Some ideas on cut-elimination for cyclic arithmetic proofs

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Cyclic arithmetic, proposed by Simpson in [6],[1] is a deduction system based in the language of arithmetic where proofs may be non-wellfounded, as opposed to usual approaches to infinitary proof theory via an omega-rule. Naturally, some form of correctness condition must be imposed to ensure sound reasoning, and this is implemented by a *trace condition* at the level of the 'flow graph' of the proof (cf. [2]). *Cyclic arithmetic* (CA) itself consists of such proofs that are regular, i.e. that have only finitely many distinct subtrees, and so may be expressed as a finite directed graph (with cycles). It was independently shown in [6] and [1] that CA and PA are equivalent, and recently by the present author that logical complexity in the two theories is similar [3].

We consider the issue of cut-elimination for CA. Such a procedure cannot preserve regularity of proofs, so the issue is to show that cut-elimination is *productive*. To this end, *continuous* cut-elimination procedures have long been studied in the proof theory of arithmetic, originating from Mints' seminal article [5]. However the difficulties arising from the *repetition* rule, ensuring continuity, and the need to preserve trace conditions seems to warrant an alternative approach.

In this work-in-progress, we show how cut-elimination can be similarly achieved by a certain reduction to finitary cut-elimination. We compute certain *runs* through a non-wellfounded proof which must be finite thanks to the trace condition, and show that these are preserved in the image of cut-elimination. Productivity follows since cut-free runs must be non-empty, and validity follows by the finiteness of runs.

The computation of runs, naively, makes use of a semantic oracle, though we believe that this can be replaced by purely syntactic concepts via a *geometry of interaction* approach to cut-elimination, cf. [4]. This would yield a novel proof of the consistency of PA indirectly via CA.

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[1] In fact, while [6] is the first publication on cylic arithmetic, Simpson proposed it and already had his main equivalence result in 2012, cf. `https://homepages.inf.ed.ac.uk/als/Talks/collog12.pdf`.

References

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