcal{I}^*$ are respectively, $O\left(\mathrm{card}(X)^2\right)$ and $O\left(\mathrm{card}(X)^5\right)$ (as described in (De Santis and Di Benedetto (2017))). Space and time complexities in computing $F_1(\mathcal{F})$ are respectively, $O\left(\mathrm{card}(X)\right)$ and $O\left(\mathrm{card}(X)^3\right)$ (as described in (Fiore et al. (2018))). Similar observations on complexity hold for computation of $\hat{F}_1(\mathcal{F})$ and $D_1(\mathcal{F})$. Therefore checking approximate predictability is a polynomial time algorithm.