The structure of non decomposable connectives of linear logic

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This paper discusses generalized multiplicative linear logic connectives. In particular, it introduces the class of *Girard connectives* which cannot be defined as combinations of the binary connectives ⅋ (*par*) and $\otimes$ (*tensor*). These new connectives are based on partitions of the cyclic permutations of an n elements set into (disjoint) subsets of size u, where $n = uv$ with u and v prime numbers.

1 Introduction

Since its inception Linear logic [4] put itself in the limelight for its ability to formulate innovative questions. Many of these, such as those on *proof-nets*, have found brilliant solutions; other issues, such as those on *generalized* (or *n-ary*) *connectives*, have received only a partially satisfactory solution. General connectives were introduced by Girard [3] but most of the results known after then are essentially due to Danos and Regnier [2]. This paper elaborates on these seminal works and brings some innovations.

Following [2], a general multiplicative connective may be defined by a pair of dual sets of partitions of a same domain, P and Q, dual in the sense that every pair of partitions, p and q s.t. $p \in P$ and $q \in Q$, must be orthogonal ($p \perp q$), where orthogonality is defined by a topological condition: the bipartite graph obtained by linking together classes of each partition sharing at least one element is acyclic and connected.

Traditionally, there are two dual ways, *sequential* and *graphical*, for viewing (pairs of) sets of partitions as defining logical connectives; this is clearly stated at page 196 of [2]:

> *"In the sequential case it seems natural to define a connective by the rules that introduce or eliminate it, that is, to describe the external situation allowing its derivation. [In the graphical case] a connective is rather defined by its internal reaction to the general situation"*.

- *Sequentially*, a partition describes a sequent calculus rule for producing the generalized formula: the domain of the partition is the set of principal formulas of the rule and each class describes one premise of the rule; so, a multiplicative rule for an n-ary connective $C(A_1,...,A_n)$ is completely characterized by the *organization* of its principal formulas $(A_1,...,A_n)$ like in the l.h.s. picture of Fig. 1. Indeed, since multiplicatives rules are *unconditional* about the context, any rule can be simply described by a *partition* over its principal formulas (their indexes), omitting the contexts $\Gamma_1,...,\Gamma_p$, like in the r.h.s. picture of Fig. 1.
- *Graphically*, a partition describes a *switching* of a generalized connective-link, in proof nets style. Given the parse tree of an MLL formula F (a formula only built by the binary multiplicative

$$\frac{\vdash \Gamma_1, A_1,...,A_{i_1} \qquad \dots \qquad \vdash \Gamma_p, A_p,...,A_{i_p}}{\vdash \Gamma_1,...,\Gamma_p, C(A_1,...,A_n)}\, C \qquad\qquad \frac{(1,...,i_1) \qquad \dots \qquad (p,...,i_p)}{C(1,...,n)}\, C$$

Figure 1: generalized sequential rule

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$$\frac{(1,2)\qquad(3,4)}{(1\parr 2)\otimes(3\parr 4)}\,p_1 \qquad \frac{(1,3)\quad(2)\quad(4)}{(1\otimes 2)\parr(3\otimes 4)}\,q_1^1 \quad \frac{(1,4)\quad(2)\quad(3)}{(1\otimes 2)\parr(3\otimes 4)}\,q_1^2 \quad \frac{(2,3)\quad(1)\quad(4)}{(1\otimes 2)\parr(3\otimes 4)}\,q_1^3 \quad \frac{(2,4)\quad(1)\quad(3)}{(1\otimes 2)\parr(3\otimes 4)}\,q_1^4$$

$$\frac{(2,3)\qquad(4,1)}{(2\parr 3)\otimes(4\parr 1)}\,p_2 \qquad \frac{(2,4)\quad(3)\quad(1)}{(2\otimes 3)\parr(4\otimes 1)}\,q_2^1 \quad \frac{(2,1)\quad(3)\quad(4)}{(2\otimes 3)\parr(4\otimes 1)}\,q_2^2 \quad \frac{(3,4)\quad(2)\quad(1)}{(2\otimes 3)\parr(4\otimes 1)}\,q_2^3 \quad \frac{(3,1)\quad(2)\quad(4)}{(2\otimes 3)\parr(4\otimes 1)}\,q_2^4$$

Figure 2: examples of generalized decomposable connectives in sequent calculus syntax

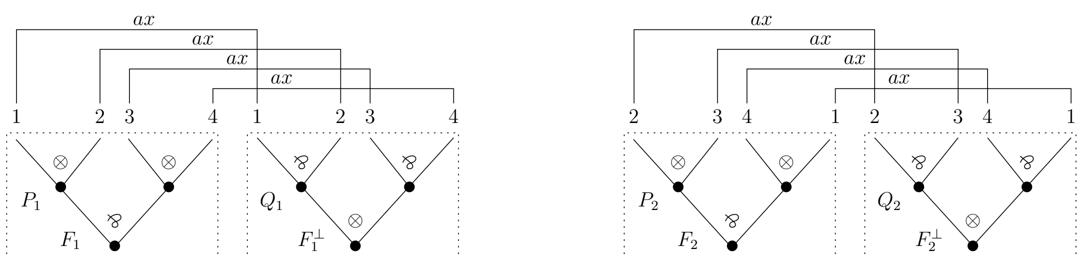

Figure 3: examples of generalized decomposable connectives in proof net syntax

connectives, "par" $\parr$ and "times or tensor" $\otimes$) a *Danos-Regnier switching* for F is the structure obtained after the mutilation of one premise of each $\parr$ link occurring in the tree. This mutilation induces a partition of the top *border* (or *frontier*) of the tree of F: all the border elements belonging to a same connected component build a *class* of the induced partition. The set of partitions induced by all the switchings of F is called the *pre-type* of F, denoted by $\mathscr{P}_F$, while $\mathscr{P}_F^{\perp\perp}$ is called the *type* of F, denoted by $\mathscr{T}_F$ (see [6]).

2 Decomposable connectives

Generalized connectives may define formulas that can already be expressed by the basic multiplicative connectives, $\otimes$ and $\parr$: these are called *decomposable* (or *definable*) generalized connectives. Sequentially, a connective P and Q is (binary) decomposable iff there exists a combination of $\otimes$ and $\parr$ having for rules the same two partitions sets. Dually, a graphical connective P and Q is (binary) decomposable iff there exists a formula F, only built with binary connectives of MLL, s.t. P is the pre-type of the parse tree of F and Q is the pre-type of the dual tree of F. E.g., the two pairs of orthogonal sets of partitions, (P_1,Q_1) and (P_2,Q_2) below, can be interpreted as two decomposable connectives, according the two dual points of view seen above: the sequential one in Fig. 2 and the graphical one in Fig. 3 (the decomposable graphical connectives are displayed as binary trees enclosed in dotted boxes):

$$P_1 = \{\{(1,2),(3,4)\}\} \text{ and } Q_1 = \{\{(1,3),(2),(4)\},\{(1,4),(2),(3)\},\{(2,3),(1),(4)\},\{(2,4),(1),(3)\}\}$$
$$P_2 = \{\{(2,3),(4,1)\}\} \text{ and } Q_2 = \{\{(2,4),(3),(1)\},\{(2,1),(3),(4)\},\{(3,4),(2),(1)\},\{(3,1),(2),(4)\}\}.$$

3 Non decomposable connectives

Surprisingly, generalized connectives allow to define new formulas that are not expressible with $\otimes$ and $\parr$. The most famous non decomposable connective (G_4, G_4^*) was discovered by Girard [3] and later reformulated by Danos and Regnier [2] as a pair of orthogonal sets of partitions over the same domain $\{1,2,3,4\}$, as below:

$$G_4 = \{\{(1,2),(3,4)\},\{(2,3),(4,1)\}\} \quad \text{and} \quad G_4^* = \{\{(1,3),(2),(4)\},\{(2,4),(1),(3)\}\}.$$

However, no characterization of (non-)decomposable connectives is known up to now. Under this respect, only [5] represents a first improvement: it defines a class of generalized connectives that are not decomposable, neither in sequential nor in graphical syntax, the so-called *entangled connectives*. A pair of partitions is entangled when each partition has at most *size* 2 (i.e., it does not contain more than two elements of the domain). Then, a connective P and Q is entangled when P or Q is an entangled pair. Intuitively, this notion of entanglement is a natural condition that can be observed as soon as we "compare" pairs of MLL proof nets (with units, eventually) having the same "bipolar skeleton" (i.e., having the same abstract graph), like e.g. the two ones of Fig. 3. Concretely, an entangled connective, $P = \{p_1, p_2\}$ and $P^\perp$, can be interpreted as union resp., intersection of types, that is: $P = \mathscr{T}_{F_1} \cup \mathscr{T}_{F_2}$ and $P^\perp = \mathscr{T}_{F_1^\perp} \cap \mathscr{T}_{F_2^\perp}$, with $\mathscr{T}_{F_1} = \{p_1\}$ and $\mathscr{T}_{F_2} = \{p_2\}$ for some *bipoles*[1] F_1 and F_2. We illustrate this fact by an example. Consider the Girard's non decomposable connective G_4 and G_4^*: it is clearly an entangled connective; in particular, G_4 (resp., G_4^*) results by the union (resp., by the intersection) of the types of bipoles $F_1 = (1 \otimes 2) \parr (3 \otimes 4)$ and $F_2 = (2 \otimes 3) \parr (4 \otimes 1)$[2] (resp., of the types of $F_1^\perp$ and $F_2^\perp$) of Fig. 3 i.e., $G_4 = \mathscr{T}_{F_1} \cup \mathscr{T}_{F_2}$, (resp., $G_4^* = \mathscr{T}_{F_1^\perp} \cap \mathscr{T}_{F_2^\perp}$). This fact is a novelty since the union of types is not in general a type while the intersection of types is always a type [6]. Indeed, entangled types are the smallest types (w.r.t. the number of partitions), if we exclude the trivial singleton types (actually, every set with a single partition is trivially a type). So, entangled connectives can be considered, in some sense, "elementary connectives", since they are the "smallest" generalized multiplicative connectives (w.r.t. the number of switchings or the number of rules), if we exclude, of course, the basic ones $(\parr, \otimes)$.

3.1 Sequentialization of non decomposable connectives

Generalized connectives are more expressive in the parallel syntax since there exist proof nets, built on pairs of non decomposable links, G and G^*, without correspondence in the sequent calculus, if we exclude the identity axioms $\vdash G, G^*$. Concretely, we show that every *proof net* built by matching (by means of binary axioms links) the dual borders of a pair of orthogonal non decomposable entangled links, G and G^* like in Fig. 4, cannot be sequentialized in the *MLL* sequent calculus extended with the general rule of Fig. 1. Actually, sequents of non decomposable entangled formulas $\vdash G, G^*$, are not provable from atomic logical axioms in the extended *MLL* sequent calculus: that is because of each rule for G (resp., G^*) is at least binary (it has at least two premises, by the entanglement) so any derivation of this sequent would build a premise with the other conclusion G^* (resp., G) together with only some (not all) of the principal formulas of the applied rule.

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[1] Naively, a *bipole* B is a MLL formula with only two layers of connectives [1]: a generalized $\parr$ of generalized $\otimes$-sub-formulas, i.e. $B = \hat{\parr}(\hat{\otimes}_1, ..., \hat{\otimes}_n)$, like e.g. F_1 and F_2 of Fig. 3 . Dually, $B^\perp = \hat{\otimes}(\hat{\parr}_1, ..., \hat{\parr}_n)$ is an *anti-bipole*, that is a MLL formula with only two layers of connectives: a generalized $\otimes$ of generalized $\parr$-sub-formulas, like e.g. $F_1^\perp$ and $F_2^\perp$ of Fig. 3 . Every bipole trivially satisfy the property that its pre-type is a type.

[2] Notice that the respective borders of F_1 and F_2 are cyclic permutations of the sequence $1 < 2 < 3 < 4$, called later *base*.

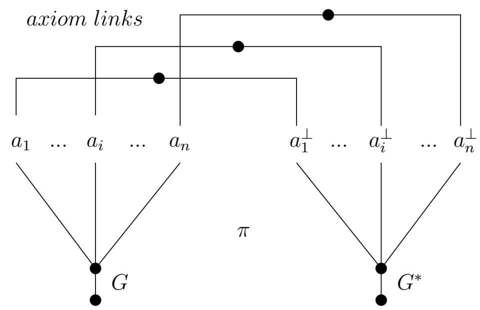

Figure 4: non sequentializable proof net with non decomposable conclusions G, G^*

3.2 Non-decomposable connectives and prime numbers

The class of entangled connectives is quite special (although minimal w.r.t. the number of partitions or switchings) and we already have examples of non decomposable connectives falling outside of it. We illustrate here a more general (although once again non exhaustive) way to build non binary decomposable connectives. We call *degree* of a partition p the number of its classes and we call *size* of a class γ the number of its elements. Whenever a partition p of degree d contains only classes with a same size s, then we say that p has *rank* $d|s$. Let $n \geq 4$ be a natural number s.t. $n = uv$ where u and v are prime numbers, then we call *base* the (strict-totally) ordered sequence $S = 1 < 2 < 3 < 4 < ... < n$. We call *basic partition of rank* $u|v$, shortly $(u|v)$-partition, the unique partition of the base $S = 1 < 2 < 3 < 4 < ... < n$ of rank $u|v$ s.t. each of the u classes consists of a sub-sequence of S with size v. E.g., if $n = 6$ then $\{(1,2,3),(4,5,6)\}$ is the unique basic partition of rank $2|3$ over the base $S = 1 < 2 < ... < 6$, while $\{(1,2),(3,4),(5,6)\}$ is the unique basic partition of rank $3|2$ over S. Now, consider the set G of basic partitions over $\{1,2,3,4,...,n\}$ with $n = uv$ (where u and v are prime numbers) built as follows: take the $(u|v)$-partition for each cyclic permutation S^i of the base $S = (1 < 2 < ... < n)$ that is: $S = S^1 = (1 < 2 < 3 < ... < n)$, $S^2 = (2 < 3 < ... < n < 1)$, $S^3 = (3 < 4 < ... < n < 1 < 2)$, ..., $S^n = (n < 1 < 2 < ... < n-1)$. We can show that every set of $(u|v)$-partitions over cyclic permutations of any base $1 < 2 < ... < n$, built like G above, is a type (i.e. $G = G^{\perp\perp}$) thus it is a generalized connective since its orthogonal is not empty; such connectives are called *Girard connectives*[3] like e.g. G_6', G_6'' and G_9:

$$G_6' = \{ \{(1,2),(3,4),(5,6)\}, \{(2,3),(4,5),(6,1)\} \}$$

$$G_6'' = \{ \{(1,2,3),(4,5,6)\}, \{(2,3,4),(5,6,1)\}, \{(3,4,5),(6,1,2)\} \}$$

$$G_9 = \{ \{(1,2,3),(4,5,6),(7,8,9)\}, \{(2,3,4),(5,6,7),(8,9,1)\}, \{(3,4,5),(6,7,8),(9,1,2)\} \}.$$

It is not difficult to show that every Girard connective is not definable by means of the binary connectives $\otimes$ and $⅋$, so we try to give in the following a proof sketch of that, assuming that Girard connectives are expressed in graphical syntax (i.e. in proof net syntax). First, observe that if a set of partitions C of $\{1,...,n\}$ (together with its orthogonal $C^\perp$) is a decomposable connective, i.e. $C = C_1 \circ C_2$ with $\circ \in \{⅋, \otimes\}$, then its cardinality $|C| = n$ is exactly the product of the cardinalities of its principal sub-components (sub-connectives) C_1 and C_2, that is $|C| = |C_1| \times |C_2|$[4] This means that $|C_1|$ or $|C_2|$ must be equal 1 as soon

[3]These connectives were suggested to the author by Jean-Yves Girard during "The Scholars chats" together with Paolo Pistone in Rome in December 2016. They represent a novelty w.r.t. the contents of [5]. Notice that, although the definition of Girard connectives makes use of cyclic permutations of the base $1 < 2 < ... < n$, these connectives do not refer to the cyclic fragment of multiplicative linear logic since there is neither order among classes neither order among the elements of each class in any partition. Cyclicity is rather a gimmick to get partitions in which some elements stay at a certain "distance".

[4]To be precise, when we represent a generalized connective in the graphical syntax, i.e. a generalized *link* in proof nets syntax, we should include (extending the base) the conclusion of the link: this fact may increase the number of partitions of

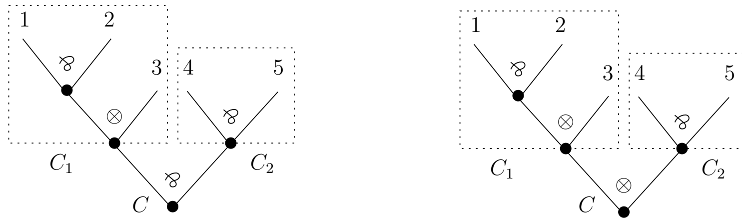

Figure 5: examples of generalized decomposable connectives in proof net syntax

as C is a Girard connective, since the cardinality of any Girard connective is always a prime number (this fact simply follows by the way we built the set of basic partitions over the cyclic permutations of the base $1 < \ldots < n$). Now, assume by absurdum that C is a Girard connective decomposable: let us say $C = C_1 \parr C_2$ with e.g. C_2 being the singleton $\{q\}$. Then C would contain only partitions given by the disjoint sum $p_i \uplus q$ for every $p_i \in C_1$ (e.g., if C is a decomposable connective like that one depicted on the l.h.s. of Fig. 5 then, classes (4) and (5) will occur in every partitions of C). This contradicts Girard connectives since by construction (relying on the cyclic permutation of the base) there are no two partitions with a class in common in any Girard connective (see e.g. G'_6, G''_6 or G_9 above). So, assume $C = C_1 \otimes C_2$ with C_2 being the singleton $\{q\}$ (as before); then the partition q must consist of a single class, i.e. $q = \{\gamma\}$, otherwise all partitions of C will result by joining together several times every partition p_i of C_1 with the unique partition q, one join for each pair of classes $\theta \in p_i$ and γ: e.g., w.r.t. the decomposable connective C on the r.h.s. of Fig. 5, element 4 (resp., element 5) of C_2 will occur both together with 1 and 3 in a class of some partition of C and together with 2 and 3 in a class of some partition of C. Then C cannot have a prime number of partitions, contradicting the assumption that C is a Girard connective. Thus C_2 must consist of a single partition q containing a single class γ with some elements of the base of C, let us say $\gamma = \{b_1, \ldots b_r\}$ for $1 \leq r < n$. This means that every element a_i in the base of C_1 must be joined together with every element b_j of the base of C_2 for some switching (or partition) in C that is, there must exist (at least) a partition $p \in C$ with a class containing both a_i and b_j: but this is not the case with Girard connectives since, by construction, there is no element x in the base of any Girard connective s.t. for every element of the base $y \neq x$, x and y occur joined together in some class of some partition of the connective; e.g., for every element x of the base $1 < 2 < \ldots < 6$ of G''_6, there always exists an element y at *distance*[5] 3 s.t. x and y never occur joined together in any class of any partition of G''_6, that is: element 1 never occurs together with 4, element 2 never occurs together with 5 and element 3 never occurs together with 6. This concludes the proof sketch.

4 Future works

All generalized connectives, decomposable or not, satisfy cut elimination. The fact that one can compute, by means of cut elimination, using non decomposable connectives, is certainly a good starting point; nevertheless, the study of their connection with *concurrency* (typically, the Pi-Calculus [7]) should be investigated, at least to answer the following two questions: *(i)* is there any *program* $[\pi]$ corresponding to the non decomposable proof net π of $\vdash G, G^*$ (when G is a Girard connective)? and in the affirmative case,

a link; e.g. in the graphical syntax, the basic binary connective $a \parr b$ should be expressed as the following set of partitions $\{\{(a,c),(b)\},\{(a),(b,c)\}\}$, where c is the conclusion, rather than the singleton $\{\{(a),(b)\}\}$ (the latter representation is more suitable with the sequential syntax).

[5] Two non null natural numbers, a and b, are at *distance* k iff $|a-b| = k$; e.g. 1 and 4 are at distance 3 since $|1-4| = 3$.

(ii) what is the correspondence between the "parallel nature" of π and the "concurrent nature" of $[\pi]$?

To conclude, non decomposable generalized connectives witness a genuine asymmetry between proof nets and sequent proofs since the former ones allow us to express a kind of parallelism of proofs (built on general connectives) that the latter ones cannot do, as as suggestively stated at page 202 of [2]:

> "*We saw with some surprise that the realm of multiplicatives became quite complex, even handled by a careful generalization. Yet the generalization seems more natural in the non sequential framework. Maybe we witness here the limits of sequential presentations of logic*".

References

[1] Andreoli, J.-M.: Focussing and Proof Construction. *APAL* 107(1), pp. 131-163, 2001.

[2] Danos, V. and Regnier, L.: The structure of multiplicatives. AML, 28, pp.181-203, 1989.

[3] Girard, J.-Y.: Multiplicatives. Rendiconti del Seminario Matematico dell'Univ. di Torino. Special Issue on Logic and Computer Science, pp. 11-33, 1987.

[4] Girard, J.-Y.: Linear Logic, Theor. Comp. Sci., London Math. vol. 50(1), pp. 1-102, 1987.

[5] Maieli, R.: Non decomposable connectives of linear logic. June 2017. Submitted to *Annals of Pure and Applied Logic* available at: `http://logica.uniroma3.it/maieli/non-decomp_MLL.pdf`.

[6] Maieli, R. and Puite, Q.: Modularity of proof nets: generating the type of a module. Archive for Math. Logic, vol. 44(2), pp. 167-193, 2005.

[7] Milner, R.: Communicating and Mobile Systems: the Pi-Calculus, Cambridge University Press, 1999.