Network-Aware Design of State-Feedback Controllers for Linear Wireless Networked Control Systems

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Abstract: We study the problem of stabilizing the origin of a plant, modeled as a discrete-time linear system, for which the communication with the controller is ensured by a wireless network. The transmissions over the wireless channel are characterized by the so-called *stochastic allowable transmission intervals* (SATI), that is a stochastic version of the maximum allowable transmission interval (MATI). Instead of deterministic transmissions, SATI gives stability conditions in terms of the cumulative probability of successful transmission over N steps. We argue that SATI is well-suited for wireless networked control systems to cope both with the stochastic nature of the communications and the design of energy-efficient communication strategies. Our objective is to synthesize a stabilizing state-feedback controller and SATI parameters simultaneously. We model the overall closed-loop system as a Markov jump linear system and we first provide linear conditions for the stability of the wireless networked control systems in a mean-square sense. We then provide linear matrix inequalities conditions for the design of state-feedback controllers to ensure stability of the closed-loop system. These conditions can be used to obtain both the controller and the SATI. A numerical example is presented to illustrate our results.

Keywords: Networked control systems, switched systems, LMIs

1. INTRODUCTION

The use of wireless networks in control systems offers many advantages in terms of ease of implementation, maintenance, flexibility, and reduced costs. However, it also generates inevitable communication imperfections and constraints such as time-varying sampling, packet drops, scheduling, delays, limited bandwidth and so on, which may deteriorate the performance of the closed-loop system. This has motivated many researchers over the two last decades to analyse and design suitable control and estimation strategies for wireless networked control systems (WNCS), see, e.g., Heemels and van de Wouw (2010); Hespanha et al. (2007); Zhang and Branicky (2001) and the references therein.

A key parameter in NCS is the *maximum allowable transmission interval* (MATI). Various works provide methodologies to compute bounds on the MATI to ensure stability, see, for instance, Nešić and Teel (2004); Carnevale et al. (2007); Omran et al. (2014); Jentzen et al. (2010); Donkers et al. (2011); Postoyan and Nešić (2016) for stabilization; Postoyan and Nešić (2012) for estimation; Postoyan et al. (2014) for tracking control. The MATI is a

deterministic constraint, which may be difficult to ensure when the network is wireless. In this case, transmissions are uncertain and subject to various external factors such as channel fading, shadowing, potential collisions, etc. Those characteristics lead naturally to the use of stochastic models for wireless channels, see, for instance, the works Tabbara et al. (2007), Quevedo et al. (2013), Speranzon et al. (2006), Fischione et al. (2011), and the references therein.

An alternative stochastic notion to the MATI was introduced in Varma et al. (2017), called the (η, δ)-*stochastic maximum allowable transmission interval*, in short, (η, δ)-SATI or simply SATI. The idea is that, for WNCS, the maximum length interval N between two successful transmissions cannot be fixed *a priori* as there is no guaranty that it will be satisfied. Stability should therefore also depend on the cumulative probability η that a transmission was successful within the interval of time of length N since the last transmission. Now, if no successful transmission occurs in N consecutive steps, then the transmitter has to use maximal power to send its packet, which is modeled via a probability δ, to be able to ensure stability. In this context, stability depends on three parameters: N, η, and δ. Alternatively, some related results can be found in the works Montestruque and Antsaklis (2004); Donkers et al. (2012); Xie and Xie (2008) through a different modelling. Even though Donkers et al. (2012); Montestruque and

Antsaklis (2004) studied also the case when the inter-transmission intervals are modeled by a Markov process, the SATI modeling in Varma et al. (2017) is different in the sense that stability depends on the aforementioned parameters N, η, and δ, see, for instance, Varma et al. (2017) for more details.

In Varma et al. (2017), the objective is to give a characterization of the SATI when a stabilizing controller (in a deterministic sense) is already known. In other words, the authors considered the emulation approach in which the controller is first designed to stabilize the origin of the plant without the network constraints, and then the network characteristics are taken into account in the stability analysis. A drawback of this approach is that the obtained values on N, δ, and η do depend on the initial choice of the controller. Thus, a "bad" choice of the controller will lead to a conservative SATI. Also, this approach does not allow to answer the important practical question: given a network and some of its characteristics in terms of N, η, and δ, how should we design a stabilizing controller, if this is possible?

To complement the existing results, we propose a method to design stabilizing state-feedback controllers, given (some of) the SATI parameters. We assume that the controller to be designed is implemented using a zero-order hold scheme. The overall system is modeled as a Markov jump linear system (MJLS), see, for instance, Costa et al. (2005), in which each mode of operation models the dynamics since the last transmission instant as in Varma et al. (2017). The extension of the results of Varma et al. (2017) to the design of state-feedback controllers is not trivial. We first have to revisit the latter by working with alternative MJLS conditions to ensure the mean-square stability of the closed-loop system. The obtained matrix inequalities are nonlinear with respect to the controller gain, and we then resort to the use of slack variables, like in de Oliveira et al. (1999), with particular structures in order to derive linear matrix inequalities suitable for the network-aware design. Our results are formulated in terms of linear matrix inequalities (LMIs), and are thus computationally oriented and easily implementable. It is noteworthy that N, η, and δ are scalar parameter, and if we find two of them, the third one can be obtained by a line search procedure. We finally present a numerical example for the unstable batch reactor commonly considered in the NCS literature.

The paper is organized as follows. The problem and the main goals are presented in Section 2. In Section 3, we provide the main results of this work, that is, conditions in the form LMIs under which the WNCS is stable in the mean-square sense for a given (η, δ)-SATI and a given controller, as well as LMIs design conditions for synthesizing state-feedback controllers such that the WNCS is stable. Finally, in Section 4, we present numerical results and we conclude with final remarks in Section 5. Due to space limitations, all proofs can be found in de Oliveira et al. (2018).

Notation. Let $\mathbb{R} := (-\infty, \infty)$, $\mathbb{N} := \{0, 1, 2, \ldots\}$, and $\mathbb{N}^+ := \mathbb{N} \setminus \{0\}$. Given $N \in \mathbb{N}$, we define the set $\mathbb{N}_N := \{0, 1, \ldots, N\}$, as well as $\mathbb{N}_N^+ = \mathbb{N}_N \setminus \{0\}$. The set $\mathbb{R}^{n \times m}$ represents the $n \times m$ real matrices, and $\mathbb{S}^{n+}$ is the set of positive definite matrices in $\mathbb{R}^{n \times n}$. For symmetric matrices, the symbol $\bullet$ stands for a symmetric block. Given $M \in \mathbb{R}^{n \times m}$, we denote by M', the transpose of M, the identity matrix of order n by I_n, the $n \times m$ zero matrix by $0_{n \times m}$, and the block diagonal matrix by $\mathbf{diag}(\cdot)$. For a square matrix M, we set $\text{Her}(M) := M + M'$. The operator E stands for the expectation taken over the relevant stochastic variables, and $\mathbf{1}(\cdot)$ for the indicator function, taking the value 1 when the condition is satisfied and 0 otherwise.

2. PROBLEM STATEMENT

2.1 Set-up description

The following discrete-time linear system is considered
$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ x_0 &= x(0) \end{aligned} \tag{1}$$
where $k \in \mathbb{N}$ is the time, $x(k) \in \mathbb{R}^n$ is the plant state, $u(k) \in \mathbb{R}^m$ is the control input, and $n, m \in \mathbb{N}^+$. Our objective is to stabilize the origin of (1) when the controller to be designed communicates with (1) via a wireless network. In particular, the sensors transmit their data to the controller via the wireless channel, and the controller is assumed to be directly connected to the actuators [1] as depicted in Figure 5.
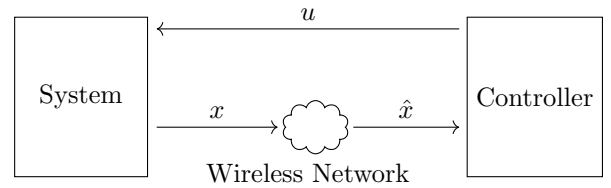

Fig. 1. Schematic of the WNCS.

Because of the network, the controller does not have access to x, but to $\hat{x}$, the network-induced version of x, which has the following dynamics
$$\hat{x}(k) = \begin{cases} x(k), & \text{successful transmission} \\ \hat{x}(k-1), & \text{otherwise.} \end{cases} \tag{2}$$
The transmission behavior in (2) is stochastic and will be precisely defined in Section 2.2. System (2) corresponds to a zero-order hold strategy, in the sense that in the case the packet is lost or not sent, the last received measurement is used. Consequently, we want to design the control law
$$u(k) = K\hat{x}(k), \tag{3}$$
to stabilize, in a stochastic sense, the origin of (1). We define the extended state
$$\chi(k)' := \begin{bmatrix} x(k)' & \hat{x}(k-1)' \end{bmatrix} \tag{4}$$
with $\hat{x}(-1)$ being chosen arbitrarily in $\mathbb{R}^n$, and we obtain the closed-loop system
$$\chi(k+1) = \begin{cases} \mathcal{A}_1 \chi(k), & \text{succesful transmission} \\ \mathcal{A}_2 \chi(k), & \text{otherwise,} \end{cases} \tag{5}$$
where
$$\mathcal{A}_1 := \begin{bmatrix} A + BK & 0 \\ I_n & 0 \end{bmatrix}, \ \mathcal{A}_2 := \begin{bmatrix} A & BK \\ 0 & I_n \end{bmatrix}. \tag{6}$$

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[1] The forthcoming results also apply when the controller is directly connected to the sensors but communicates with the actuators via the network.

The main goal of this work is to provide conditions to design the control law (3) and some of the network characteristics such that (1) is stable in some stochastic sense.

2.2 The SATI

To model transmissions over the network, we introduce the clock $\tau(k) \in \mathbb{N}^+$ with the dynamics

$$\tau(k+1) = \begin{cases} 1, & \text{successful communication at } k \\ \tau(k) + 1, & \text{failed communication at } k, \end{cases} \tag{7}$$

which counts the number of time instants since the last successful transmission. Given that, we assume that the transmission is modeled by a Bernoulli process and the probability of successful communication at time $k \in \mathbb{N}$ is given by $f(\tau(k)) \in [0, \delta]$, where $\delta > 0$ is the maximum success probability representing the physical limitation of the communication system with regard to packet success, see the work Theodore (1996), for instance. Thus, the successful communication probability at time k depends on the current state of the clock $\tau(k)$, due to factors that vary over the time, like the channel quality and the used transmission power. The cumulative probability that a transmission is successful within the interval of length N since the last transmission instant, represented by η, is

$$\eta := 1 - \prod_{i=1}^{N}(1 - f_i), \tag{8}$$

where we use $f_i = f(i)$ for the ease of presentation. We assume that the probabilities f_i for $i \in \mathbb{N}_N^+$ are not known *a priori*, but satisfy (8). The motivation for this policy is that as long as the clock $\tau(k)$ is less than or equal to N, communication can be tried with lower resource consumption such as bandwidth and radio-transmit power. On the other hand, for $\tau(k) > N$, we have to enforce transmission by using the maximum power so that the transmission probability is set to its maximum value, i.e. δ. This requires that a simple acknowledge scheme is implemented, which is common in wireless networks.

As a result, the transmission policy is characterized by the SATI, which involves three parameters: N the interval length since the last successful transmission, η as defined in (8), and the maximum probability δ.

2.3 The MJLS model

As in Varma et al. (2017), we model the overall closed-loop system as a discrete-time Markov chain $\theta(k)$ composed by $N + 1$ virtual nodes, in which the first mode $\theta(k) = 1$ represents the successful communication, and the remaining modes represent the transmission failures. Formally, we have that

$$\theta(k) = \begin{cases} \tau(k), & \tau(k) \leq N, \\ N + 1, & \tau(k) > N, \end{cases}$$

and we define the set of Markov states as $\mathbb{M} := \mathbb{N}_{N+1}^+$. The motivation for this model is that each mode of operation represents the system dynamics for the corresponding clock $\tau(k) \leq N$ in which the success probabilities $f(\tau(k))$ are different, as explained in Section 2.2. For $\tau(k) > N$, then it is only necessary to use one virtual mode $\theta(k) =$
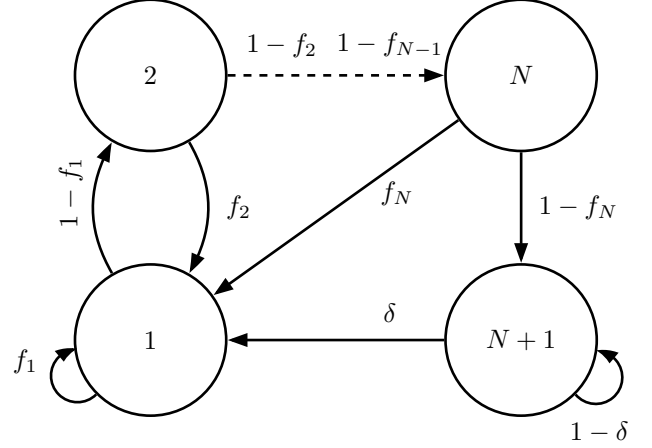

Fig. 2. The Markov chain for the SATI framework, adapted from Varma et al. (2017).

$N + 1$, as the success probabilities are all set to δ. The state diagram for this Markov chain of $N + 1$ modes of operation is shown in Figure 2. The transition probability matrix Π is given by

$$\Pi := \begin{bmatrix} f_1 & 1 - f_1 & 0 & \dots & 0 \\ f_2 & 0 & 1 - f_2 & \dots & 0 \\ \vdots & & & & \\ f_N & 0 & 0 & \dots & 1 - f_N \\ \delta & 0 & 0 & \dots & 1 - \delta \end{bmatrix}. \tag{9}$$

In short, transmission is attempted with probabilities f_i, $i \in \mathbb{N}_N^+$, such that (8) is respected, and with a fixed success probability δ afterwards. It is important to notice that the transition probability matrix in (9) is not known, *a priori*. Only η and δ are known and these are related to the unknown set of probabilities $\{f_1, \dots, f_N\}$ by (8).

In view of the above Markov chain, we define the following MJLS from the closed-loop system in (5)

$$\begin{aligned} \chi(k+1) &= \mathcal{A}_{1+\mathbf{1}(\theta(k)>1)}\chi(k), \\ \chi(0) &= \chi_0, \ \theta(0) = \theta_0, \end{aligned} \tag{10}$$

where χ, $\mathcal{A}_1$, and $\mathcal{A}_2$ are defined in (4) and (6), respectively. Thus, for $\theta(k) = 1$, the dynamic matrix in (10) becomes $\mathcal{A}_{1+\mathbf{1}(\theta(k)>1)} = \mathcal{A}_1$, that is, the stable mode of the Markov chain presented in Figure 2, and for $\theta(k) > 1$, $\mathcal{A}_{1+\mathbf{1}(\theta(k)>1)} = \mathcal{A}_2$, representing the unstable mode of operation.

Our aim is to propose a methodology to design K in (3), given N, η, and δ to ensure that system (10) is mean-square stable (MSS) as defined next.

Definition 1. (Geromel et al. (2009)). System (10) is said to be MSS if, for every initial state (χ_0, θ_0),

$$\lim_{k \to \infty} E\left[\chi(k)'\chi(k)\right] = 0. \tag{11}$$

□

The set of admissible controllers K is represented by

$$\mathcal{K}(N, \eta, \delta) := \left\{K \in \mathbb{R}^{m \times n} : (10) \text{ is MSS}\right\}. \tag{12}$$

The next lemma, adapted from Costa et al. (2005), establishes necessary and sufficient conditions for the MSS of (10).

Lemma 1. (Costa et al. (2005)). Given K, the following statements are equivalent.

(1) $K \in \mathcal{K}(N, \eta, \delta)$, for $\mathcal{K}(N, \eta, \delta)$ defined in (12).

(2) There exists $P_i \in \mathbb{S}^{2n+}$, $i \in \mathbb{M}$, such that

$$P_i - \mathcal{A}'_{1+\mathbf{1}(i>1)} \sum_{j \in \mathbb{M}} \Pi_{ij} P_j \mathcal{A}_{1+\mathbf{1}(i>1)} > 0 \qquad (13)$$

holds for all $i \in \mathbb{M}$, where Π is in (9) under (8). $\qquad \square$

The result in Lemma 1 cannot be used for our purpose since it involves the probabilities Π_{ij} in (9) which we do not know. Hence, we first provide in the next section LMI conditions that depend only on the SATI parameters N, η, and δ that ensure that the closed-loop is MSS for a given K. Although these new conditions are linear in the decision variables for a given controller gain, they become nonconvex when K is a variable. To overcome this issue, we derive LMI conditions for the design of K such that the closed-loop is MSS by imposing some structures on the problem variables, as well as exploring the form of the closed-loop matrices in (6).

3. MAIN RESULTS

3.1 First stability result

The next theorem provides sufficient conditions for the MSS of system (10), given K, N, η, and δ. Those conditions are different with respect to the ones given in Varma et al. (2017) in the sense that they are related to a "primal" MJLS Lyapunov-like inequality set, whereas in Varma et al. (2017) a "dual" formulation was used, see, for instance, Costa et al. (2005). This difference is essential for the forthcoming design results.

Theorem 1. Given the controller gain K, $N \in \mathbb{N}^+$, $\eta \in [0, 1]$, $\delta \in [0, 1]$, if there exists $P_1, P_{N+1}, S \in \mathbb{S}^{2n+}$ such that

$$P_{N+1} > \mathcal{A}'_2 \left[(1 - \delta) P_{N+1} + \delta P_1 \right] \mathcal{A}_2, \qquad (14)$$

$$P_1 > \mathcal{A}'_1 \left[(1 - \eta)(\mathcal{A}'_2)^{N-1} P_{N+1} (\mathcal{A}_2)^{N-1} + \eta S \right] \mathcal{A}_1, \quad (15)$$

$$S > (\mathcal{A}'_2)^i P_1 \mathcal{A}^i_2 \quad \forall i \in \mathbb{N}_{N-1}, \qquad (16)$$

holds, then $K \in \mathcal{K}(N, \eta, \delta)$. $\qquad \square$

The proof of this theorem can be found in de Oliveira et al. (2018). We show that (14)-(16) yield the set of coupled Lyapunov-like inequalities in (13) that implies the MSS of (10) by Lemma 1.

3.2 Conditions for network-aware design

The result in Theorem 1 is useful for studying the MSS of (10) for a given state-feedback controller K and SATI constraints N, η, and δ. However, when we want to design K, the set of solutions of (14)-(16) becomes non-convex due to the products between K and P_1, P_{N+1}, and S via the closed-loop matrices (6). Also, the structure of (6) along with the powers of $\mathcal{A}_2$ in (16) adds difficulties. We overcome these obstacles by using slack variables that consider the change of structure in (6), and also by noting that $\mathcal{A}^i_2$ is indeed linear in K. Considering that, we provide LMI conditions in Theorem 2 below for obtaining K such that the closed-loop system (10) is MSS for given SATI constraints N, η, and δ. In order to ease the exposition, we set

$$\tilde{\mathcal{A}}_1 := \begin{bmatrix} A & 0 \\ I_n & 0 \end{bmatrix}, \ \tilde{\mathcal{A}}_2 := \begin{bmatrix} A & 0 \\ 0 & I_n \end{bmatrix}, \ \tilde{\mathcal{B}} := \begin{bmatrix} B \\ 0 \end{bmatrix}, \qquad (17)$$

as well as

$$\tilde{\mathcal{S}}_i := \begin{bmatrix} \Phi_i & 0 \\ 0 & I_n \end{bmatrix} \qquad (18)$$

where $\Phi_i := \sum_{j=0}^{i-1} A^j$, $i \in \mathbb{N}_{N-1}$.

Furthermore, we define the following partitions for the variables $\tilde{G}_1, \tilde{G}_2, \tilde{Y}$ arising in Theorem 2 below

$$\tilde{G}_1 := \begin{bmatrix} G & G \\ G_3 & G_4 \end{bmatrix}, \tilde{G}_2 := \begin{bmatrix} G_1 & G_2 \\ G & G \end{bmatrix}, \tilde{Y}' := \begin{bmatrix} Y' \\ Y' \end{bmatrix} \qquad (19)$$

along with $\tilde{R}$ and $\tilde{H}_i$,

$$\tilde{R} := \begin{bmatrix} R_1 & R_2 \\ G & G \end{bmatrix}, \tilde{H}_i := \begin{bmatrix} H_{i1} & H_{i2} \\ G & G \end{bmatrix}, \qquad (20)$$

for all $i \in \mathbb{N}_{N-1}$. Matrices $\tilde{G}_1, \tilde{G}_2, \tilde{R}$, and $\tilde{H}_i$ are slack variables from which the controller may be obtained. The next theorem presents the main result of this paper.

Theorem 2. For given $N \in \mathbb{N}^+$, $\eta \in [0, 1]$, and $\delta \in [0, 1]$, if there exist $Q_1, Q_{N+1}, M, X \in \mathbb{S}^{2n+}$, $\tilde{G}_1, \tilde{G}_2, R \in \mathbb{R}^{2n \times 2n}$, $\tilde{Y} \in \mathbb{R}^{m \times 2n}$, and $\tilde{H}_i \in \mathbb{R}^{2n \times 2n}$, $i \in \mathbb{N}_{N-1}$, partitioned as in (19)-(20) such that

$$\begin{bmatrix} \mathrm{Her}(\tilde{G}_2) - Q_{N+1} & \bullet & \bullet \\ \delta_d(\tilde{\mathcal{A}}_2 \tilde{G}_2 + \tilde{\mathcal{B}} \tilde{Y}) & Q_{N+1} & \bullet \\ \delta_n(\tilde{\mathcal{A}}_2 \tilde{G}_2 + \tilde{\mathcal{B}} \tilde{Y}) & 0 & Q_1 \end{bmatrix} > 0 \qquad (21)$$

$$\begin{bmatrix} \mathrm{Her}(\tilde{G}_1) - Q_1 & \bullet & \bullet \\ \eta_d(\tilde{\mathcal{A}}_1 \tilde{G}_1 + \tilde{\mathcal{B}} \tilde{Y}) & M & \bullet \\ \eta_n(\tilde{\mathcal{A}}_1 \tilde{G}_1 + \tilde{\mathcal{B}} \tilde{Y}) & 0 & X \end{bmatrix} > 0 \qquad (22)$$

$$\begin{bmatrix} \mathrm{Her}(\tilde{R}) - M & \bullet \\ (\tilde{\mathcal{A}}_2)^{N-1} \tilde{R} + \tilde{\mathcal{S}}_{N-1} \tilde{\mathcal{B}} \tilde{Y} & Q_{N+1} \end{bmatrix} > 0 \qquad (23)$$

$$\begin{bmatrix} \mathrm{Her}(\tilde{H}_i) - X & \bullet \\ (\tilde{\mathcal{A}}_2)^i \tilde{H}_i + \tilde{\mathcal{S}}_i \tilde{\mathcal{B}} \tilde{Y} & Q_1 \end{bmatrix} > 0, \quad \forall i \in \mathbb{N}_{N-1}, \qquad (24)$$

holds, where $\eta_n = \sqrt{\eta}$, $\delta_n = \sqrt{\delta}$, $\eta_d = \sqrt{1 - \eta}$, and $\delta_d = \sqrt{1 - \delta}$. Then, $K = Y G^{-1} \in \mathcal{K}(N, \eta, \delta)$. $\qquad \square$

The proof of this result can be found in de Oliveira et al. (2018). Basically, the idea is to show that, if (21)-(24) hold, then by taking $K = Y G^{-1}$, we have that (14)-(16) is also satisfied. Fixing the structure of the variables as in (19)-(20) has two roles. The first one is that we are able to have a one-to-one relation between the problem variables and the control gain K. The second one is that, by fixing the structure of the slack variables instead of Q_{N+1}, Q_1, and S, we have less conservative results. Finally, the key point is that we can rewrite (22) by considering that

$$\tilde{\mathcal{A}}_1 \tilde{G}_1 + \tilde{\mathcal{B}} \tilde{Y} = \begin{bmatrix} AG + BY & AG + BY \\ G & G \end{bmatrix}$$

$$= \begin{bmatrix} A + BYG^{-1} & 0 \\ I & 0 \end{bmatrix} \begin{bmatrix} G & G \\ G_3 & G_4 \end{bmatrix}$$

$$= \mathcal{A}_1 \tilde{G}_1.$$

Furthermore, for (21), we have that

$$\tilde{\mathcal{A}}_2 \tilde{G}_2 + \tilde{\mathcal{B}} \tilde{Y} = \begin{bmatrix} AG_1 + BY & AG_2 + BY \\ G & G \end{bmatrix}$$

$$= \mathcal{A}_2 \tilde{G}_2,$$

and similarly for (23) and (24), for all $i \in \mathbb{N}_{N-1}$,

$$\tilde{\mathcal{A}}^i_2 \tilde{H}_i + \tilde{\mathcal{S}}_i \tilde{\mathcal{B}} \tilde{Y} = \begin{bmatrix} A^i & \Phi_i BYG^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} H_{i1} & H_{i2} \\ G & G \end{bmatrix}$$

$$= (\mathcal{A}_2)^i \tilde{H}_i,$$

where $\Phi_i = \sum_{j=0}^{i-1} A^j$, and $\mathcal{A}_1$ and $\mathcal{A}_2$ are the closed-loop matrices given in (6). It follows that the powers of $\mathcal{A}_2$ are linear with respect to K.

The result in Theorem 2 may provide a stabilizing state-feedback controller K for a given maximum packet rate success δ, and parameters N and η. As we are going to see in the example in Section 4, condition (21)-(24) guarantees the MSS of the WNCS for given SATI parameters, but the resulting controller may also stabilize the plant for greater values of N and smaller η. Besides, since N, η, and δ are scalar parameters, we can obtain the controller and the SATI by means of simple search procedures.

4. NUMERICAL EXAMPLES

We consider the unstable batch reactor as in Walsh et al. (2002). The system is exactly discretized by means of a zero order hold with rate $T = 50$ ms, leading to the following values for A and B

$$A = \begin{bmatrix} 1.0795 & -0.0045 & 0.2896 & -0.2367 \\ -0.0272 & 0.8101 & -0.0032 & 0.0323 \\ 0.0447 & 0.1886 & 0.7317 & 0.2354 \\ 0.0010 & 0.1888 & 0.0545 & 0.9115 \end{bmatrix}$$

$$B' = \begin{bmatrix} 0.0006 & 0.2567 & 0.0837 & 0.0837 \\ -0.0239 & 0.0002 & -0.1346 & -0.0046 \end{bmatrix}.$$

We study the network-aware design for different characteristics of the (η, δ)-SATI by means of Theorem 2. For that, we set the maximum packet rate success to $\delta = 0.2$ and use the conditions of Theorem 1 to construct a controller, if possible, for different values of N and η. For solving the LMIs in Theorem 1 and 2, we used the parser YALMIP, see Löfberg (2004), along with the solver MOSEK Aps 8.1, see MOSEK (2017). For every LMI L in Theorems 1 and 2, we impose the restriction $L \geq I\epsilon$, for $\epsilon = 1e^{-4}$. Figure 3 shows whether a stabilizing controller was found by Theorem 2, represented by circles, or not, represented by crosses. The solver can find stabilizing controllers up
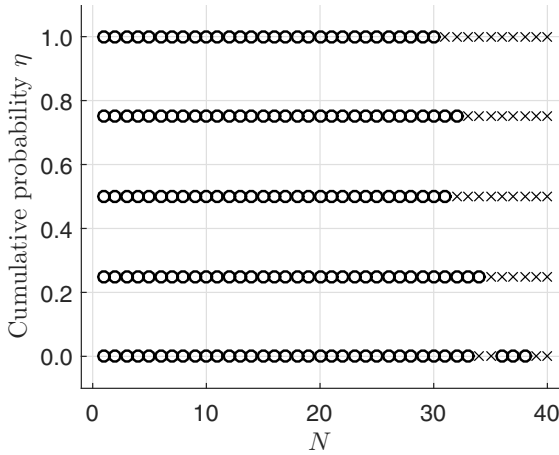

Fig. 3. Solution (circles) or unfeasible (crosses) pairs (N, η) of the conditions of Theorem 2, for $\delta = 0.2$.

to $N = 30$ for a maximum packet rate success of $\delta = 0.2$, even for the worst-case scenario defined by $\eta = 0$, that is, when no communication is attempted. Even though the extreme situations of Figure 3 can be achieved, the choice of η must also be motivated by the system performance

over the time, which will be addressed on future work. On the other hand, feasible solutions for (21)-(24) becomes rarer for $N > 30$, and some numerical inconsistencies can be found in terms of η. For instance, the solver is able to find a controller for $N = 31$ and $\eta = 0.5$, but it did not find a solution for the pair $N = 31$ and $\eta = 1.0$, which is not intuitive in the sense that we would expect to obtain solutions for larger values of the cumulative probability η. This may be explained by the summation Φ_i in (23)-(24) whose elements can grow arbitrarily large, leading to numerically ill-posed problems for great values of N.

Next we construct two controllers with the goal of comparing their performance in terms of N and η. The first one, with gain K_{SATI}, is obtained by means of Theorem 2 with $N = 2$, a cumulative probability of success of $\eta = 0.2$, and a maximum packet rate success of $\delta = 0.4$. The second one, with gain K_{LQR}, is designed by solving a discrete-time linear quadratic regulator problem for $Q = I_4$ and $R = I_2$. The control gains for both cases are given by

$$K_{\text{SATI}} := \begin{bmatrix} 0.3330 & -0.6397 & 0.1613 & -0.6449 \\ 1.9403 & 0.1250 & 1.6043 & -0.8239 \end{bmatrix}$$

$$K_{\text{LQR}} := \begin{bmatrix} 0.0153 & -0.8159 & -0.2394 & -0.7515 \\ 2.3250 & 0.0801 & 1.6225 & -1.0657 \end{bmatrix}.$$

We then study the impact on the stability through Theorem 1 for $\delta = 0.4$ by increasing N and obtaining the smallest η possible such that the closed-loop is still MSS for both controllers. The result is shown in Figure 4. Two
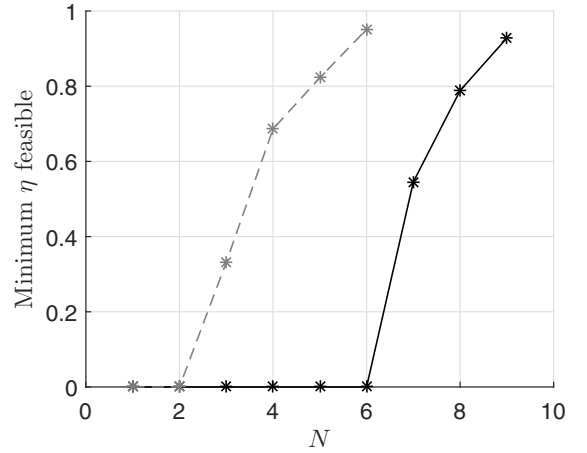

Fig. 4. Minimum η feasible for the SATI controller (black line) and the LQR controller (dashed grey line).

interesting behaviors can be noted, the first one is that for both controllers the system is still MSS for $\eta = 0$ until some N, that is precisely the case where no communication is attempted (lower values of η means that fewer power resources are used to communicate). Above this limit, it is necessary to increase the cumulative probability η in order to get the MSS of the WNCS, where the maximum value of N in which the WNCS is still MSS is given by $N_{max} = 9$ for the SATI controller and $N_{max} = 6$ for the LQR controller. Thus, the control given by the LQR problem for this choice of Q and R performs worse in terms of the maximum length N than the one calculated via Theorem 2. This is intuitive, as the conditions shown in Theorem 2 are optimized for the $(\eta, \delta)-$SATI restrictions. Note that Q and R could be chosen in order to improve

the behavior in terms of N and η, even though there is no systematic procedures to tune Q and R for a given SATI. Besides the SATI controller can achieve greater values of N and smaller values of η compared to the ones used in the design

We also briefly discuss the behavior of the closed-loop states over the time in a network modeled by a SATI policy. By calculating a controller via Theorem 2 with $N = 1$, $\delta = 0.2$, and $\eta = 0$, the closed-loop system will be stable for a SATI of $\delta = 0.2$ and $\eta = 0$ from $N = 1$ up to $N = 13$. Conversely, a controller obtained with the same values of δ and η, but for $N = 14$ would guarantee the stability from $N = 1$ up to $N = 42$.

5. CONCLUSION

In this work, we have extended the results of Varma et al. (2017), which deals with stabilization of linear WNCS using emulated controllers under conditions on the SATI co-design, to co-design. For that purpose, we have reworked the stability analysis in Varma et al. (2017) and then derived suitable linear matrix inequalities for analyzing the stability of the closed-loop system. The latter may then be used to construct a mean-square stabilizable controller given constraints on the SATI.

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