Observability of Linear Hybrid Systems with unknown inputs and discrete dynamics modeled by Petri nets.

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Abstract: This work deals with the observability analysis for LHS's considering both known and unknown inputs and constrained discrete dynamics, modeled by Petri nets. For this, the concept of *eventual observability* is recalled as the possibility of uniquely determining both the discrete and the continuous states after a finite number of switchings. In this way, the information provided by the continuous and the discrete outputs of the LHS can be combined to determine the discrete state after a finite number of switchings. Next, based on the knowledge of the visited locations, a continuous observer can estimate the continuous state. It is shown that under this approach the observability conditions are greatly relaxed with respect to other approaches in the literature, in particular, neither the observability of the linear systems nor the observability of the underlying discrete event system are required.

Keywords: Hybrid systems; Observers; Petri nets.

1. INTRODUCTION

A Linear Hybrid System (LHS) can be defined as a collection of Linear Systems (LS's) and a switching signal determining the LS structure that rules the behavior of the LHS. The switching signal is a function of the time that takes discrete values, thus the LHS has a continuous state and a discrete state. There are interesting applications in which the LHS is not allowed to commute from a given LS to any other in the collection. We refer to this property as *constrained discrete dynamics*. For instance, in process systems it frequently occurs that discrete actuators, leading to different LS's, must be activated according to certain sequence (e.g., Balluchi et al. (2005)). Nonlinear models are frequently approximated by LS's operating at different operation points, leading to autonomous switchings (e.g., Song et al. (2015)) where a LS can only switch to its neighbors. In traffic systems in urban areas, when considering a fluid flow approach for the traffic behavior, traffic lights lead to different LS's (e.g., Vázquez et al. (2010)) that are commuted according to a predefined light sequence.

The observability is a fundamental property of any dynamic system, defined as the possibility to recover its internal state from the knowledge of its dynamic model, external observations and the known applied inputs. The observability in LHS's has been studied considering different assumptions and approaches. For instance, in Sun et al. (2002) and Martínez-Martínez et al. (2014) the switching signal is designed to improve the observability. In Tanwani et al. (2011, 2013) it is assumed that the switching signal is known, thus the continuous state is estimated by the combination of partial observations obtained when visiting each LS. In Vidal et al. (2003); Santis et al. (2003); Gómez-Gutiérrez et al. (2012) the switching signal is unknown, but the switching from any LS to any other LS is allowed. It is known that without any information about the switching sequence, for observability it is required that each LS is *observable* and each pair of LS's is *distinguishable* (Vidal et al. (2003); De Santis (2011); Gómez-Gutiérrez et al. (2012)), i.e., two different LS's must not produce the same output trajectory in order to be able to determine the discrete state. Another studied assumption is the existence of unknown inputs, which can represent faults, state perturbations and parametric variations. This assumption was considered in Gómez-Gutiérrez et al. (2012).

Most of the observability and observer synthesis works consider that the LHS can switch from any LS to any other LS. There are few works in the literature addressing the case in which the switching is constrained. In Arichi et al. (2014) a LHS is defined such that the switching signal is generated by a Petri net (PN), assuming that the continuous systems are observable and the PN is observable after each switching. In Balluchi et al. (2002) the observer synthesis problem is addressed considering that the switching signal is produced by a finite automaton. There, the discrete location is determined by computing residuals from Luenberger observers. In Petreczky and van Schuppen (2010), the observability of LHS where the discrete dynamic is determined by a Moore automaton is addressed. There, each pair of states with the same discrete

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output are required to be distinguishable, which requires their associate LS to be observable.

In this work the observability analysis for LHS is addressed, considering both known and unknown inputs and constrained discrete dynamics, modeled by PNs. This work is an extension to Vázquez et al. (2017), in which the observability and observer design for *autonomous* LHS was considered. For this, the concept of *eventual observability* is recalled as the possibility of uniquely determining both the discrete and the continuous states after a finite number of switchings. The information provided by the continuous and the discrete outputs of the LHS is combined to determine the discrete state after a finite number of switchings. Next, based on the knowledge of the visited locations, the continuous state can be reconstructed. This approach greatly relaxes the observability conditions, since neither the observability of the LS's nor the observability of the PN are required.

This paper is organized as follows. In Section 2 some basic concepts about LHS's and PN's are provided. In Section 3, the eventual observability problem is defined and the proposed approach is introduced. Section 4 introduces the observability analysis of LHS's, presenting Theorem 5, in which sufficient conditions of eventual observability of LHS's are provided. The application of this result is illustrated trough an example. Finally, some conclusions are presented in Section 5.

2. BASIC CONCEPTS AND DEFINITIONS

In the sequel, given a vector a, its j-th entry is denoted as $a[j]$. Similarly, given a matrix A, $A[i,j]$ denotes its entry at the i-th row and j-th column.

2.1 Petri nets

Definition 1. An Interpreted Petri net (IPN) system is a discrete event system described by the tuple $\langle \mathcal{N}, M_0, \Sigma_o, \Phi \rangle$, where $\mathcal{N}$ is a PN structure, i.e., a bipartite directed graph $\mathcal{N} = \langle P, T, Pre, Post \rangle$ where places (P) and transitions (T) are disjoint sets of nodes and $Post$ and Pre are $|P| \times |T|$ natural valued incidence matrices that describe the arcs from nodes in T to nodes in P, and from nodes in P to nodes in T, respectively. $M_0 \in \mathbb{N}^{|P|}$ is a vector, named *initial marking*, that encodes the number of marks or tokens that initially reside in the net places, Σ_o is an alphabet of observable label types and $\Phi : P \rightarrow 2^{\Sigma_o}$ is a mapping that associates a subset of labels to each place. A label type can be associated to different places. The *state* in a PN system is defined as the distribution of marks in the places, codified in the *marking vector* $M \in \mathbb{N}^{|P|}$. The evolution of the IPN system is described as follows:

(1) A transition $t_i \in T$ is said to be *enabled* at the marking $M \in \mathbb{N}^{|P|}$ if $M[j] \geq Pre[j,i]\ \forall p_j \in P$.
(2) The occurrence or *firing* of an enabled transition t_i leads to a new marking distribution $M' \in \mathbb{N}^{|P|}$ that is computed by using $M' = M + W \cdot e_i$, where $W = Post - Pre$ is named the *incidence matrix* and e_i denotes the i-th column vector of the identity matrix of dimension $|T|$.
(3) If there is a token at place $p_j \in P$ and the set of labels $\{l_i, ..., l_k\} \in 2^{\Sigma_o}$ are associated to p_j then all the labels $l_i, .., l_k$ will be *concurrently* observed.

Here we consider strongly connected safe *state machine* PN's, i.e., for each pair of nodes $a, b \in P \cup T$ there is a directed path from a to b, there will be only one token in the net and each transition has one input arc and one output arc of unitary weight. Consequently, the IPN output can be characterized by a matrix Φ defined as: $\Phi[i,j] = 1$ if $l_i \in \Sigma_o$ is associated to $p_j \in P$, and $\Phi[i,j] = 0$ otherwise. Thus, the discrete output vector y_d can be computed as

$$y_d = \Phi \cdot M$$

where $y_d[i] = 1$ iff the label l_i is observed, otherwise $y_d[i] = 0$. Thus, observing labels is equivalent to observing y_d. Nevertheless, notice that observing labels does not necessarily allow the determination of the marking, for instance, if all the places are associated to the unique label type l_1, an external observer will observe l_1 at any time, thus being unable to determine the current marking.

A sequence of transitions $\sigma = t_1...t_k$ is said to be a *fireable* sequence from a marking M_1 if there are markings $M_2, ...M_{k+1}$ such that t_i is enabled at M_i and its firing leads to the marking M_{i+1}. The *length* of a sequence σ, denoted as $|\sigma|$, is defined as the number of transition firings in σ. The marking reached after the firing of σ is given by the so called *fundamental equation*

$$M' = M + W \cdot \boldsymbol{\sigma} \tag{1}$$

where $\boldsymbol{\sigma} = \sum_{i \in \{1,...,k\}} e_i$ is called the *firing count vector* of σ (notice σ is a sequence while $\boldsymbol{\sigma}$ is a vector). In the sequel, the column of W related to the transition t_i is denoted as $W(\bullet, t_i)$. Given a transition $t_i \in T$, the pre-set of t_i is defined as ${}^\bullet t_i = \{p_j \in P | Pre[j,i] \neq 0\}$. Two transitions t_i and t_j are said to be in *conflict* relation if ${}^\bullet t_i = {}^\bullet t_j$, in such case, $\{p\} = {}^\bullet t_i$ is said to be a *choice place.*

Definition 2. Given a PN, a *T-semiflow* is a vector $X \geq \mathbf{0}$ (comparison is component-wise) such that $W \cdot X = \mathbf{0}$ and $X \neq \mathbf{0}$. The PN is said to be *consistent* if there exists a T-semiflow $X > \mathbf{0}$. The firing of a sequence σ_X, such that $\boldsymbol{\sigma_X}$ is a T-semiflow, will lead to the same marking. In such case, we say that σ_X *describes* the T-semiflow X. A T-semiflow X is said to be *minimal* if there does not exist another T-semiflow such that $Y \leq X$.

In strongly connected safe state machines, the length of any sequence describing a minimal T-semiflow is lower or equal than $|T|$. See Colom and Silva (1987) for computation methods of minimal T-semiflows.

2.2 Linear Hybrid Systems

Definition 3. A Linear Hybrid System (LHS) $\Sigma_{\alpha(\tau)}$ is a collection of linear systems (LS's) $\mathcal{F} = \{\Sigma_1, \ldots, \Sigma_m\}$, each one defined in the state space $\mathcal{X} = \mathbb{R}^n$, and a switching signal $\alpha(\tau)$, taking values in $\{1, \ldots, m\}$, that determines the active linear system. $\Sigma_{\alpha(\tau)}$ evolves according to

$$
\begin{aligned}
&\dot{x}(\tau) = A_{\alpha(\tau)}x(\tau) + B_{\alpha(\tau)}u(\tau) + F_{\alpha(\tau)}v(\tau), \quad x(\tau_0) = x_0 \\
&y(\tau) = C_{\alpha(\tau)}x(\tau) \qquad (2)\\
&\alpha(\tau) \in \{1, \dots, m\}.
\end{aligned}
$$

where $u \in \mathbb{R}^q$ is the known input, $v \in \mathbb{R}^r$ is an unknown input (disturbance) and $y \in \mathbb{R}^s$ is the continuous output. The evolving LS at a mode $i \in \{1, \dots, m\}$, i.e. $\alpha(\tau) = i$, is represented by $\Sigma_i(A_i, B_i, F_i, C_i)$ or simply by Σ_i, where A_i, B_i, F_i and C_i are matrices of appropriate dimensions. The initial mode is assumed to be fixed but unknown. Neither resets nor state-jumps are considered in this work, thus, if a switching occurs at time τ_i then $x(\tau_i^-) = x(\tau_i^+)$. Here, $\alpha(\tau)$ is generated by a strongly connected safe state machine *IPN*, as in Definition 1, having m places and one token at the initial marking. Thus, $\alpha(\tau) = i$ if there is a token at place p_i at time τ.

The choice for a state machine IPN for the discrete dynamics allows us to use existing results for the observability analysis. Nevertheless, the analysis herein presented can be easily extended to consider any other class of bounded IPN or Moore automata.

2.3 Observability of LS's with unknown inputs

The observability of a LS under unknown inputs was investigated in Basile and Marro (1969) from a geometrical perspective. Let us recall the main result.

Theorem 1. Basile and Marro (1969) The observability subspace of a LS $\Sigma = (A, 0, F, C)$ with unknown inputs v is the least $(A^T, (Im(F))^\perp)$-conditioned invariant containing $Im(C^T)$, which can be recursively computed as:

$$
\begin{aligned}
&\mathcal{Y}_0 = Im(C^T) \\
&\mathcal{Y}_k = Im(C^T) + A^T(\mathcal{Y}_{k-1} \cap (Im(F))^\perp), \; k = 1, ..., n-1.
\end{aligned} \qquad (3)
$$

By duality, the unobservable subspace of $\Sigma = (A, 0, F, C)$ is the greatest $(A, Im(F))$-controlled invariant contained in $ker(C)$, which can be iteratively computed as follows:

$$
\begin{aligned}
&\mathcal{V}^0 = kerC \\
&\mathcal{V}^k = kerC \cap A^{-1}\left(Im(F) + \mathcal{V}^{k-1}\right), \; k = 1, ..., n-1.
\end{aligned} \qquad (4)
$$

2.4 Distinguishability concepts

An important concept in the observability of LHS is the distinguishability property (Vidal et al. (2003); Gómez-Gutiérrez et al. (2012)) which deals with determining the currently evolving LS from the continuous output.

Definition 4. Denote as $u_{[0,\tau]}$ and $v_{[0,\tau]}$ the known input and the unknown input trajectories during a proper time interval $[0, \tau]$, respectively. A linear system Σ_i is said to be distinguishable from Σ_j if

$$
\begin{aligned}
&\forall x_0, \, u_{[0,\tau]}, \, v_{[0,\tau]}, \qquad \nexists x_0', \, v_{[0,\tau]}' \text{ such that} \\
&y_i(x_0, u_{[0,\tau]}, v_{[0,\tau]}) = y_j(x_0', u_{[0,\tau]}, v_{[0,\tau]}'),
\end{aligned} \qquad (5)
$$

where $y_i(x_0, u_{[0,\tau]}, v_{[0,\tau]})$ denotes the output trajectory during $[0, \tau]$, described by the evolution of the LS Σ_i from the initial state x_0, when the known and unknown inputs $u_{[0,\tau]}$ and $v_{[0,\tau]}$ are applied. If condition (5) is not fulfilled then Σ_i is said to be *indistinguishable* from Σ_j.

It may occur that a LS Σ_i is distinguishable from another Σ_j but Σ_j is indistinguishable from Σ_i. In such case, if the LHS is evolving at Σ_i we can determine from the known input and output measures that the LHS is not at Σ_j, on the contrary, if the LHS is at Σ_j we cannot determine that the system is not at Σ_i. The following distinguishability result is recalled from Gómez-Gutiérrez et al. (2012).

Proposition 2. Consider two LS's $\Sigma_i = (A_i, B_i, F_i, C_i)$ and $\Sigma_j = (A_j, B_j, F_j, C_j)$. Define $A_{ij} = diag(A_i, A_j)$, $C_{ij} = [Ci, -C_j]$, $\mathcal{B}_{ij} = [B_i^T \; B_j^T]^T$, $\mathcal{F}_i = [F_i^T \; \mathbf{0}]^T$ and $\mathcal{F}_j = [\mathbf{0} \; F_j^T]^T$. Let W_{ij} be the indistinguishable subspace of Σ_i and Σ_j, computed as the greatest $(A_{ij}, \mathcal{B}_{ij})$-controlled invariant contained in $ker(C_{ij})$ and let $\mathcal{V}$ be the greatest $(A_{ij}, \mathcal{F}_j)$-controlled invariant contained in $ker(C_{ij})$. Let $Q : \mathbb{R}^{2n} \to \mathbb{R}^n$ be a natural projection. The system Σ_i is distinguishable from Σ_j for almost every initial state iff one of the following conditions hold:

(1) $dim(QW_{ij}) < dim(\mathcal{X})$
(2) $Im[B_{ij}, \mathcal{F}_i] \not\subseteq \mathcal{V} + \mathcal{F}_j$

The first condition implies that for almost any initial condition $x_0 \notin QW_{ij}$ and then the LS Σ_i can be distinguished from Σ_j. This condition is referred as *generic* (in terms of algebraic geometry, the exceptional set is an algebraic set defined as an affine variety of lower dimension). The second condition states that the combination of the known input $u(\tau)$ and the unknown input $v(\tau)$ does not lie neither in the indistinguishable subspace nor the image of $\mathcal{F}_j$, thus, the inputs make possible to distinguish Σ_i from Σ_j.

If the initial condition is known, Proposition 2 can be used with W_{ij}^c instead of W_{ij} in condition (1), where

$$
W_{ij}^c = Im\left(\begin{bmatrix} I \\ I \end{bmatrix}\right) \cap W_{ij} \qquad (6)
$$

3. OBSERVABILITY APPROACH

Let us introduce the observability notion considered in this work, adapted from Vázquez et al. (2017).

Definition 5. A Linear Hybrid System $\Sigma_{\alpha(\tau)}$ is said to be *eventually-observable* if there exists a finite integer k such that, after the occurrence of k switchings from the initial state, the current continuous state x and the current marking of the underlying IPN M can be exactly determined by using the knowledge of the continuous output $y(\tau)$, the known continuous input $u(\tau)$ and the discrete output $y_d(\tau)$ during the evolution of the system (i.e., from the initial time until the k-th switching), and the continuous and discrete states can be exactly determined for future time.

For the observability analysis, we will consider the strategy proposed in Vázquez et al. (2017) for autonomous systems, which consists of the following steps:

I. The distinguishability of the LS's is captured as labels that are added to the places of the IPN.
II. The observability of the IPN is investigated. If it is observable then the discrete state can be uniquely determined in a finite number of switchings, by detecting the occurrence of a minimal T-semiflow.

III. The possibility to determine the continuous state is analyzed, assuming the switching sequence is known, for any sequence describing a minimal T-semiflow.
IV. The possibility to track both the discrete and continuous states for further time is analyzed.

In the sequel, the following assumptions will be made.

(1) The firing of each transition produces a change in the output, i.e., $\Phi W(\bullet, t_i) \neq \mathbf{0}$, $\forall t_i \in T$.
(2) The firing of the transitions of the IPN are neither controllable nor known (measured).
(3) There are no continuous state jumps at switchings, i.e., for each switching time τ it holds $x(\tau^-) = x(\tau^+)$.

4. OBSERVABILITY ANALYSIS

4.1 Labeling of distinguishable linear systems

In the following algorithm, new labels are added to the places based on the distinguishability between pairs of LS's with unknown input.

Algorithm 1: **Labelling I**

(1) Initialize the distinguishability matrix D as a $|P|\times|P|$ matrix with null entries.
(2) For any $p_i \in P$ do:
  (a) For any $p_j \in P$ do:
    (i) If p_i is distinguishable from p_j according to Proposition 2, then set $D[i,j] = 0$, otherwise set $D[i,j] = 1$.
(3) The augmented output matrix for the IPN is:

$$\Phi_I = \begin{bmatrix} \Phi \\ D^T \end{bmatrix} \tag{7}$$

The previous algorithm adds $|P|$ new label types or outputs. For instance, suppose that the LHS is evolving at Σ_i, which is distinguishable from any other Σ_j in the collection, thus $D[i,j] = 0$ and $(D^T)[j,i] = 0$. On the other hand, $D[i,i] = 1$ by definition. Now, since the LHS is evolving at Σ_i then M has only one mark at p_i. Therefore, $(D^T \cdot M)[j] = 0$ $\forall j \neq i$ and $(D^T \cdot M)[i] = 1$, i.e., only the $i - th$ new output is 1, while the others are 0, thus it is possible to determine the marking from the new outputs when p_i is marked. This agrees with the possibility to determine that Σ_i is the active LS based on the continuous output and the distinguishability of the LS's.

In other case, if Σ_i is indistinguishable from a LS Σ_k, thus $D[i,k] = 1$, $(D^T)[k,i] = 1$ and then $(D^T \cdot M)[k] = 1$, i.e., the $k - th$ new output is 1. Moreover, $(D^T \cdot M)[i] = 1$, meaning that it may be possible that either p_k or p_i is marked. Of course, since the LHS is evolving at Σ_i and this LS is indistinguishable from Σ_k it is expected that, by only observing the continuous output, we cannot determine which one of them is active, we just can say that either Σ_i or Σ_k is active.

4.2 Estimation of the discrete state

Once the IPN is enriched with additional labels, the following step is to investigate the observability of the IPN. This property was studied in Nuño-Sánchez et al. (2015). From there, the following result can be derived:

Corollary 3. Let $< \mathcal{N}, M_0, \Phi >$ be a live, binary and pure state machine. Assume that the firing of any transition makes a change at the output, i.e., $\Phi W(\bullet, t_i) \neq \mathbf{0}$, $\forall t_i \in T$. After a finite number of firings, the marking can be inferred from the output iff

(1) *Distinguishability of sequences describing minimal T-semiflows:* For each pair of firing sequences $\sigma_1 = t_1^1, t_2^1 ... t_h^1$ and $\sigma_2 = t_1^2, t_2^2 ... t_h^2$ that describe minimal T-semiflows, it holds $\Phi W(\bullet, t_j^1) \neq \Phi W(\bullet, t_j^2)$ for some $j \in \{1, .., h\}$.
(2) *Preservation of the marking estimation at choices:* $\forall t_i, t_j \in T$ with ${}^\bullet t_i = {}^\bullet t_j$ it holds $\Phi W(\bullet, t_i) \neq \Phi W(\bullet, t_j)$.

Condition 1) implies that, after a finite number of firings, the current marking can be uniquely computed (any infinite sequence must describe T-semiflows, thus if the firing of T-semiflows produce different outputs then the marking can be computed at some moment). Condition 2) implies that, once the current marking is computed, it will be possible to track the token trajectory (notice that if two transitions t_i and t_j are in conflict and produce the same output change, their firings will be confused).

In our case, the information provided by the continuous and discrete outputs will be combined in order to estimate the marking. This will be achieved by considering Φ_I instead of Φ, where Φ_I is the output matrix obtained from the algorithm *Labelling I.*

4.3 Estimation of the continuous state

Once the marking is determined, it is required to estimate the continuous state. In this work, each LS may be unobservable, then, the information provided by the continuous output when the LHS is switching through different LS's must be collected and combined for the continuous state estimation. The following proposition gives conditions for the continuous state estimation by computing the unobservable subspace through a switching sequence.

Proposition 4. Consider a LHS as in Definition 3. Let $\tau_1, \ldots, \tau_j$ be a sequence of switching times such that $\tau_1 < \ldots < \tau_k$ and $\alpha(\tau) = q_i \in \{1, \ldots, m\}$ for all $\tau \in [\tau_{i-1}, \tau_i)$, i.e. the switching signal $\alpha(\tau)$ produces the mode sequence $q_1, \ldots, q_k$. Let $\bar{\mathcal{V}}_0^k$ be the unobservable subspace through the sequence recursively computed by the Algorithm 2. The continuous state of the LHS can be uniquely computed for the switching signal $\alpha(\tau)$ if $\bar{\mathcal{V}}_0^k = \{0\}$.

Algorithm 2 : **Unobservable subspace through a sequence $\alpha = q_1, ..., q_k$.**

(1) Initialize: $\bar{\mathcal{V}}_0^0 := \mathbb{R}^n$.
(2) For $i = 1$ to k do:
  (a) Initialize: $\mathcal{W}_i^0 := ker(C_{q_i}) \cap \bar{\mathcal{V}}_0^{i-1}$
  (b) For $j = 1$ to $n-1$ do:
    (i) Compute: $\mathcal{W}_i^j := ker(C_{q_i}) \cap A_{q_i}^{-1}(Im(F_{q_i}) + \mathcal{W}_i^{j-1})$
  (c) Define: $\bar{\mathcal{V}}_0^i = \mathcal{W}_i^{n-1}$

Proposition 4 must be verified for each firing (mode) sequence that describes a minimal T-semiflow.

4.4 Tracking the state

Once the marking and the continuous state are estimated, it is required to keep track of both. The marking estimation can only be lost after visiting a choice place. As in the first stage, the continuous output can be used to determine the marking immediately after visiting a choice place, by adding labels to distinguishable places as in the algorithm *Labelling I*, but in this case the continuous state is known before the last switching. This is considered in the following labelling algorithm:

Algorithm 3: **Labelling II**

(1) Initialize the distinguishability matrix for conflicts D_c as $|P| \times |P|$ matrix with null entries.
(2) For any $p_i \in P$ do:
    (a) For any $p_j \in P$ do:
        (i) If p_i is distinguishable from p_j with known input, i.e., according to Proposition 2 with W_{ij}^c instead of W_{ij}, then set $D_c[i,j] = 0$, otherwise set $D_c[i,j] = 1$.
(3) The augmented output matrix for the IPN is:

$$\Phi_{II} = \begin{bmatrix} \Phi \\ D_c^T \end{bmatrix} \tag{8}$$

Therefore, to maintain the marking estimation, for each choice place p_o it must hold $\forall t_i, t_j \in p_o^\bullet$, $\Phi_{II} W(\bullet, t_i) \neq \Phi_{II} W(\bullet, t_j)$, where Φ_{II} is the output matrix of the IPN obtained after the algorithm *Labelling II*. Additionally, to maintain the continuous state estimation despite the presence of the unknown inputs, it is required that, in each LS, the unknown input does not affect the unobservable subspace, i.e., $Im(F_i) \cap \mathcal{V}_i = \{0\}$, where $\mathcal{V}_i$ is the unobservable subspace of Σ_i as described in Theorem 1.

4.5 Observability conditions

The previous discussion results in sufficient conditions for the eventual-observability of LHS's with constrained discrete dynamics and unknown inputs.

Theorem 5. Consider a LHS. Consider the assumptions enumerated in Subsection 3. The system is *eventually-observable* (Definition 5) if the following conditions hold:

(1) *Distinguishability of sequences describing minimal T-semiflows:* Given two firing sequences $\sigma_1 = t_1^1, t_2^1 ... t_h^1$ and $\sigma_2 = t_1^2, t_2^2 ... t_h^2$ that describe minimal T-semiflows (i.e., $W\boldsymbol{\sigma_1} = W\boldsymbol{\sigma_2} = \mathbf{0}$), it holds $\Phi_I W(\bullet, t_j^1) \neq \Phi_I W(\bullet, t_j^2)$ for some $j \in \{1,..,h\}$, where Φ_I is the output matrix of the IPN obtained after applying algorithm *Labelling I*.
(2) *Unique estimation of the continuous state:* Proposition 4 holds for any mode sequence resulting from a firing sequence describing a minimal T-semiflow.
(3) *Preservation of the discrete state estimation:* $\forall t_i, t_j \in T$ with ${}^\bullet t_i = {}^\bullet t_j$, $\Phi_{II} C(\bullet, t_i) \neq \Phi_{II} C(\bullet, t_j)$, where Φ_{II} is the output matrix of the IPN obtained after algorithm *Labelling II*.
(4) *Preservation of the continuous state estimation:* for every LS, $Im(F_i) \cap \mathcal{V}_i = \{0\}$, where $\mathcal{V}_i$ is the unobservable subspace as described in Theorem 1.

4.6 Example

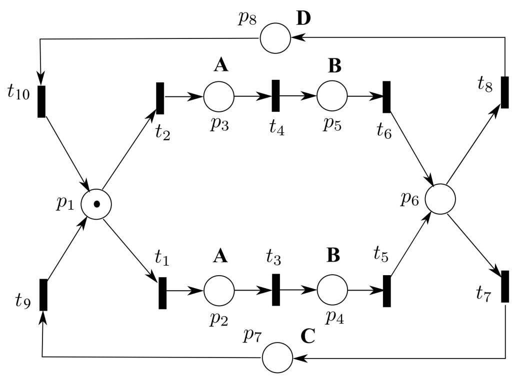

Fig. 1. Underlying IPN of the LHS. The firing sequences $t_1 t_3 t_5 t_7 t_9$ and $t_2 t_4 t_6 t_7 t_9$ are indistinguishable.

Consider a LHS that consists of the IPN of fig. 1, with label types $\{A, B, C, D\}$, and the collection of the following LS's (each place p_i is associated to the LS Σ_i).

	A_i	B_i	F_i	C_i
Σ_1	$\begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$
Σ_2	$\begin{bmatrix} 0.2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -0.1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$
Σ_3	$\begin{bmatrix} 0.2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -0.1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0.5 \\ -1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 2 & 1 & 0 \end{bmatrix}$
Σ_4	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$
Σ_5	$\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -0.01 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$
Σ_6	$\begin{bmatrix} -0.25 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -0.01 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$
Σ_7	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -0.1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$
Σ_8	A_7	B_7	F_7	C_7

Let us firstly investigate the observability of the IPN. The following sequences and their rotations describe all the minimal T-semifows: $\sigma_1 = t_1 t_3 t_5 t_7 t_9$, $\sigma_2 = t_2 t_4 t_6 t_7 t_9$, $\sigma_3 = t_1 t_3 t_5 t_8 t_{10}$ and $\sigma_4 = t_2 t_4 t_6 t_8 t_{10}$. The sequences σ_1 and σ_3 produce the same output trajectories that σ_2 and σ_4, respectively. Furthermore, the firings of the transitions in the conflict $\{t_1, t_2\}$ produce the same output change. Thus, the IPN of fig. 1 is unobservable.

On the other hand, almost all the LS's are indistinguishable between them, excepting Σ_5 and Σ_6 in which $F_i = \mathbf{0}$. Consequently, it is not possible to determine the active

LS by only using the continuous output information. Even more, none of the LS's is observable. The distinguishability relations of the LS's are described by the following distinguishability matrices.

$$D = \begin{bmatrix} 1&1&1&1&0&0&1&1 \\ 1&1&1&1&0&0&1&1 \\ 1&1&1&1&0&0&1&1 \\ 1&1&1&1&0&0&1&1 \\ 1&1&1&1&1&0&1&1 \\ 1&1&1&1&0&1&1&1 \\ 1&1&1&1&0&0&1&1 \\ 1&1&1&1&0&0&1&1 \end{bmatrix}, \quad D_c = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&1&1 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&1&1 \\ 1&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&1&1 \end{bmatrix}.$$

Now, let us demonstrate that the LHS is observable. First, the algorithm *Labelling I* results in $\Phi_I = [\Phi^T \, D^T]$. The firing of all the sequences that describe minimal T-semiflows provide different output trajectories with Φ_I, thus the *first condition* of the Theorem 5 holds. On the other hand, the unobservable subspace of the mode sequences $\Gamma_1 = \Sigma_1\Sigma_2\Sigma_4\Sigma_6\Sigma_7$, $\Gamma_2 = \Sigma_1\Sigma_3\Sigma_5\Sigma_6\Sigma_7$, $\Gamma_3 = \Sigma_1\Sigma_2\Sigma_4\Sigma_6\Sigma_8$ and $\Gamma_4 = \Sigma_1\Sigma_3\Sigma_5\Sigma_6\Sigma_8$, associated to the firing sequences σ_1, σ_2, σ_3 and σ_4, was computed with the Algorithm 2, obtaining $\overline{\mathcal{V}}_0^5 = \{0\}$ for all the cases. Thus, the *second condition* is verified. Next, let us analyze the conflicts. The firing of any transitions in the conflict t_7t_8 can be distinguished from the original discrete output. On the other hand, for the conflict t_1t_2, the algorithm *Labelling II* is applied obtaining $\Phi_{II} = [\Phi \, D_c^T]$, with which the firings of t_1 and t_2 produce different output changes. Thus, the *third condition* is verified. Finally, in order to verify the fourth condition, the unobservable subspaces of the LS's are computed as described in Theorem 1, obtaining that $Im(F_i) \cap \mathcal{V}_i = \{0\}$ for all the LS's. Then, the *fourth condition* is verified. Therefore, the LHS is eventually observable by combining the discrete and continuous outputs information.

5. CONCLUSIONS

In this work, sufficient conditions are provided for the eventual observability of LHS in which the discrete dynamics are constrained by a PN and both known and unknown continuous inputs are applied to the LSs. In the proposed approach, the continuous and discrete outputs information can be combined to determine the discrete location. After that, the continuous state can be reconstructed by partial observations in the visited LS's. It is shown through an example that a LHS can be observable even if the LSs and the PN are unobservable.

REFERENCES

Arichi, F., van Gorp, J., Djemai, M., Defoort, M., and Cherki, B. (2014). Hybrid state estimation for a class of switched system using Petri nets. In *European Control Conference*, 2534–2539.

Balluchi, A., Benvenuti, L., Benedetto, M.D.D., and Sangiovanni-Vincentelli, A.L. (2002). Design of observers for hybrid systems. In *Hybrid systems: Computation and Control*, 76–89.

Balluchi, A., Benvenutti, L., Engell, S., Geyer, T., Johansson, K.H., Lamnabhi-Lagarrigue, F., Lygeros, J., Morari, M., Papafotiou, G., Sangiovanni-Vicentelli, A.L., Santucci, F., and Stursberg, O. (2005). Hybrid control of networked embedded systems. *European Journal of Control*, 11, 478–508.

Basile, G. and Marro, G. (1969). On the observability of linear, time-invariant systems with unknown inputs. *Journal of optimization theory and applications*, 3, 410–415.

Colom, J.M. and Silva, M. (1987). Convex geometry and semiflows in P/T nets. In K. Voss, H. Genrich, and G. Rozenberg (eds.), *Concurrency and Nets, Advances in Petri Nets*, 435–460. Springer-Verlag, Berlin.

De Santis, E. (2011). On location observability notions for switching systems. *Systems & Control Letters*, 60(10), 807–814.

Gómez-Gutiérrez, D., Ramírez-Treviño, A., Ruiz-León, J., and Di Gennaro, S. (2012). On the observability of continuous-time switched linear systems under partially unknown inputs. *IEEE Transactions on Automatic Control*, 57, 732–738.

Martínez-Martínez, S., Messai, N., Hamelin, F., Manamanni, N., and Boukhobza, T. (2014). Graphic approach for the determination of the existence of sequences guaranteeing observability of switched linear systems. *Automatica*, 50, 584–590.

Nuño-Sánchez, S.A., Ramírez-Treviño, A., and Ruiz-León, J. (2015). Structural sequence detectability in free choice interpreted Petri nets. *IEEE Transactions on Automatic Control*, 16, 198–203.

Petreczky, M. and van Schuppen, J.H. (2010). Realization theory for linear hybrid systems. *IEEE Transactions on Automatic Control*, 55(10), 2282–2297.

Santis, E.D., Di Benedetto, M.D., and Pola, G. (2003). On observability and detectability of continuous-time linear switching systems. In *42nd IEEE Conference on Decisions and Control*, 5777–5782.

Song, C., Wu, B., Zhao, J., and Li, P. (2015). An integrated state space partition and optimal control method of multi-model for nonlinear systems based on hybrid systems. *Journal of Process Control*, 25, 59–69.

Sun, Z., Ge, S.S., and Lee, T.H. (2002). Controllability and reachability criteria for switched linear systems. *Automatica*, 38, 775–786.

Tanwani, A., Shim, H., and Liberzon, D. (2011). Observability implies observer design for switched linear systems. In *Hybrid systems: Computation and Control*, 3–12.

Tanwani, A., Shim, H., and Liberzon, D. (2013). Observability for switched linear systems: Characterization and observer design. *IEEE Transactions on Automatic Control*, 58, 891–904.

Vázquez, C.R., Gómez-Gutiérrez, D., and Ramírez-Treviño, A. (2017). Observer synthesis for linear hybrid systems with constrained discrete dynamics. *Nonlinear Analysis: Hybrid Systems*, 26, 254–273.

Vázquez, C.R., Sutarto, H.Y., Boel, R., and Silva, M. (2010). Hybrid Petri net model of a traffic intersection in an urban network. In *IEEE International Conference on Control Applications*, 658–664.

Vidal, R., Chiuso, A., Soatto, S., and Sastry, S. (2003). Observability of linear hybrid systems. In *Hybrid systems: Computation and Control*, 526–539.