Stability of switched systems on non-uniform time domains with non commuting matrices

Fatim Z. Taousser * **M. Defoort** ** **M. Djemai** **
Seddik M. Djouadi * **Kevin Tomsovic** *

* *Department of Electrical engineering (CURENT), University of Tennessee, USA ; (e-mail: ftaousse@utk.edu, taoussfatima@gmail.com, mdjouadi@utk.edu, ktomsovi@tennessee.edu)*
** *LAMIH UVHC UMR CNRS 8201, 59313 Valenciennes, France; (e-mail: {mohamed.djemai, michael.defoort}@univ-valenciennes.fr).*

Time scales; Stability analysis; Switched systems; Exponential stability; Non-uniform time domain.

Abstract: The time scale theory is introduced to study the stability of a class of switched linear systems on non-uniform time domains, where the dynamical system commutes between a continuous-time linear subsystem and a discrete-time linear subsystem during a certain period of time. Without assuming that matrices are pairwise commuting, some conditions are derived to guarantee the exponential stability of the switched system by considering that the continuous-time subsystem is stable and the discrete-time subsystem can be stable or unstable. Some examples show the effectiveness of the proposed scheme.

1. INTRODUCTION

Switched systems are systems involving both continuous and discrete dynamics. They consists of a finite number of subsystems and a discrete rule that dictates switching between them. Stability of these systems have been widely studied during these two last decades Liberzon [2003], Liberzon and Morse [1999], Lin and Antsaklis [2009], Sun and Ge [2005] because they can describe a wide range of physical and engineering applications. Most of the existing methods can be categorized into two separated directions depending on whether each subsystem is continuous-time Geromel and Colaneri [2006a] or discrete-time Geromel and Colaneri [2006b], Zhang and Yu [2009].

Motivated by this observation, in this paper, the stability of switched linear systems on non-uniform time domains is analyzed. There are many interesting applications involving non-uniform time domains. For example, impulsive systems (which are a relevant class of switched systems, in which the state jumps occur only at some time instants Amato et al. [2013]) with non-instantaneous state jumps, indeed, their temporal nature cannot be represented by the continuous or the discrete line. Cooperative control over network under intermittent information transmissions is another example Guo et al. [2014], Taousser et al. [2016].

The theory of system dynamics on an arbitrary time scale is promising because it demonstrates the interplay between the theory of continuous-time and discrete-time systems Bohner and Peterson [2001, 2003], Hilger [1990]. It leads to a new understanding and analysis of dynamical systems on any non-uniform time domains that are closed subsets of $\mathbb{R}$. As expected, time scale dynamic equations reduce to standard continuous differential equations (resp. standard difference equations) when the time scale is the continuous line (resp. homogeneous discrete domain). Exponential stability was derived for linear systems using the time scale exponential function in DaCuhna [2005], Doan et al. [2010], Peterson and Raffoul [2005], Potzsche et al. [2003]. Dynamic equations with general structured perturbations Du et al. [2011] and nonlinear finite-dimensional control systems Bartosiewicz and Piotrowska [2013] on time scales have also been investigated. The dynamical systems evolving on a discrete time scale are studied by using a stochastic approach in D. R. Poulsen and Gravagne [2013], D. R. Poulsen and Gravagne [2017].

In general, the stability analysis of switched systems is based on the existence of a common (or multiple) Lyapunov function which imposed some algebraic conditions on the matrices of subsystems. In Zhai et al. [2006], some stability conditions were derived for switched normal linear systems, which are given by two subsystems evolving on continuous-time and discrete uniform time domains with fixed sampling periods. However, the extension to a larger class of systems evolving on a non-uniform time domain is not trivial. To solve this issue, the theory of system dynamics on an arbitrary time scale $\mathbb{T}$ seems to be appropriate. The stability of a class of switched linear systems on time scale which consists of a set of stable linear continuous-time and stable linear discrete-time subsystems was studied in Taousser et al. [2016, 2015a], Davis et al. [2010], Taousser et al. [2014, 2017]. However, in these papers, the matrices of each subsystem were assumed to be pairwise commuting. In Eisenbarth et al. [2014], the stability of simultaneously triangularizable switched systems on arbitrary time scale was analyzed using a common Lyapunov function. One can note that the approaches given in Taousser et al. [2015a], Gravagne et al. [2011], Davis et al. [2010], Eisenbarth et al. [2014] do not work if one individual subsystem is not asymptotically stable. In Taousser et al. [2016, 2014, 2017] the studied system switches between a continuous-time and a discrete-time dynamic subsystem with bounded graininess function where the matrices of the two subsystems are pairwise commuting. Using the time scale exponential function properties, some sufficient conditions were derived to guarantee the exponential stability of this class of switched systems when the subsystems are possibly stable or unstable using the spectrum of the system matrices. In Taousser et al. [2015b], a necessary and sufficient conditions of exponential stability of this class of switched systems are derived by determining a region of exponential stability.

In this paper, we are interested in extending the existing results for stability of switched systems on non-uniform time domains formed by a union of disjoint intervals with variable length and variable gap Taousser et al. [2016, 2014, 2015a]. Without assuming restrictive conditions on the subsystems (pairwise commuting or simultaneosly triangularizable conditions, etc.), sufficient conditions are derived to guarantee the exponential stability of this class of switched systems under bounded graininess condition by supposing that the continuous-time subsystem is stable and the discrete-time subsystem can be stable or instable. Notice that in this work as in Taousser et al. [2016, 2014, 2015a], we suppose that the time scale $\mathbb{T}$ is given in advance (i.e the switching times is known) and the conditions of stability are derived with respect it. Some examples are presented to validate the results.

The paper is organized as fellows. A preliminary on time scale theory are presented in Section II. Conditions of exponential stability of the proposed switched system are derived in Section III. Numerical examples which validate the proposed scheme are given in Section IV.

2. PROBLEM STATEMENT

2.1 Preliminaries on time scale theory

In this subsection, we recall some basics on time scale theory Bohner and Peterson [2001, 2003]. A *time scale* $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. For $t \in \mathbb{T}$, the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$. The mapping $\mu : \mathbb{T} \to \mathbb{R}^+$, called the *graininess function*, is defined by $\mu(t) = \sigma(t) - t$. A point $t \in \mathbb{T}$ is called *right-scattered* if $\sigma(t) > t$ and *right-dense* if $\sigma(t) = t$. The set $\mathbb{T}^\kappa$ is defined as follows: if $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$; otherwise $\mathbb{T}^\kappa = \mathbb{T}$. Let $f : \mathbb{T} \to \mathbb{R}$ be Δ-*differentiable* on $\mathbb{T}^\kappa$. The Δ-derivative of f at $t \in \mathbb{T}^\kappa$ is

$$f^\Delta(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} \tag{1}$$

One can notice that if $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = \dot{f}(t)$, which is the usual derivative of f; and if $\mathbb{T} = h\mathbb{Z}$, then $f^\Delta(t) = \frac{f(t+h)-f(t)}{h}$ (using the time scale theory, the theory of differential and difference equations is unified).

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *rd-continuous*, if it is continuous at right-dense points in $\mathbb{T}$ and its left-hand limit exists at left-dense points in $\mathbb{T}$. A function $p : \mathbb{T} \to \mathbb{R}$ is *regressive* if $1 + \mu(t)p(t) \neq 0, \forall t \in \mathbb{T}^\kappa$. We denote the set of regressive and rd-continuous functions by $\mathcal{R}$ and by $\mathcal{R}^+$ if they satisfy $1 + \mu(t)p(t) > 0$, $\forall t \in \mathbb{T}^\kappa$ (i.e *positively regressive functions*). Similarly, a matrix function $A : \mathbb{T} \to \mathbb{R}^n$ is called *regressive*, if $\forall t \in \mathbb{T}^\kappa$, $I + \mu(t)A(t)$ is invertible, where I is the identity matrix. Equivalently, $A(t)$ is regressive if and only if all its eigenvalues are regressive.

The *generalised exponential function* of $p \in \mathcal{R}$ on time scale $\mathbb{T}$ is expressed by

$$e_p(t,s) = \begin{cases} e^{\int_s^t \frac{\log(1+\mu(\tau)p(\tau))}{\mu(\tau)} \Delta\tau} & \text{if } \mu(t) \neq 0,\ s,t \in \mathbb{T} \\ e^{\int_s^t p(\tau)\Delta\tau} & \text{if } \mu(t) = 0,\ s,t \in \mathbb{T} \end{cases} \tag{2}$$

where log is the principal logarithm function. Let p be constant, so for $\mathbb{T} = \mathbb{R}$, $e_p(t,t_0) = e^{p(t-t_0)}$ and for $\mathbb{T} = h\mathbb{Z}$, $e_p(t,s) = \prod_{\tau=s}^{t-h}(1 + hp(\tau))$.

Theorem 1. Bohner and Peterson [2003] Let a regressive constant matrix $A \in \mathbb{R}^{n\times n}$, the unique solution of

$$x^\Delta(t) = Ax(t), \quad x(t_0) = I, \quad t_0 \in \mathbb{T} \tag{3}$$

is the generalized exponential function $e_A(t,t_0)$.

The definitions of stability of dynamical systems on time scales are achieved by modifications of the standard stability concepts for continuous and discrete dynamical systems.

System (3) is exponentially stable on a time scale $\mathbb{T}$, if there exists a constant $\beta \geq 1$ and a negative constant $\alpha \in \mathcal{R}^+$, such that the corresponding solution satisfies

$$\|x(t)\| \leq \beta\|x_0\|e_\alpha(t,t_0), \quad \forall t \in \mathbb{T}.$$

This characterization of exponential stability is a generalization of the definition of exponential stability for systems defined in $\mathbb{R}$ or $h\mathbb{Z}$. More specifically, the condition that $\alpha < 0$ and $\alpha \in \mathcal{R}^+$ in the characterization of exponential stability is reduced to $\alpha < 0$ for $\mathbb{T} = \mathbb{R}$ and to $0 < 1+h\alpha < 1$ for $\mathbb{T} = h\mathbb{Z}$ (in this case $e_\alpha(t,t_0) > 0$, $\forall t \geq t_0,\ t, t_0 \in \mathbb{T}$). In Potzsche et al. [2003], the exponential stability of dynamical system (3) is given by determining a region of exponential stability $\mathcal{S}_\mathbb{C}$ on an arbitrary, unbounded, time scale $\mathbb{T}$. To get around the computational difficulties of this region of exponential stability, the authors showed in Gard and Hoffacker [2003], that for any $\mathbb{T}$, the *Hilger circle* at time t defined as

$$\mathcal{H}_{\mu(t)} = \left\{ z \in \mathbb{C} : |z + \frac{1}{\mu(t)}| < \frac{1}{\mu(t)} \right\}$$

is a subset of $\mathcal{S}_\mathbb{C}$. When the graininess function is bounded (i.e. $\mu_{\max} = \sup_{t\in\mathbb{T}} \mu(t)$), the smallest Hilger circle (denoted $\mathcal{H}_{\min}$) is the Hilger circle associated with $\mu_{\max}$. When $\mu(t) = 0$, the Hilger circle is defined as $\mathcal{H}_0 = \{z \in \mathbb{C} : Re(z) < 0\}$, the open left-half complex plane. Furthermore, a regressive constant matrix A is said *Hilger stable* if $spec(A) \subset \mathcal{H}_{\min}$ (i.e all eigenvalues of A are in $\mathcal{H}_{\min}$) Eisenbarth et al. [2014], Potzsche et al. [2003].

2.2 Problem statement

In this paper, time scale theory is introduced to study the stability of a special class of switched systems where the dynamical system commutes between a continuous-time linear subsystem and a discrete-time linear subsystem during a certain period of time (which may correspond to the time needed for state jump such as an interruption of the information transmissions for instance). Consider the particular time scale $\mathbb{T} = \mathbb{P}_{\{\sigma(t_k),t_{k+1}\}} = \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}]$ where $t_0 = \sigma(t_0) = 0$ is the initial time, and t_{k+1}, $(k \in \mathbb{N})$, are the switching times. Let $\{A_c, A_d\}$ be a set of two constant regressive matrices of appropriate dimensions. The eigenvalues of A_c (resp. A_d) are denoted $\lambda_c^j \in spec(A_c)$ (resp. $\lambda_d^j \in spec(A_d)$). The studied switched linear system on time scale $\mathbb{T} = \mathbb{P}_{\{\sigma(t_k),t_{k+1}\}}$ is written as

$$x^{\Delta}(t) = \begin{cases} A_c x(t) \text{ for } t \in \cup_{k=0}^{\infty}[\sigma(t_k), t_{k+1}) \\ \\ A_d x(t) \text{ for } \quad t \in \cup_{k=0}^{\infty}\{t_{k+1}\} \end{cases} \tag{4}$$

The first equation of (4) describes the continuous-time linear dynamics of the system and the second one can be seen as the non instantaneous jumps. We will consider in this paper that the dynamical system commutes between a stable continuous-time linear subsystem and a possibly instable linear discrete-time subsystem during a certain period of time.

3. STABILITY ANALYSIS OF SWITCHED LINEAR SYSTEMS ON TIME SCALES

The stability of (4) with non pairwise commuting matrices is discussed by using the properties of the generalized exponential function. Let us derive the explicit solution of (4). Using time scale theory, one can obtain, $\forall k \in \mathbb{N}$, if $t \in [\sigma(t_k), t_{k+1})$, then the corresponding solution is

$$x(t) = e^{A_c(t-\sigma(t_k))}x(\sigma(t_k))$$

For $t = t_{k+1}$, we have

$$x(\sigma(t_{k+1})) = (I + \mu(t_{k+1})A_d)\ x(t_{k+1})$$

Therefore, for $\sigma(t_k) \le t \le t_{k+1}$, $k \in \mathbb{N}$, the solution of (4) is given by (see Taousser et al. [2014] for more details)

$$\begin{aligned} x(t) &= e^{A_c(t-\sigma(t_k))}(I + \mu(t_k)A_d)\ \ldots \\ &\quad \times (I + \mu(t_1)A_d)e^{A_c t_1}x_0 \\ &= e^{A_c(t-\sigma(t_k))}e_{A_d}(\sigma(t_k), t_k)\ \ldots \\ &\quad \times e_{A_d}(\sigma(t_1), t_1)e^{A_c t_1}x_0 \end{aligned} \tag{5}$$

In the following, we will investigate the case on whether continuous-time subsystem is stable and the discrete-time subsystem can be stable or instable.

Let us assume that:

(i) Matrix A_c is stable.

(ii) The graininess function is bounded i.e., $0 < \mu_{\min} \le \mu(t) \le \mu_{\max}$ for all $t \in \cup_{k=0}^{\infty}\{t_{k+1}\}$

(iii) Let us define constants $\alpha_c < 0$, $\alpha_d \in \mathcal{R}^+$ and corresponding constants $\beta_c, \beta_d \ge 1$ such that $\forall k \in \mathbb{N}^*$ and for $t, s \in [\sigma(t_k), t_{k+1}]$, $t \ge s$,

$$\begin{aligned} \|e_{A_c}(t, s)\| &\le \beta_c\ e_{\alpha_c}(t, s) \\ \|e_{A_d}(\sigma(t_k), t_k)\| &\le \beta_d\ e_{\alpha_d}(\sigma(t_k), t_k) \end{aligned} \tag{6}$$

And one of the following conditions is fulfilled:

$$\max_{1\le i\le k}(1 + \mu(t_i)\alpha_d) < \frac{1}{\beta^2} \tag{7}$$

or

$$\max_{1\le i\le k}(1 + \mu(t_i)\ \alpha_d) < e^{-[\alpha_c\ \ \min_{1\le i\le k}(t_i-\sigma(t_{i-1}))\ +\log(\beta^2)]} \tag{8}$$

with $\beta = \max\{\beta_c, \beta_d\}$.

Remark 1.
There always exist constants $\alpha_c < 0$, $\alpha_d \in \mathcal{R}^+$ and $\beta_c, \beta_d \ge 1$, such that $\alpha_c \ge Re(\lambda_c) = \max_j\{Re(\lambda_c^j),\ \lambda_c^j \in spec(A_c)\}$ and $(1+\mu(t)\alpha_d) \ge \max_{1\le j\le n}\{|1+\mu(t)\lambda_d^j|,\ \lambda_d^j \in spec(A_d)\}$, $\forall\ t \in \cup_{k=1}^{\infty}\{t_k\}$.
If A_c (resp. A_d) are diagonalizable, so the above two inequalities can be replaced by equalities.

Remark 2.
Condition (7) means that A_d is stable (i.e the eigenvalues of A_d lie strictly within the Hilger circle). If condition (7) does not hold, one may check condition (8). Roughly speaking, this condition means that the effect of the discrete-time subsystem (stable or instable) is less significant than the effect of the continuous-time subsystem to guarantee the exponential stability of the switched system.

Theorem 2.
Under Assumptions (i)-(iii), the switched system (4) is exponentially stable.

Proof.
According to Assumption (i), the state transition matrix of the continuous-time subsystem satisfies

$$\|e_{A_c}(t, s)\| = \|e^{A_c(t-s)}\| \le \beta_c\ e^{\alpha_c(t-s)}$$

for $t, s \in [\sigma(t_k), t_{k+1}[$, $t \ge s$ with $\alpha_c < 0$.

Therefore, on $[\sigma(t_k), t_{k+1}]$, $k \in \mathbb{N}$, one can derive an upper bound of solution (5) as follows

$$\begin{aligned} \|x(t)\| &\le \|e^{A_c(t-\sigma(t_k))}\|\|e_{A_d}(\sigma(t_k), t_k)\|\|e^{A_c(t_k-\sigma(t_{k-1}))}\| \\ &\quad \ldots \|e_{A_d}(\sigma(t_1), t_1)\|\|e^{A_c t_1}\|\ \|x_0\| \\ &\le \beta_c e^{\alpha_c(t-\sigma(t_k))}\ \beta_d(1 + \mu(t_k)\alpha_d)\ \beta_c e^{\alpha_c(t_k-\sigma(t_{k-1}))} \\ &\quad \ldots \beta_d(1 + \mu(t_1)\alpha_d)\ \beta_c e^{\alpha_c t_1}\ \|x_0\| \\ &\le \beta_c^{k+1}\ \beta_d^k e^{\alpha_c(t-\sum_{i=1}^k \mu(t_i))}\prod_{i=1}^{k}(1 + \mu(t_i)\alpha_d) \\ &\qquad \times \|x_0\| \\ &\le \beta^{2k+1}e^{\alpha_c(t-\sum_{i=1}^k \mu(t_i))}(\max_{1\le i\le k}(1 + \mu(t_i)\alpha_d))^k \\ &\qquad \times \|x_0\| \\ &\le \beta e^{\alpha_c(t-\sum_{i=1}^k \mu(t_i))} \\ &\qquad \times e^{k[\log(\max_{1\le i\le k}(1+\mu(t_i)\alpha_d))+\log(\beta^2)]}\ \|x_0\| \end{aligned} \tag{9}$$

with $\beta = \max\{\beta_c, \beta_d\}$.

Suppose that condition (7) is satisfied, i.e.

$$\log(\max_{1\leq i\leq k}(1+\mu(t_i)\alpha_d))+\log(\beta^2)<0$$

Assumption (ii) yields

$$k \geq \frac{\sum_{i=1}^{k}\mu(t_i)}{\mu_{\max}}$$

Then, the upper bound of solution (5) becomes

$$\|x(t)\| \leq \beta \ e^{\lambda t} \ \|x_0\|$$

with $\lambda = \max\{\alpha_c, \frac{\log(\max_{1\leq i\leq k}(1+\mu(t_i)\alpha_d))+\log(\beta^2)}{\mu_{\max}}\} < 0$. In this case, system (4) is exponentially stable.

Let us now consider that condition (7) of Assumption (iii) is not satisfied. Hence, one has

$$\log(\max_{1\leq i\leq k}(1+\mu(t_i)\alpha_d))+\log(\beta^2)>0,$$

one can derive, for $t\in[\sigma(t_k), t_{k+1}]$,

$$k \leq \frac{t-\sum_{i=1}^{k}\mu(t_i)}{\min_{1\leq i\leq k}(t_i-\sigma(t_{i-1}))}$$

Then, the upper bound of solution (5) becomes

$$\|x(t)\| \leq \beta \ e^{(t-\sum_{i=1}^{k}\mu(t_i))\left(\alpha_c+\frac{\log(\max_{1\leq i\leq k}(1+\mu(t_i)\alpha_d))+\log(\beta^2)}{\min_{1\leq i\leq k}(t_i-\sigma(t_{i-1}))}\right)} \ \|x_0\| \quad (10)$$

Suppose that condition (8) is satisfied. Therefore, one can obtain

$$\log(\max_{1\leq i\leq k}(1+\mu(t_i)\alpha_d))+\log(\beta^2) < -\alpha_c \ \min_{1\leq i\leq k}(t_i-\sigma(t_{i-1}))$$

It means that

$$\alpha_c + \frac{\log(\max_{1\leq i\leq k}(1+\mu(t_i)\alpha_d))+\log(\beta^2)}{\min_{1\leq i\leq k}(t_i-\sigma(t_{i-1}))} < 0 \quad (11)$$

From Eqs. (10) , (11), the general solution of (4) given by (5) converges exponentially to zero.

■

4. NUMERICAL EXAMPLES

Let us consider the following example using the time scale $\mathbb{T}=\bigcup_{k=0}^{\infty}\left[2k+\frac{1.5k}{k+1.25} \ , \ 2(k+1)\right]$

$$x^{\Delta}=\begin{cases}\begin{pmatrix}\frac{-3}{2} & 1\\ 1 & -1\end{pmatrix}x, \ t\in\bigcup_{k=0}^{\infty}\left[2k+\frac{1.5k}{k+1.25}, 2(k+1)\right)\\ \begin{pmatrix}\frac{-1}{2} & \frac{1}{10}\\ 0 & -1\end{pmatrix}x, \ t\in\bigcup_{k=0}^{\infty}\{2(k+1)\}\end{cases} \quad (12)$$

System (12) can be written as (4) with $t_k=2k$, $\sigma(t_k)=2k+\frac{1.5k}{k+1.25}$, $\frac{2}{3}\leq\mu(t_k)=\sigma(t_k)-t_k=\frac{1.5k}{k+1.25}\leq\frac{3}{2}$, and $\frac{2}{3}\leq(t_{k+1}-\sigma(t_k))\leq\frac{3}{2}$, $k\in\mathbb{N}^*$. $\lambda_c^1=-2.2808, \lambda_c^2=-0.2192, \lambda_d^1=-0.5$ and $\lambda_d^2=-1$. Hence, the dynamical system (12) commutes between a stable continuous-time linear subsystem and a stable discrete-time linear subsystem.

Conditions (i)-(iii) are satisfied. Indeed, condition (7) is fulfilled with $\lambda_d^1=\frac{-1}{2}$ and $\beta=1.2198$, such that

$$\begin{aligned}\max_{1\leq i\leq k}(1+\mu(t_i)\alpha_d) &= \max_{1\leq i\leq k}|1+\mu(t_i)\lambda_d^1|\\ &= 0.6667\\ &< \frac{1}{\beta^2}=0.6721.\end{aligned}$$

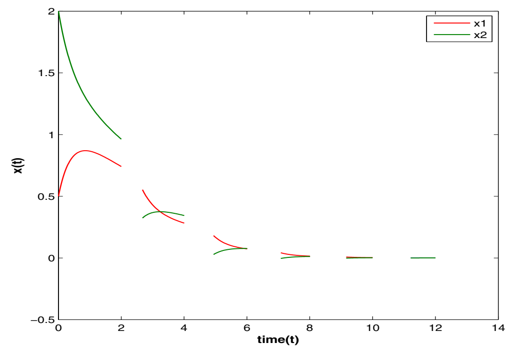

Fig. 1. Trajectories of the switched system (12) with stable subsystems and condition (7) is satisfied. $x_0=[0.5 \ \ 2]^T$

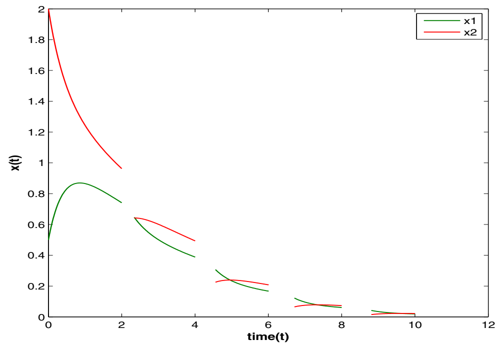

Fig. 2. Trajectories of the switched system (12) with stable subsystems and condition (8) is satisfied. $x_0=[0.5 \ \ 2]^T$

Fig. 1 illustrate the result with initial state $x_0=[0.5 \ \ 2]^T$.

Let us consider the same system (12) using the time scale $\mathbb{T}=\mathbb{P}_{\{\sigma(t_k),t_{k+1}\}}=\bigcup_{k=0}^{\infty}\left[\frac{5}{2}k+\frac{3k}{2k+7}, \ \frac{5}{2}(k+1)\right]$, with $t_k=\frac{5}{2}k$, $\sigma(t_k)=\frac{5}{2}k+\frac{3k}{2k+7}$, $\frac{1}{3}\leq\mu(t_k)=\sigma(t_k)-t_k=\frac{3k}{2k+7}\leq\frac{3}{2}$ and $1\leq(t_i-\sigma(t_{i-1}))\leq\frac{39}{28}$, $k\in\mathbb{N}^*$.
The discrete subsystem is stable on this time scale, but condition (7) is not satisfied whereas condition (8) is verified for $\beta=1.2198$, $\alpha_c=\lambda_c^2=-0.2192$ and $\lambda_d^1=\frac{-1}{2}$, such that

$$\begin{aligned}\max_{1\leq i\leq k,1\leq j\leq n}|1+\mu(t_i)\lambda_d^j| &= 0.8333\\ &< e^{[-\alpha_c \ \min_{1\leq i\leq k}(t_i-\sigma(t_{i-1}))-\log(\beta^2)]}\\ &= e^{[0.2192-\log((1.2198)^2)]}\\ &= 0.8368.\end{aligned}$$

Hence the exponential stability of the solution holds. The result is illustrated in Fig. 2 with initial state $x_0=[0.5 \ \ 2]^T$.

Let us now consider that the dynamical system (12) commutes between a stable continuous-time and an instable discrete-time subsystems on the same time scale $\mathbb{T}=\bigcup_{k=0}^{\infty}\left[2k+\frac{1.5k}{k+1.25} \ , \ 2(k+1)\right]$ with $A_c=\begin{pmatrix}-6 & 4\\ 4 & -4\end{pmatrix}$

and $A_d = \begin{pmatrix} \frac{1}{6} & \frac{-1}{30} \\ 0 & \frac{1}{3} \end{pmatrix}$.

$\lambda_c^1 = -9.1232$, $\lambda_c^2 = -0.8768$, $\lambda_d^1 = \frac{1}{6}$ and $\lambda_d^2 = \frac{1}{3}$. Conditions (i)-(iii) are satisfied. Indeed, condition (7) cannot be fulfilled. Instead, inequality (8) is verified with $\alpha_d = \lambda_d^2 = \frac{1}{3}$, $\alpha_c = \lambda_c^2 = -0.877$ and $\beta = 1.2198$, such that

$$\begin{aligned}\max_{1\le i\le k} |1 + \mu(t_i)\lambda_d^2| &= 0.8333 \\ &< e^{[-\alpha_c \ \min_{1\le i\le k}(t_i-\sigma(t_{i-1})) \ -\log(\beta^2)]} \\ &= e^{[0.2192-\log((1.2198)^2)]} \\ &= 0.8368.\end{aligned}$$

The corresponding trajectories are given in Figs. 3, where the initial state is $x_0 = [0.5 \quad 2]^T$. One can see the exponential stability of the switched system on time scale $\mathbb{T}$.

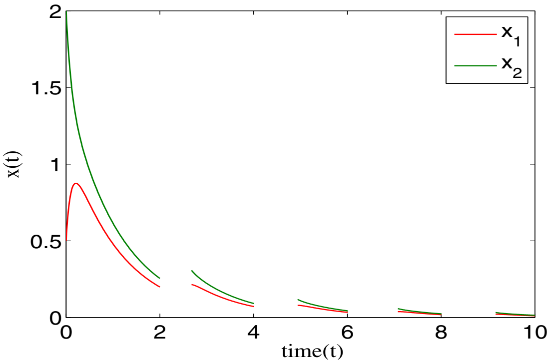

Fig. 3. Trajectories of the switched system (12) with stable continuous-time and instable discrete-time subsystem.

5. CONCLUSION

In this paper, time scale theory is introduced to study the stability of a special class of switched linear systems which evolve on non-uniform time domains. The considered dynamical system commutes between a continuous-time linear subsystem which is supposed stable and a discrete-time linear subsystem during a certain period of time and can be stable or instable. Without assuming that matrices are pairwise commuting, some conditions are derived to guarantee the exponential stability of the switched system by using the general solution of the switched system and the properties of the generalized exponential function. The effectiveness of the proposed scheme is illustrated in numerical examples.

6. ACKNOWLEDGMENTS

This work has been supported by International Campus on Safety and Intermodality in Transportation, the European Community, the Regional Delegation for research and Technology, the Nord-Pas-de-Calais Region, the Ministry of Higher Education and Research, the National Center for Scientific Research under the UVHC BI-CFNes and PHC NUSANTARA project and in part by the National Science Foundation under the Grant ECCS-1711432, and in part by the Engineering Research Center Program of the National Science Foundation and the Department of Energy under NSF Award Number EEC-1041877.

REFERENCES

F. Amato, G. De Tommasi, and A. Pironti. Necessary and sufficient conditions for finite-time stability of impulsive dynamical linear systems. *Automatica*, 49:2546–2550, 2013.

Z. Bartosiewicz and E. Piotrowska. On stabilisability of nonlinear systems on time scales. *Int. J. Control*, 86: 139–145, 2013.

M. Bohner and A. Peterson. *Dynamic Equations on Time Scales, An Introduction with Applications.* Birkhauser, Boston, 2001.

M. Bohner and A. Peterson. *Advances in Dynamic Equations on Time Scales.* Birkhauser, Boston, 2003.

J. M. Davis D. R. Poulsen and I. A. Gravagne. Stochastic time scales: quadratic lyapunov functions and probabilistic regions of stability. In *IEEE Southeastern Symposium on System Theory*, 2013.

J. M. Davis D. R. Poulsen and I. A. Gravagne. Optimal control on stochastic time scales. In *IFAC World Congress, Toulouse*, 2017.

J. DaCuhna. Stability for time varying linear dynamic systems on time scales. *J. Comput. Appl. Math.*, 176: 381–410, 2005.

J. M. Davis, I.A. Gravagne, and A.A. Ramos. Stability of switched linear systems on non-uniform time domains. In *IEEE Southeastern Symp. Systems Theory, Texas*, 2010.

T. Doan, A. Kalauch, S. Siegmund, and F. Wirth. Stability radii for positive linear time-invariant systems on time scales. *Syst. Control Lett.*, 59:173–179, 2010.

N. H. Du, D.D. Thuan, and N.C. Liem. Stability radius of implicit dynamic equations with constant coefficients on time scales. *Syst. Control Lett.*, 60:596–603, 2011.

G. Eisenbarth, J. Davis, and I. Gravagne. Stability of simultaneously triangularizable switched systems on hybrid domains. *Electron. J. Differ. Eq.*, 63:1–19, 2014.

T. Gard and J. Hoffacker. Asymptotic behavior of natural growth on time scales. *Dynamic Systems and Applications*, 12:131–147, 2003.

J. Geromel and P. Colaneri. Stability and stabilization of continuous-time switched linear systems. *SIAM J. Control Optim.*, 45:1915–1930, 2006a.

J. Geromel and P. Colaneri. Stability and stabilization of discrete-time switched systems. *Int. J. Control*, 79: 719–728, 2006b.

I. A. Gravagne, E. Johns, and J. M. Davisi. Switched linear systems on time scales with relaxed commutativity constraints. In *IEEE Southeastern Symposium on Systems Theory, Auburn University*, 2011.

G. Guo, L. Ding, and Q. Han. A distributed event-triggered transmission strategy for sampled-data consensus of multi-agent systems. *Automatica*, 50:1489–1496, 2014.

S. Hilger. Analysis on measure chains- aunified approach to continuous and discret calculus. *Results in Mathematics*, 18:18–56, 1990.

D. Liberzon. *Switching in systems and control.* Birkhauser, Boston, 2003.

D. Liberzon and A. Morse. Basic problems in stability and design of switched systems. *IEEE Contr. Syst. Mag.*, 19: 59–70, 1999.

H. Lin and P. Antsaklis. Stability and stabilizability of switched linear systems: A survey of recent results. *IEEE T. Automat. Contr.*, 54:308–332, 2009.

A. Peterson and Y. Raffoul. Exponential stability of dynamic equations on time scales. *Adv. Differ. Equ-Ny*, 2:133–144, 2005.

C. Potzsche, S. Siegmund, and F. Wirth. A spectral characterization of exponential stability for linear time-invariant systems on time scales. *Discrete Contin. Dyn. Syst.*, 9:1223–1241, 2003.

Z. Sun and S. Ge. Analysis and synthesis of switched linear control systems. *Automatica*, 41:181–195, 2005.

F. Z. Taousser, M. Defoort, and M. Djemai. Stability analysis of a class of switched linear systems on non-uniform time domains. *Syst. Control Lett.*, 74:24–31, 2014.

F. Z. Taousser, M. Defoort, and M. Djemai. Stability analysis of a class of uncertain switched systems on time scale using lyapunov functions. *Nonlinear Anal.-Hybri*, 16:13–23, 2015a.

F. Z. Taousser, M. Defoort, and M. Djemai. Region of exponential stability of switched linear systems on time scales. In 5*th IFAC Conference on Analysis and Design of Hybrid Systems, Atlanta University, USA*, 2015b.

F. Z. Taousser, M. Defoort, and M. Djemai. Consensus for linear multi-agent system with intermittent information transmissions using the time-scale theory. *International Journal of Control*, 89:210–220, 2016.

F. Z. Taousser, M. Defoort, and M. Djemai. Stability analysis of switched linear systems on non-uniform time domains with unstable subsystems. In *IFAC World Congress, Toulouse*, 2017.

G. Zhai, X. Xu, H. Lin, and A. Michel. Analysis and design of switched normal systems. *Nonlinear analysis*, 65:2248–2259, 2006.

W. Zhang and L. Yu. Stability analysis for discrete-time switched time-delay systems. *Automatica*, 45:2265–2271, 2009.