Razumikhin-type Theorems on Practical Stability of Dynamic Equations on Time Scales

Bacem Ben Nasser * **Michael Defoort** ** **Mohamed Djemai** **
Taous-Meriem Laleg-Kirati *

* *Computer, Electrical and Mathematical Sciences and Engineering Division (CEMSE), King Abdullah University of Science and Technology (KAUST), Thuwal 23955-6900, Kingdom of Saudi Arabia (e-mail:bacem.bennasser@kaust.edu.sa, taousmeriem.laleg@kaust.edu.sa)*
** *LAMIH, CNRS, UMR-8201, UVHC, F-59313 Valenciennes, France (e-mail: michael.defoort@univ-valenciennes.fr, mohamed.djemai@univ-valenciennes.fr)*

Abstract: In this work, we investigate some Razumikhin-type criteria for the uniform global practical asymptotic stability on arbitrary time domains, for time-varying dynamic equations. Using Lyapunov-type functions on time scales, we develop appropriate inequalities ensuring that trajectories decay to the neighborhood of the trivial solution asymptotically. Some numerical examples are discussed to illustrate our results.

Keywords: Dynamic equations on time scales, Practical stability, Lyapunov-Razumikhin techniques, Non-uniform time domains.

1. INTRODUCTION

Recently, there has been a big interest in the analysis of dynamic equations on time scales (Hilger, 1990; Bohner and Peterson, 2001, 2003) which unifies the traditional and well-covered theories of continuous and discrete systems respectively. The use of Lyapunov-type functions plays a fundamental role in investigating the stability properties of dynamic systems. Our focus in this paper is to analyze the well-know Lyapunov-Razumikhin method in the time scale framework. This method has been previously considered by (Peterson and Tisdell, 2004; Peterson and Raffoul, 2005; Liu, 2006; Akin-Bohner et al., 2010) to study boundedness, uniqueness of solutions and exponential stability of dynamic equations on arbitrary time domains.

The main description of the mentioned approach is that the candidate Lyapunov function must be decrescent and positive definite. As well, from its first time derivative type condition, along trajectories, such function decreases over time. Hereby, the important role of the comparison functions, i. e., $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{KL}$ function classes (introduced in (Massera, 1956; Hahn, 1967; Sontag, 1989)), arises. The Lyapunov-Razumikhin method uses the comparison functions and a great effort was made to extend these concepts to the time scale field in order to study some stability properties as seen in (Kaymakçalan, 1993; Lakshmikantham and Sivasundaram, 1998; Hoffacker and Tisdell, 2005; Bohner and Martynyuk, 2007; Messina et al., 2015; Yakar and Oğur, 2015; Martynyuk, 2016; Ogulenko, 2017).

Further, an interesting objective of the use of Lyapunov theory is to prove global stability of a dynamic system. Nevertheless, in some cases it is impossible that all states of the considered system reach the equilibrium state. In those cases, the trajectories may be confined in a ball around the equilibrium point. A basic concept which matches to such trajectory aspect is the practical stability. In this context, the practical stability of continuous and discrete dynamics has been widely studied and several stability criteria based on Lyapunov functions and Razumikhin techniques have been obtained as in (Raffoul, 2003, 2004, 2007; Chaillet and Loría, 2008; Song et al., 2008; Wangrat and Niamsup, 2016; Wei and Lin, 2016; Mironchenko, 2017).

It should be noted that the Lyapunov-Razumikhin process can be applied for studying the practical stability of dynamic systems evolving on arbitrary time domains, which is theoretically challenging and of fundamental importance to some applications (Wang and Wu, 2007; Chernetskaya et al., 2013; Wang and Sun, 2014; Ben Nasser et al., 2016). The main contribution of the paper is to study the uniform global practical asymptotic stability of dynamic equations on arbitrary time scales. We will derive some sufficient conditions for uniform global practical asymptotic stability in terms of Lyapunov-type functions based on Razumikhin techniques. Analytically, we will derive interesting estimates of trajectories using the weak triangular inequality given in (Jiang et al., 1994). Also, we will introduce some numerical results illustrating the feasibility of the proposed approach.

The organization of the paper is as follows. Some preliminaries highlighting the time scale theory are stated in Section 2. The problem formulation and stability definitions are given in Section 3. The main result, that is the sufficient criteria for practical stability of dynamic systems on time scales, are formulated and proved in Section 4. In addition, some numerical examples show the effectiveness of the presented approach in Section 5. At the end, concluding remarks are given.

2. NOTATIONS AND PRELIMINARY FACTS

Throughout this work, we use rather standard notation. Denote real numbers by $\mathbb{R}$, natural numbers by $\mathbb{N}$ and integers by $\mathbb{Z}$. Also, $\mathbb{R}_+$ (resp. $\mathbb{R}_-$) indicates the set of non-negative (resp. negative) real numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $n \in \mathbb{N}$, for a vector $\mathcal{U} = (u_1, \ldots, u_n) \in \mathbb{R}^n$, $\|.\|$ stands its Euclidean norm, i. e., $\|\mathcal{U}\| = \left(\sum_{i=1}^{n} |u_i|^2\right)^{\frac{1}{2}}$. We will denote identity function by id, i. e., $\mathrm{id}(r) = r$ for all $r \in \mathbb{R}$.

For convenience, we include hereafter main characteristics and some basic results of time scale theory (Bohner and Peterson, 2001, 2003). We mean by a time scale, denoted by symbol $\mathbb{T}$, every arbitrary nonempty closed subset of $\mathbb{R}$ with the ordering induced from $\mathbb{R}$. If $a \in \mathbb{R}$, then $\mathbb{T}_a^+$ designates the trace of $\mathbb{T}$ on the ordinary interval $[a, +\infty[$. Similarly, closed and open intervals are defined. This theory includes the most familiar scales such as: the continuous time scale $\mathbb{R}$, closed intervals in $\mathbb{R}$, the discrete sets $h\mathbb{Z}$ for $h \in \mathbb{R}$, the so-called quantum calculus $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$, the space of harmonic numbers $\left\{H_m; m \in \mathbb{N}_0, H_0 = 0,\ H_m = \sum_{k=0}^{m} \frac{1}{k}\right\}$, the hybrid scales as the finite union of closed subintervals of $\mathbb{R}$ and the Cantor set. Topologically, closed subset elements of $\mathbb{R}$ may or may not be connected. The link is through two operators characterizing the theory as follows. For $t \in \mathbb{T}$, the forward jump operator σ (resp. the backward jump operator ρ) is defined by $\sigma(t) := \inf\{s \in \mathbb{T}; s > t\}$ (resp. $\rho(t) := \sup\{s \in \mathbb{T}; s < t\}$). Those operators ensure the classification of time scale elements: If $t < \sup(\mathbb{T})$ and $t = \sigma(t)$ then, t is called right dense point. While, t is called a left dense element when $t > \inf(\mathbb{T})$ and $\rho(t) = t$. Moreover, the distance between the time scale elements is measured by the granulation operator μ given by $\mu(t) := \sigma(t) - t$. Also, we define the set of all non-degenerate points of $\mathbb{T}$ by

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T}\backslash]\rho(\sup(\mathbb{T})), \sup(\mathbb{T})] & \text{if } \sup(\mathbb{T}) < \infty \\ \mathbb{T} & \text{otherwise.} \end{cases}$$

The following definition characterizes the differentiability on time scales.

Definition 1. (Hilger, 1990)
Let $h : \mathbb{T} \to \mathbb{R}$ be a function and $t \in \mathbb{T}^\kappa$. We define $h^\Delta(t)$ to be the real number, provided it exists, with the property that for all $\varepsilon > 0$, there exists a neighborhood U_t of t, i. e., $U_t =]t - \delta, t + \delta[_\mathbb{T}$ for some $\delta > 0$, such that

$$|[h(\sigma(t)) - h(s)] - h^\Delta(t)[\sigma(t) - s]| \le \varepsilon|\sigma(t) - s|,\ \forall s \in U_t.$$

$h^\Delta(t)$ is called the delta-derivative of h at t. We say that $h : \mathbb{T} \to \mathbb{R}$ is delta-differentiable on $\mathbb{T}^\kappa$, if $h^\Delta(t)$ exists for every $t \in \mathbb{T}^\kappa$. A function $h : \mathbb{T} \to \mathbb{R}$ is called rd-continuous, if it is continuous in right-dense elements and if the left-sided limits exist in left-dense points. $\mathcal{C}_{rd}(\mathbb{T}, \mathbb{R})$ denotes the set of rd-continuous functions. A function $H : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $h : \mathbb{T} \to \mathbb{R}$ provided $H^\Delta = h$ on $\mathbb{T}^\kappa$. In this case, we define the Cauchy integral by $\int_s^t h(\tau)\Delta\tau = H(t) - H(s)$ for all $s, t \in \mathbb{T}$. Moreover, every rd-continuous function $h : \mathbb{T} \to \mathbb{R}$ has an antiderivative. In particular, if $t_0 \in \mathbb{T}$, then H defined by $H(t) = \int_{t_0}^t h(\tau)\Delta\tau$, for $t \in \mathbb{T}$, is an antiderivative of h. Now, if $a \in \mathbb{T}$, $\sup(\mathbb{T}) = +\infty$ and h is rd-continuous on $\mathbb{T}_a^+$, then we define the improper integral by $\int_a^{+\infty} h(t)\Delta t = \lim_{b \to +\infty} \int_a^b h(t)\Delta t$ provided this limit exists, and we say that the improper integral converges. In contrary, the improper integral diverges. For more details about the delta integral, one can refer to (Bohner and Peterson, 2003, Section 1.4). One can note, there exist more general definitions for time scale integrals including notions of Riemann or Lebesgue delta integrability (Bohner and Peterson, 2001, Chapter 5).

A function $h : \mathbb{T} \to \mathbb{R}$ is called regressive if $1 + \mu h \neq 0$ on $\mathbb{T}^\kappa$. $\mathcal{R}(\mathbb{T}, \mathbb{R})$ denotes the set of regressive and rd-continuous functions and $\mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \left\{h \in \mathcal{R} : (1 + \mu h)_{|\mathbb{T}^\kappa} > 0\right\}$ is the set of positively regressive elements of $\mathcal{C}_{rd}(\mathbb{T}, \mathbb{R})$. The set $\mathcal{R}(\mathbb{T}, \mathbb{R})$ (resp. $\mathcal{R}^+(\mathbb{T}, \mathbb{R})$) together with the circle addition $\overset{\mu}{\oplus}$ defined by $p \overset{\mu}{\oplus} q = p + q + \mu pq$ is an Abelian group. For $p \in \mathcal{R}$, the inverse element is given by $\overset{\mu}{\ominus} p = \frac{-p}{1 + \mu p}$. If the circle substraction is defined as $\overset{\mu}{\ominus}\left(p \overset{\mu}{\ominus} q\right) = p \overset{\mu}{\oplus}\overset{\mu}{\ominus} q$, then $p \overset{\mu}{\ominus} q = \frac{p - q}{1 + \mu q}$.

The time scale exponential function is introduced as follows.

Definition 2. (Hilger, 1990)
Let $p \in \mathcal{R}(\mathbb{T}, \mathbb{R})$. The Cauchy problem

$$\begin{cases} u^\Delta(t) = p(t)u(t) \\ u(t_0) = 1 \end{cases}$$

has a unique solution, the so-called exponential function, denoted by $e_p(t, t_0)$.

Let $t, s, r \in \mathbb{T}$ and $p, q \in \mathcal{R}(\mathbb{T}, \mathbb{R})$. Those useful relationships hold $e_p(t, s)e_p(s, r) = e_p(t, r)$, $\frac{e_p(t, s)}{e_q(t, s)} = e_{p \overset{\mu}{\ominus} q}(t, s)$, $e_p(\sigma(t), s) = \big(1 + \mu(t)p(t)\big)e_p(t, s)$, $e_p(s, t) = e_p(t, s)^{-1} = e_{\overset{\mu}{\ominus} p}(t, s)$ and $[e_p(r, .)]^\Delta = -pe_p(r, \sigma(.))$. Moreover, if $p \in \mathcal{R}^+(\mathbb{T}, \mathbb{R})$, then $e_p(t, s) > 0$. More if $p \le q$, then $e_p(t, s) \le e_q(t, s)$.
For more details on the time scale exponential function properties, one can refer to (Bohner and Peterson, 2003, Chapter 2).

3. PROBLEM FORMULATION AND PRACTICAL STABILITY DEFINITION

In this paper, the problem of uniform global practical asymptotic stability is addressed to dynamic equations on time scale $\mathbb{T}$ of the form

$$\begin{cases} x^\Delta(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases} \tag{1}$$

where $t \in \mathbb{T}^+_{t_0}$, $t_0 \in \mathbb{T}^+_a$, $\sup(\mathbb{T}) = +\infty$, $a \in \mathbb{T}$, $x_0, x \in \mathbb{R}^n$ and $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is a rd-continuous vector function. Denote by $x(t) := x(t, t_0, x_0)$ an arbitrary solution of system (1) initiated at (t_0, x_0). It is assumed that the conditions for the existence of a unique solution of system (1) on $\mathbb{T}^+_{t_0}$ are satisfied for all initial states x_0. Hereafter, we introduce for system (1) some basic definitions of comparison functions and stability and a technical lemma.

Comparison functions is an important and useful tool for stability analysis. Over the last decades, some significant results are developed highlighting the computational efficiency of this process in studying time scale stability properties, as in (Kaymakçalan, 1993; Lakshmikantham and Sivasundaram, 1998; Hoffacker and Tisdell, 2005; Wang and Wu, 2007; Bohner and Martynyuk, 2007; Messina et al., 2015; Yakar and Oğur, 2015; Ogulenko, 2017). Hahn has first defined functions of class-$\mathcal{K}$ in (Hahn, 1967) after their use in stability studies by Massera (Massera, 1956). A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class-$\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is continuous, zero at zero and strictly increasing. To englobe radially unbounded functions, Sontag (Sontag, 1989) introduced, class-$\mathcal{K}_\infty$ functions, a characterization if α is also unbounded. So, a function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class-$\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if $\alpha \in \mathcal{K}$, and in addition, $\lim_{r \to +\infty} \alpha(r) = +\infty$.

To characterize attractivity of zero-solutions, we will introduce a time scale version for class-$\mathcal{L}$ as given in (Hahn, 1967).

Definition 3.
A function $\psi : \mathbb{T} \to \mathbb{R}_+$ is said to belong to class $\mathcal{L}_\mathbb{T}$ ($\alpha \in \mathcal{L}_\mathbb{T}$) if it is rd-continuous, strictly decreasing and $\lim_{t \to +\infty} \psi(t) = 0$.

For a constant $\delta \in \mathbb{R}_- \cap \mathcal{R}^+$, $t \mapsto e_\delta(t, a)$, $t \in \mathbb{T}^+_a$, is of class-$\mathcal{L}_\mathbb{T}$. After, we state a time scale version of class-$\mathcal{KL}$ functions.

Definition 4.
An rd-continuous function $\beta : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R}_+$ is said to belong to class-$\mathcal{KL}_\mathbb{T}$ ($\beta \in \mathcal{KL}_\mathbb{T}$) if it is of class-$\mathcal{K}$ in its first argument and of class-$\mathcal{L}_\mathbb{T}$ in its second argument.

In other words, it means that, for each fixed s, the mapping $\beta(., s) \in \mathcal{K}$ and, for each fixed r, the mapping $\beta(r, .) \in \mathcal{L}_\mathbb{T}$. One can see that the last characterization is similar and more general to that introduced by Hahn (Hahn, 1967). As an example, for a constant $\delta \in \mathbb{R}_- \cap \mathcal{R}^+$, $(w, t) \mapsto w e_\delta(t, a)$, $w \geq 0$, $t \in \mathbb{T}^+_a$, is of class-$\mathcal{KL}_\mathbb{T}$.

Now, practical stability is characterized here using class-$\mathcal{KL}_\mathbb{T}$ functions.

Definition 5.
The dynamic system (1) is said to be uniformly globally practically asymptotically stable if and only if there exist $\beta \in \mathcal{KL}_\mathbb{T}$ and a nonnegative constant r such that

$$\|x(t)\| \leq r + \beta\big(\|x(t_0)\|, t\big) \qquad \text{for all } (t_0, t) \in \mathbb{T}^+_a \times \mathbb{T}^+_{t_0} \quad (2)$$

Furthermore, if $r = 0$, then (1) is uniformly globally asymptotically stable.

The following auxiliary lemma gives the weak triangular inequality, which will be useful in relaxing estimation for trajectories of system (1) while developing practical stability criteria.

Lemma 1. (Jiang et al., 1994)
For any functions $\alpha \in \mathcal{K}$ and $\rho \in \mathcal{K}_\infty$ such that $\rho - \mathrm{id} \in \mathcal{K}_\infty$, and any nonnegative real numbers a and b we have

$$\alpha(a + b) \leq \alpha \circ \rho(a) + \alpha \circ \rho \circ (\rho - \mathrm{id})^{-1}(b) \qquad (3)$$

4. RAZUMIKHIN THEOREMS

In this section, using Lyapunov-Razumikhin method, we will develop some results which provide sufficient criteria for uniform global practical asymptotic stability of system (1) using the so-called Lyapunov-type functions on time scales as studied in (Liu, 2006; Peterson and Raffoul, 2005; Peterson and Tisdell, 2004). They are the mapping $V : \mathbb{T}^+_a \times \mathbb{R}^n \to \mathbb{R}_+$, verifying $V(t, 0) = 0$ for all $t \in \mathbb{T}^+_a$, delta-differentiable in its first variable and continuously differentiable in variable x, $x(t)$ denotes a solution of (1). At first, the study is done on an arbitrary time domain.

Theorem 1.
Assume there exists a Lyapunov-type function $V : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}_+$ satisfying the following conditions

($\mathcal{F}_1$) There exist a nondecreasing rd-continuous positive function λ_1, positive bounded rd-continuous function λ_2 with bound $\bar{\lambda}_2$ and class-$\mathcal{K}$ functions α_1, α_2 such that
$$\lambda_1(t)\alpha_1(\|x\|) \leq V(t, x) \leq \lambda_2(t)\alpha_2(\|x\|).$$

($\mathcal{F}_2$) There exist a positive rd-continuous function λ_3 and a class-$\mathcal{K}$ function α_3 such that the derivative of V with respect to (1) verifies
$$V^\Delta(t, x) \leq \frac{-\lambda_3(t)\alpha_3(\|x\|)}{1 + \lambda_3(t)\mu(t)}.$$

($\mathcal{F}_3$) $MV(t, x) - \lambda_3(t)\alpha_3 \circ \alpha_2^{-1}\Big(\frac{V(t, x)}{\lambda_2(t)}\Big) \leq \gamma$, for a constant $\gamma \geq 0$, with $M = \sup_{t \in \mathbb{T}^+_a} \lambda_3(t)$.

Then, the dynamical system (1) is uniformly globally practically asymptotically stable.

Proof.
Let $\varepsilon > 1$ and define $Q_1(t, x) := V(t, x)e_M(t, t_0)$. Then,

$$\begin{aligned} Q_1^\Delta(t, x) &= [Q_1(t, x(t))]^\Delta \\ Q_1^\Delta(t, x) &= MV(t, x)e_M(t, t_0) + V^\Delta(t, x)e_M(\sigma(t), t_0) \\ &= \Big(MV(t, x) + V^\Delta(t, x)\big(1 + M\mu(t)\big)\Big)e_M(t, t_0) \\ &\overset{(\mathcal{F}_2)}{\leq} \big(MV(t, x) - \lambda_3(t)\alpha_3(\|x\|)\big)e_M(t, t_0) \\ &\overset{(\mathcal{F}_1)}{\leq} \bigg(MV(t, x) - \lambda_3(t)\alpha_3 \circ \alpha_2^{-1}\Big(\frac{V(t, x)}{\lambda_2(t)}\Big)\bigg) \\ &\quad e_M(t, t_0) \\ &\overset{(\mathcal{F}_3)}{\leq} \gamma e_M(t, t_0). \end{aligned}$$

A simple integration of the last inequality gives
$$V(t, x(t))e_M(t, t_0) \leq V(t_0, x(t_0)) + \frac{\gamma}{M}\big(e_M(t, t_0) - 1\big).$$

As $M > 0$, $e_M(t, t_0)$ is well-defined and positive and one can see that
$$V(t, x(t)) \leq V(t_0, x_0)e_{\ominus M}{}_{\mu}(t, t_0) + \frac{\gamma}{M}.$$

By manipulating the upper and the lower bound in ($\mathcal{F}_1$), we get,

$$\|x(t)\| \overset{(\mathcal{F}_1)}{\leq} \alpha_1^{-1}\left(\frac{\bar{\lambda}_2}{\lambda_1(a)}\alpha_2(\|x_0\|)e_{\underset{\ominus M}{\mu}}(t,t_0) + \frac{\gamma}{M\lambda_1(a)}\right) \tag{4}$$

Applying the weak triangular inequality (3) to (4) with $\rho(r) = \varepsilon.r$,
i. e., $\rho \circ (\rho - \mathrm{id})^{-1} = \frac{\varepsilon}{\varepsilon - 1}\,\mathrm{id}$, one can obtain

$$\|x(t)\| \leq \underbrace{\alpha_1^{-1}\left(\varepsilon\frac{\bar{\lambda}_2}{\lambda_1(a)}\alpha_2(\|x_0\|)e_{\underset{\ominus M}{\mu}}(t,t_0)\right)}_{:=\beta_1(\|x_0\|,t)} + \underbrace{\alpha_1^{-1}\left(\frac{\varepsilon\gamma}{(\varepsilon-1)M\lambda_1(a)}\right)}_{:=r_1}$$

for all $t \geqslant t_0$ and $x_0 \in \mathbb{R}^n$, proving uniform global practical asymptotic stability for solutions of system (1).

Hereafter, the study will be reduced over a time scale $\mathbb{T}$ with bounded graininess, i. e., $\mu(t) \leq \mu_{\mathbb{T}} < +\infty$. We will relax hypothesis $(\mathcal{F}_2)$ by adding a term in the delta-derivative Lyapunov-type function bound.

Theorem 2.
Suppose that there exists a Lyapunov-type function $V : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}_+$ such that the following statements hold.

$(\mathcal{H}_1)$ There exist a nondecreasing rd-continuous positive function λ_1, positive bounded rd-continuous function λ_2 with bound $\bar{\lambda}_2$ and α_1, $\alpha_2 \in \mathcal{K}$ such that
$$\lambda_1(t)\alpha_1(\|x\|) \leq V(t,x) \leq \lambda_2(t)\alpha_2(\|x\|).$$

$(\mathcal{H}_2)$ There exist a rd-continuous positive function λ_3, $\alpha_3 \in \mathcal{K}$, a nonnegative rd-continuous function λ_4 and a positive constant $\delta > m = \inf_{t \in \mathbb{T}_a^+} \lambda_3(t)$ such that
$$V^{\Delta}(t,x) \leq \frac{-\lambda_3(t)\alpha_3(\|x\|) + \lambda_4(t)}{1 + m\mu(t)}$$
$$\text{and} \int_a^{+\infty} \lambda_4(s)e_m(s,a)\Delta s \leq \bar{\lambda}_4.$$

$(\mathcal{H}_3)$ $V(t,x) - \alpha_3 \circ \alpha_2^{-1}\left(\frac{V(t,x)}{\lambda_2(t)}\right) \leq \gamma e_{\underset{\ominus\delta}{\mu}}(t,a)$, for a nonnegative constant γ.

Then, the dynamic system (1) is uniformly globally practically asymptotically stable.

Proof.
Let $\varepsilon > 1$ and denote $Q_2(t,x) := V(t,x)e_m(t,t_0)$. Δ-differentiating Q_2, one can see that

$$\begin{aligned}
Q_2^{\Delta}(t,x) &= mV(t,x)e_m(t,t_0) + V^{\Delta}(t,x)e_m(\sigma(t),t_0) \\
&= \Big(mV(t,x) + (1+m\mu(t))V^{\Delta}(t,x)\Big)e_m(t,t_0) \\
&\overset{(\mathcal{H}_2)}{\leq} \big(mV(t,x) - \lambda_3(t)\alpha_3(\|x\|) + \lambda_4(t)\big)e_m(t,t_0) \\
&\overset{(\mathcal{H}_1)}{\leq} \left(mV(t,x) - \lambda_3(t)\alpha_3 \circ \alpha_2^{-1}\left(\frac{V(t,x)}{\lambda_2(t)}\right)\right) \\
&\quad\; e_m(t,t_0) + \lambda_4(t)e_m(t,t_0) \\
&\overset{(\mathcal{H}_3)}{\leq} m\gamma e_{\underset{m\ominus\delta}{\mu}}(t,a) + \lambda_4(t)e_m(t,a).
\end{aligned}$$

Integrating both sides from t_0 to t, we get

$$V(t,x(t))e_m(t,t_0) \leq V(t_0,x(t_0)) + \int_{t_0}^{t} m\gamma e_{\underset{m\ominus\delta}{\mu}}(s,a) + \lambda_4(s)e_m(s,a)\Delta s.$$

Since $m > 0$, $e_m(t,t_0)$ is well-defined and positive and one can deduce

$$V(t,x(t)) \leq V(t_0,x_0)e_{\underset{\ominus m}{\mu}}(t,t_0) + \left(\frac{\gamma\delta(1+m\mu_{\mathbb{T}})}{\delta - m} + \bar{\lambda}_4\right)e_{\underset{\ominus m}{\mu}}(t,t_0)$$

and

$$\|x(t)\| \overset{(\mathcal{H}_1)}{\leq} \alpha_1^{-1}\left(\frac{\bar{\lambda}_2}{\lambda_1(a)}\alpha_2(\|x_0\|)e_{\underset{\ominus m}{\mu}}(t,t_0) + \frac{\gamma\delta(1+m\mu_{\mathbb{T}})}{\lambda_1(a)(\delta-m)} + \frac{\bar{\lambda}_4}{\lambda_1(a)}\right).$$

Using Lemma 1 with $\alpha = \alpha_1^{-1}$ and $\rho(r) = (1+\varepsilon)r$, one can obtain

$$\|x(t)\| \leq \underbrace{\alpha_1^{-1}\left(\frac{(1+\varepsilon)\bar{\lambda}_2}{\lambda_1(a)}\alpha_2(\|x_0\|)e_{\underset{\ominus m}{\mu}}(t,t_0)\right)}_{:=\beta_2(\|x_0\|,t)} + \underbrace{\alpha_1^{-1}\left(\frac{(1+\varepsilon)\gamma\delta(1+m\mu_{\mathbb{T}})}{\varepsilon\lambda_1(a)(\delta-m)} + \frac{(1+\varepsilon)\bar{\lambda}_4}{\varepsilon\lambda_1(a)}\right)}_{:=r_2}.$$

This completes the proof.

Remark 1.
It should be mentioned that Lyapunov's approach using Razimikhin's techniques is properly addressed in the past for the study of some stability properties on arbitrary time domains as boundedness of solutions, exponential stability and uniform exponential stability in references (Peterson and Tisdell, 2004; Peterson and Raffoul, 2005; Liu, 2006). Our contribution extends this method for studying the uniform global practical asymptotic stability on time scales in a more general context which may include the stability properties already mentioned.

5. NUMERICAL EXAMPLES

To demonstrate the applicability of the stability criteria given in Theorem 1, we first propose a numerical example in the plan tested for a variety of time domains.

Example 1.
Consider the following planar linear system on time scale $\mathbb{T}_0^+$

$$\begin{cases} x_1^{\Delta} = -x_1 - e^{-t}x_2 \\ x_2^{\Delta} = e^{-t}x_1 - x_2 \end{cases} \tag{5}$$

where $x = (x_1, x_2)^{\top} \in \mathbb{R}^2$ denotes the state.

Let us consider the Lyapunov-type function $V(t,x) = x_1^2 + x_2^2$. Hypothesis $(\mathcal{F}_2)$ is satisfied with $\alpha_1(r) = \alpha_2(r) = r^2$ and $\lambda_1(t) = \lambda_2(t) = 1$. Calculating the upper right derivative of V along the solution of (5), it follows

$$\begin{aligned}
V^{\Delta}(t,x) &= -2\big(x_1^2 + x_2^2\big) + \mu(t)\big(x_1^2 + e^{-2t}x_2^2 + e^{-2t}x_1^2 \\
&\quad + x_2^2\big) = -\big(2 - (1+e^{-2t})\mu(t)\big)\|x\|^2 \\
&\overset{\lambda_3(t)=\lambda}{\leq} \frac{-\lambda}{1+\lambda\mu(t)}\|x\|^2.
\end{aligned}$$

If as requested in the last inequality λ_3 is a positive constant λ, $\alpha_3(r) = r^2$, $(\mathcal{F}_3)$ is fulfilled with $\gamma = 0$ and the statement $(\mathcal{F}_2)$ is achieved as long as such constant solves the following relationship

$$\lambda\big[(1+e^{-2t})\mu^2(t)-2\mu(t)+1\big] \leq -(1+e^{-2t})\mu(t)+2, \quad t \geq 0 \tag{6}$$

This will be discussed for certain time domains.

Case 1: If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$ and (6) is satisfied if $\lambda \in]0, 2]$.

Case 2: If $\mathbb{T} = \mathbb{T}_h := h\mathbb{Z}$ with a positive constant $h \leq \frac{1}{2}$, then $\mu(t) = h$. If we can find a constant $\lambda > 0$ such that

$$0 < \lambda\big[(1+e^{-2t})h^2 - 2h+1\big] \leq -(1+e^{-2t})h+2, \quad t \geq 0.$$

then condition (6) holds. Thus, we can choose $\lambda \in]0, \dfrac{2}{3}]$.

Case 3: If $\mathbb{T} = \mathbb{T}_{h,l} := \cup_{k=0}^{\infty}[k(h+l), k(h+l)+l]$, where $l > 0$ and $0 < h \leq \frac{1}{2}$, then $\mu(t) \in \{0, h\}$. Inequality (6) gives

$$\begin{cases} 0 < \lambda\big[(1+e^{-2t})h^2 - 2h + 1\big] \leq -(1+e^{-2t})h + 2 \\ \lambda \leq 2, \quad t \geq 0 \end{cases}$$

So, λ can be chosen in $]0, \dfrac{2}{3}]$.

From the above discussion, we can see that all conditions in Theorem 1 hold and the original state of (5) is uniformly globally asymptotically stable.

The numerical simulation for this example is given in Fig. 1 made for $\mathbb{T}_{0.3}$ and $\mathbb{T}_{\frac{1}{2},1}$ respectively with initial state $x_0 = (-1, 3)^\top$. It shows the exponential decay of the trajectory components over time.

For testing the applicability of Theorem 2, we choose a nonlinear system in the plan.

Example 2.
Let consider on a time scale $\mathbb{T}_0$ with $\mu_{\mathbb{T}} = \frac{3}{4}$ the following system

$$\begin{cases} x_1^{\Delta} = -\dfrac{1}{2}x_1 + \dfrac{1}{2}x_2 + \dfrac{1}{4}\phi \\ x_2^{\Delta} = -\dfrac{1}{2}x_1 - \dfrac{1}{2}x_2 + \dfrac{1}{4}\phi \end{cases} \tag{7}$$

where $x = (x_1, x_2)^\top \in \mathbb{R}^2$ denotes the state, $\phi(t) = ce_{\ominus_{\mu}\delta}(t, 0)$, $c \leq 0$ and $\delta > \frac{1}{5}$.

We choose the Lyapunov-type function as $V(t, x) = x_1^2 + x_2^2$. It is clear that $(\mathcal{H}_1)$ and $(\mathcal{H}_3)$ hold with $\alpha_i(r) = r^2$ and $\lambda_1(t) = \lambda_2(t) = 1$. It is then straightforward to show that

$$V^{\Delta}(t,x) = 2\Big(-\frac{1}{2}\|x\|^2 + \frac{1}{4}\phi(t)\big(x_1+x_2\big)\Big) + \mu(t)\Big(\frac{1}{2}\|x\|^2 - \frac{1}{2}\phi(t)x_1 + \frac{1}{8}\phi(t)^2\Big)$$

We apply Cauchy's inequality to get

$$\begin{aligned} V^{\Delta}(t,x) &\leq 2\Big(-\frac{1}{2}\|x\|^2 + \frac{1}{8}\|x\|^2 + \frac{1}{4}\phi(t)^2\Big) + \mu(t) \\ &\Big(\frac{1}{2}\|x\|^2 + \frac{1}{4}\|x\|^2 + \frac{1}{4}\phi(t)^2 + \frac{1}{8}\phi(t)^2\Big) \\ &\leq -\frac{3}{4}\Big(1-\mu(t)\Big)\|x\|^2 + \frac{1}{2}\Big(1+\frac{3}{4}\mu(t)\Big)\phi(t)^2 \\ &\leq \frac{-\lambda_3(t)\|x\|^2 + \lambda_4(t)}{1+\mu(t)m}. \end{aligned}$$

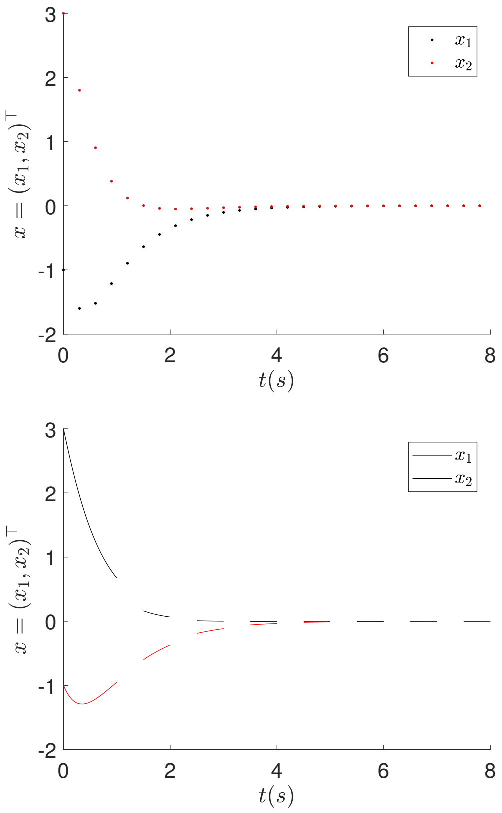

Fig. 1. State trajectories for system (5) with time scales $\mathbb{T}_{0.3}$ and $\mathbb{T}_{\frac{1}{2},1}$ respectively.

For $c = -5$, $(\mathcal{H}_2)$ is validated with $\lambda_3(t) = m = \dfrac{1}{5}$, $\lambda_4(t) = c\phi(t)$. It can be seen that the uniform global practical asymptotic stability of system (7) is achieved according to Theorem 2 as long as the requirements for the relaxing derivative term are satisfied. The numerical simulation of this example is given in Fig. 2 made with $\mu(t) = \frac{3}{4}e^{-\frac{1}{2}t}$, $\delta = \frac{2}{3}$ and initial condition $x_0 = (-2, 4)^\top$. It shows the convergence of the solution in the neighborhood of the origin.

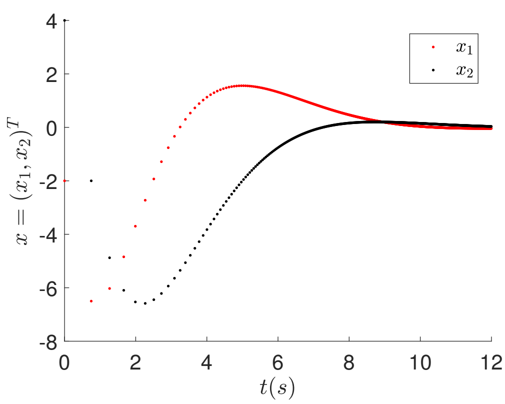

Fig. 2. Dynamics behavior of system (7) with time scale $\mathbb{T}_0^+$ and $x_0 = (-2, 4)^\top$.

6. CONCLUSION

In this paper we have considered the problem of practical stability for dynamic equations on time scales. By the tools of Lyapunov-Rzumikhin method, we have obtained sufficient criteria under which the trivial solution is uniformly globally practically asymptotically stable. The given numerical examples corroborate our analytical findings. One future study is to extend the current Lyapunov-stability approach, while considering the trajectory tracking problem on arbitrary time domains.

ACKNOWLEDGEMENTS

Research reported in this publication has been supported by the King Abdullah University of Science and Technology (KAUST). Also, this work has been supported by the European Community, the Regional Delegation for Research and Technology, the Haut de France Region and the National Center for Scientific Research under the projects ELSAT VUMOPE and the UVHC BI-CFNes.

REFERENCES

Akin-Bohner, E., Raffoul, Y.N., and Tisdell, C. (2010). Exponential stability in functional dynamic equations on time scales. *Commun. Math. Anal.*, 9(1), 93–108.

Ben Nasser, B., Boukerrioua, K., Defoort, M., Djemai, M., and Hammami, M.A. (2016). State feedback stabilization of a class of uncertain nonlinear systems on non-uniform time domains. *Systems Control Lett.*, 97, 18–26.

Bohner, M. and Martynyuk, A.A. (2007). Elements of stability theory of A. M. Liapunov for dynamic equations on time scales. *Nonlinear Dyn. Syst. Theory*, 7(3), 225–251.

Bohner, M. and Peterson, A. (2001). *Dynamic equations on time scales. An introduction with applications.* Birkhäuser Boston, Inc., Boston, MA.

Bohner, M. and Peterson, A. (2003). *Advances in dynamic equations on time scales.* Birkhäuser Boston, Inc., Boston, MA.

Chaillet, A. and Loría, A. (2008). Uniform semiglobal practical asymptotic stability for non-autonomous cascaded systems and applications. *Automatica J. IFAC*, 44(2), 337–347.

Chernetskaya, L.N., Martynyuk, A.A., and Rasshyvalova, S.N. (2013). Practical stability of dynamic equations. *Commun. Appl. Anal.*, 17(3-4), 503–510.

Hahn, W. (1967). *Stability of motion.* Springer-Verlag New York, Inc., New York.

Hilger, S. (1990). Analysis on measure chains–a unified approach to continuous and discrete calculus. *Results Math.*, 18(1-2), 18–56.

Hoffacker, J. and Tisdell, C.C. (2005). Stability and instability for dynamic equations on time scales. *Comput. Math. Appl.*, 49(9-10), 1327–1334.

Jiang, Z.P., Teel, A.R., and Praly, L. (1994). Small-gain theorem for ISS systems and applications. *Math. Control Signals Systems*, 7(2), 95–120.

Kaymakçalan, B. (1993). Stability analysis in terms of two measures for dynamic systems on time scales. *J. Appl. Math. Stochastic Anal.*, 6(4), 325–344.

Lakshmikantham, V. and Sivasundaram, S. (1998). Stability of moving invariant sets and uncertain dynamic systems on time scales. *Comput. Math. Appl.*, 36(10-12), 339–346.

Liu, A.L. (2006). Boundedness and exponential stability of solutions to dynamic equations on time scales. *Electron. J. Differential Equations*, 2006(12), 1–14.

Martynyuk, A.A. (2016). *Stability theory for dynamic equations on time scales.* Systems & Control: Foundations & Applications. Birkhäuser/Springer, [Cham].

Massera, J.L. (1956). Contributions to stability theory. *Ann. of Math. (2)*, 64, 182–206.

Messina, E., Russo, E., and Vecchio, A. (2015). Volterra integral equations on time scales: stability under constant perturbations via Liapunov direct method. *Ric. Mat.*, 64(2), 345–355.

Mironchenko, A. (2017). Uniform weak attractivity and criteria for practical global asymptotic stability. *Systems Control Lett.*, 105, 92–99.

Ogulenko, A. (2017). Asymptotical properties of social network dynamics on time scales. *J. Comput. Appl. Math.*, 319, 413–422.

Peterson, A.C. and Raffoul, Y.N. (2005). Exponential stability of dynamic equations on time scales. *Adv. Difference Equ.*, (2), 133–144.

Peterson, A.C. and Tisdell, C.C. (2004). Boundedness and uniqueness of solutions to dynamic equations on time scales. *J. Difference Equ. Appl.*, 10(13-15), 1295–1306.

Raffoul, Y.N. (2003). Boundedness in nonlinear differential equations. *Nonlinear Stud.*, 10(4), 343–350.

Raffoul, Y.N. (2004). Boundedness in nonlinear functional differential equations with applications to Volterra integrodifferential equations. *J. Integral Equations Appl.*, 16(4), 375–388.

Raffoul, Y.N. (2007). Boundedness and exponential asymptotic stability in dynamical systems with applications to nonlinear differential equations with unbounded terms. *Adv. Dyn. Syst. Appl.*, 2(1), 107–121.

Song, X., Li, S., and Li, A. (2008). Practical stability of nonlinear differential equation with initial time difference. *Appl. Math. Comput.*, 203(1), 157–162.

Sontag, E.D. (1989). Smooth stabilization implies coprime factorization. *IEEE Trans. Automat. Control*, 34(4), 435–443.

Wang, P. and Sun, W. (2014). Practical stability in terms of two measures for set differential equations on time scales. *The Scientific World Journal*, 2014, 1–7.

Wang, P. and Wu, M. (2007). Practical ϕ_0-stability of impulsive dynamic systems on time scales. *Appl. Math. Lett.*, 20(6), 651–658.

Wangrat, S. and Niamsup, P. (2016). Exponentially practical stability of impulsive discrete time system with delay. *Adv. Difference Equ.*, 2016, 1–10.

Wei, Y. and Lin, Z. (2016). Stabilization of exponentially unstable discrete-time linear systems by truncated predictor feedback. *Systems Control Lett.*, 97, 27–35.

Yakar, C. and Oğur, B. (2015). Stability of perturbed dynamic system on time scales with initial time difference. *Turkish J. Math.*, 39(1), 1–15.