Admissible tools in the kitchen of intuitionistic logic

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1 Introduction

Axiomatic proof systems are presented by giving axioms and rules of inference, which are respectively the ingredients and the tools for cooking new proofs. For example, when presenting *classical propositional logic* (**CPC**) in *natural deduction*, for each of the usual connectives $\wedge, \vee, \neg, \rightarrow, \bot$ one gives a set of standard tools to introduce or remove that connective from a formula in order to obtain a proof.

Rules have the form $\varphi_1, \ldots, \varphi_n / \psi$ (read "from $\varphi_1, \ldots, \varphi_n$ infer ψ") where $\varphi_1, \ldots, \varphi_n, \psi$ are schemata of logic formulas. A rule $\varphi_1, \ldots, \varphi_n / \psi$ is said to be *admissible* in a proof system if it is in a way redundant, *i.e.* whenever φ is provable, then ψ is already provable without using that rule. Adding or dropping rules may increase or decrease the amount of proofs we can cook in a proof system. The effect can be dramatic: for example, classical logic can be obtained by simply adding the rule of *double negation elimination* to *intuitionistic propositional logic* (**IPC**). Admissible rules are all the opposite: if we decide to utilize one in order to cook something, then we could have just used our ingredients in a different way and obtain the same result.

One appealing feature of **CPC** is being *structurally complete*: all admissible rules are *derivable*, in the sense that whenever $\varphi_1, \ldots, \varphi_n / \psi$ is admissible, then also $\varphi_1 \wedge \cdots \wedge \varphi_n \rightarrow \psi$ is provable [3] – *i.e.* the system acknowledges that there's no need for that additional tool. This is not the case in intuitionistic logic: the mere fact that we *know* that the tool was not needed, doesn't give us any way to show inside the system *why* is that. On the other hand, **IPC** has other wonderful features. Relevant here is the *disjunction property*, fundamental for a constructive system: when a disjunction $\varphi \vee \psi$ is provable, then one of the disjuncts φ or ψ is provable as well.

Our interest is in the intuitionistic admissible rules that are not derivable, the computational principles they describe, and the logic systems obtained by adding such rules to **IPC**

Can one effectively identify all intuitionistic admissible rules? The question of whether that set of rules is recursively enumerable was posed by Friedman in 1975, and answered positively by Rybakov in 1984. It was then de Jongh and Visser who exhibited a numerable set of rules (now known as *Visser's rules*) and conjectured that it formed a basis for all the admissible rules of **IPC**. This conjecture was later proved by Iemhoff in the fundamental [4]. Rozière in his Ph.D. thesis [5] reached the same conclusion with a substantially different technique, independently of Visser and Iemhoff. These works elegantly settled the problem of identifying and building admissible rules. However our question is different: *why* are these rules superfluous, and what reduction steps can eliminate them from proofs?

Rozière first posed the question of finding a computational correspondence for his basis of the admissible rules in the conclusion of his thesis, but it looks to us that no work has been done on this ever since.

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$$\begin{array}{rcl} t,u,v & ::= & x \text{ (variable)} \\ & \mid & u\,v \text{ (application)} \\ & \mid & \lambda x.t \text{ (abstraction)} \\ & \mid & \texttt{efq}\,t \text{ (exfalso)} \\ & \mid & \langle u,v\rangle \text{ (pair)} \\ & \mid & \texttt{proj}_i\,t \text{ (projection)} \\ & \mid & \texttt{inj}_i\,t \text{ (injection)} \\ & \mid & \texttt{case}[t \parallel y.u \mid y.v] \text{ (case)} \\ & \mid & \texttt{V}_n[\vec{x}.t \parallel \vec{y}.u_1 \mid \vec{y}.u_2 \parallel z.\vec{v}] \\ & & \text{(Visser)} \end{array}$$	Evaluation contexts for **IPC**: $$\begin{array}{rcl} H & ::= & [\cdot] \\ & \mid & H\,t \mid \texttt{efq}H \mid \texttt{proj}_i H \\ & \mid & \texttt{case}[H \parallel - \mid -] \end{array}$$ Evaluation contexts for **KP**: $$\begin{array}{rcl} E & ::= & [\cdot] \\ & \mid & H[E] \\ & \mid & \texttt{V}_n[\vec{x}.E \parallel - \mid - \parallel \cdots] \end{array}$$

Figure 1: Proof terms (left) and evaluation contexts (right)

Our plan is to understand the phenomenon of admissibility by equipping proofs with lambda terms and associated reductions in the spirit of the Curry-Howard correspondence. The detour removal procedure will show explicitly what is the role that admissible rules play in a proof.

2 Term calculus for the admissible rules

The family of Visser's rules, forming a basis for all the admissible rules of **IPC**, is the following succession of rules:

$$V_n: \quad (\alpha_i \to \beta_i)_{i=1\ldots n} \to \gamma \vee \delta \,/\, \begin{cases} \bigvee_{j=1}^{n}((\alpha_i \to \beta_i)_{i=1\ldots n} \to \alpha_j) \\ \vee \\ ((\alpha_i \to \beta_i)_{i=1\ldots n} \to \gamma) \\ \vee \\ ((\alpha_i \to \beta_i)_{i=1\ldots n} \to \delta) \end{cases}$$

It is an infinite family, as none of the rules can derive a succedent rule [5]. We propose a way to uniformly assign a term to all these inferences. First, we describe a set of rules in natural deduction style corresponding to the V_i. Since the conclusions of the rules is a disjunction, we can model our rules with the shape of a generalized disjunction elimination; however, the main premise will be the disjunction in the antecedent of the V_i, under i implicative assumptions. The form of the rules is therefore

$$\dfrac{\begin{array}{c}[\alpha_i \to \beta_i]_{i=1\ldots n} \\ \vdots \\ \gamma \vee \delta\end{array} \qquad \begin{array}{c}[(\alpha_i \to \beta_i)_{i=1\ldots n} \to \alpha] \\ \vdots \\ \psi\end{array} \qquad \begin{array}{c}[(\alpha_i \to \beta_i)_{i=1\ldots n} \to \beta] \\ \vdots \\ \psi\end{array} \qquad \begin{array}{c}[(\alpha_i \to \beta_i)_{i=1\ldots n} \to \alpha_j] \\ (j = 1\ldots n) \\ \psi\end{array}}{\psi}$$

In order to treat the admissibility of the rules, we need to restrict the allowed proofs of the main premise $\gamma \vee \delta$ to be only closed proofs: otherwise we would clearly be able to prove the implication corresponding to the rule, and thus we would go beyond **IPC**. On the other side, it is not too difficult to see that our rules directly correspond to the V_n, and they adequately represent admissibility. The proof term associated with this inference is again modeled on a case analysis, the difference being the number of assumptions that are bound and the number of possible cases. We use the vector notation $\vec{x}.t$ on

- Rules for **IPC**:
  - *Beta:* $(\lambda x.t)\,u \mapsto t\{u/x\}$
  - *Projection:* $\mathtt{proj}_i\,\langle t_1,t_2\rangle \mapsto t_i$
  - *Case:* $\mathtt{case}[\mathtt{inj}_i t \parallel y\,.\,u_1 \mid y\,.\,u_2] \mapsto u_i\{t/y\}$
- Additional rules for Visser terms:
  - *Visser-inj:* $\mathtt{V}_n[\vec{x}\,.\,\mathtt{inj}_i t \parallel y\,.\,u_1 \mid y\,.\,u_2 \parallel z\,.\,\vec{v}] \mapsto u_i\{\lambda\vec{x}.t/y\} \quad (i=1,2)$
  - *Visser-app:* $\mathtt{V}_n[\vec{x}\,.\,H[x_j t] \parallel y\,.\,u_1 \mid y\,.\,u_2 \parallel z\,.\,\vec{v}] \mapsto v_j\{\lambda\vec{x}.t/z\} \quad (j=1\dots n)$

Figure 2: Reduction rules

variables to indicate that a list of (indexed) variables x_i is bound, and on terms as $z\,.\,\vec{v}_n$ to indicate a list of (indexed) terms v_i on each of which we are binding a variable z.

All the other proof terms of the calculus are the ones from a usual Curry-Howard correspondence for intuitionistic logic (see for example [6]); the grammar is presented in figure 1.

The reduction rules for the proof terms are given in Figure 2: once again the first block contains the usual ones for **IPC**, and the second block contains reduction rules for the new construct $\mathtt{V}[\vec{x}\,.\,t \parallel \vec{y}\,.\,u_1 \mid y\,.\,u_2 \parallel z\,.\,\vec{v}]$, depending on two shapes that t might have. Let us explain the intuition. In the first case, the term is the injection $\mathtt{inj}_i t$ with possibly some free variable x_i of type $\alpha_i \to \beta_i$; in that branch one has clearly chosen to prove one of the two disjuncts γ or δ; we may just reduce to the corresponding proof u_i, in which we plug the proof t but after binding the free variables $\vec{x}$. In the second case, the term is an application with one of the variables bound by the Visser rule on the left hand side, *i.e.* the proof uses one of the Visser assumptions to prove the disjunction. We reduce to the corresponding case v_j, where $\lambda\vec{x}.t$ is substituted for the assumption of type $(\alpha_i \to \beta_i)_{i=1\dots n} \to \alpha_j$.

Finally, let's turn to reduction. Contexts are defined intuitively as proof terms with a hole; the hole is denoted by $[\cdot]$, and $E[t]$ means replacing the hole in the context E with the term t. Reduction contexts are defined by the grammar in Figure 1 on the right. Reduction is obtained as usual from reduction rules $\mapsto$ as the contextual closure under evaluation contexts: if $t \mapsto u$ then $E[t] \to E[u]$. We chose carefully the evaluation contexts in order to simplify normal forms: instead of full reduction, we use *weak head reduction*, *i.e.* reductions are performed only in the head of terms, and not under abstractions.

3 A first application: independence of premise

The simplest and oldest studied admissible rule of **IPC** is the Independence of premise rule, or Harrop's rule [3]:

$$\neg\psi \to \alpha \vee \beta / (\neg\psi \to \alpha) \vee (\neg\psi \to \beta)$$

The logic that arises by adding it to **IPC** has also been studied, and is known as Kreisel-Putnam logic (**KP**). Historically, it was the first logic stronger than **IPC** with the disjunction property to be found.

We derived a Curry-Howard calculus for **KP** as a particular case of the system we presented. It suffices to see that Harrop's rule is a particular case of V_1 where β_1 is taken to be always $\perp$. Then we get the following natural deduction rule:

$$\text{Harrop: } \frac{\begin{array}{c}[\neg\psi]\\ \vdots\\ \alpha\vee\beta\end{array} \qquad \begin{array}{c}[\neg\psi\to\alpha]\\ \vdots\\ \varphi\end{array} \qquad \begin{array}{c}[\neg\psi\to\beta]\\ \vdots\\ \varphi\end{array}}{\varphi}$$

If we lift the restriction on the proofs of the main premise and allow open proofs, we can prove the new axiom in our system:

$$\dfrac{\text{Harrop } \dfrac{\dfrac{[\neg\psi\to\gamma\vee\delta]_{(2)} \qquad [\neg\psi]_{(1)}}{\alpha\vee\beta} \qquad \dfrac{[\neg\psi\to\alpha]_{(1)}}{(\neg\psi\to\alpha)\vee(\neg\psi\to\beta)} \qquad \dfrac{[\neg\psi\to\beta]_{(1)}}{(\neg\psi\to\alpha)\vee(\neg\psi\to\beta)}}{(\neg\psi\to\alpha)\vee(\neg\psi\to\beta)}\,(1)}{(\neg\psi\to\gamma\vee\delta)\to(\neg\psi\to\alpha)\vee(\neg\psi\to\beta)}\,(2)$$

The resulting proof term annotation is

$$\frac{\Gamma,x\colon \neg\psi\vdash t\colon \alpha\vee\beta \qquad \Gamma,y\colon \neg\psi\to\alpha\vdash u_1\colon \varphi \qquad \Gamma,y\colon \neg\psi\to\beta\vdash u_2\colon \varphi}{\Gamma\vdash \mathtt{hop}[x.t \mid\mid y.u_1 \mid y.u_2]\colon \varphi}$$

By looking at the reduction rules, we can see that the case of *Visser-app* is now simplified: since the Visser assumption x has negated type, the application xt has type $\bot$, and is thus succeded by an *ex-falso*. The *Visser-app* rule will simply use *ex-falso* reasoning to directly conclude on (either) one of the disjuncts.

We proved the usual properties of Subject Reduction and Normalization for the system. Then, we obtained the following classification of normal forms:

Lemma 1 (Classification)**.** *Let $\Gamma_\neg\vdash t\colon \tau$ for t in n.f. and t not an exfalso:*

- Implication*: if $\tau=\varphi\to\psi$, then t is an abstraction or a variable in $\Gamma_\neg$;*
- Disjunction*: if $\tau=\varphi\vee\psi$, then t is an injection;*
- Conjunction*: if $\tau=\varphi\wedge\psi$, then t is a pair;*
- Falsity*: if $\tau=\bot$, then $t=xv$ for some v and some $x\in\Gamma_\neg$;*

From this, we obtained

Theorem 2 (Disjunction property)**.** *If $\vdash t\colon \varphi\vee\psi$, then $\vdash t\colon \varphi$ or $\vdash t\colon \psi$.*

4 Conclusions and future work

Our system provides a meaningful explanation of the admissible rules in terms of normalization of natural deduction proofs. In addition, by simply lifting the condition of having closed proofs on the main premise, we can study intermediate logics characterized by the axioms corresponding to some admissible rules; the study of the Kreisel-Putnam logic exemplifies this approach.

We believe that our presentation is well-suited to continue the study of admissibility in intuitionistic systems, a subject that is currently mostly explored with semantic tools. We conclude with some remarks on future generalizations:

The logic AD

Rozière showed that by adding V_1 as an axiom to **IPC** (he called such logic AD), all other rules of the Visser base are derivable. By the results of Rybakov this means that AD has no admissible rules, and is thus the minimal structurally complete intermediate logic. AD has appearently never been studied, and although it cannot have the disjunction property, we believe that it is an interesting subject to be studied within our system.

First-order logic

Many first-order admissible rules directly correspond to propositional rules. For example the first-order version of *Independence of premise* is:

$$(\neg A \to \exists x.\, B(x)) \to \exists x.\, (\neg A \to B(x))$$

As expected, it is an admissible but not derivable rule of intuitionistic logic, and our framework can be easily extended to handle it as well.

Arithmetic

Since its inception with Harrop [3], the motivation for studying admissible rules of **IPC** was to understand arithmetical systems. A famous theorem of de Jongh states that the propositional formulas whose arithmetical instances are provable in *intuitionistic arithmetic* (**HA**) are exactly the theorems of **IPC**, and many studies of the admissible rules of **HA** (like Visser [7], Iemhoff and Artemov [1]) originated from it.

The Independence of Premise has an important status in the theory of arithmetic, and was given a constructive interpretation for example by Gödel [2]. Many other admissible rules of **HA**, such as Markov's principle, have been studied for a long time. Therefore, we believe that a substantial field of application of our technique is the constructive study of admissible rules of **HA**.

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