(Short Paper) Towards a Dualized Sequent Calculus with Canonicity

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In pursuit of a logic that has canonicity and is comprised of dualized proof rules, we introduce a sequent calculus system, $\mathbf{2Int}^x$, that is inspired by Wansing's bi-intuitionistic propositional logic **2Int**. Though **2Int** has canonicity and duality, it defines only natural deduction proof rules and employs an unintuitive Kripke semantics that allows atomic formulas to be both true and false. In addition to defining the sequent calculus rules of $\mathbf{2Int}^x$, we also define a Kripke semantics that only admits models in which atomic formulas are either true or false, but not both. Finally, we prove soundness of $\mathbf{2Int}^x$.

1 Motivation: Non-canonicity in BiInt

We are seeking a canonical sequent calculus with dualized proof rules (as in [1]) because this results in fewer rules, can be useful for induction and coinduction, and is relevant to our interest in constructive control operators (analogous to the classical control operators found in the $\lambda\mu$-calculus[2]).

Though a bi-intuitionistic (or **BiInt**) sequent calculus may consist of dualized proof rules, the logic lacks the property of canonicity. For example, the formula $A \vee (\top \prec A)$ is valid in **BiInt**, but the **BiInt** derivation [1] shown in Figure 1 is not a canonical proof of $A \vee (\top \prec A)$ because it represents an object that is not necessarily either a proof of A, or a proof of $\top \prec A$. Here the connective $\prec$ denotes dual-intuitionistic implication (the dual connective of $\rightarrow$), and the formula $\top \prec A$ is called the dual-intuitionistic negation of A.[2]. The dual of intuitionistic implication is also sometimes called "subtraction", "co-implication", and "exclusion".[3]

The derivation of $A \vee (\top \prec A)$ demonstrates that **BiInt** does not have canonicity, so we must investigate alternative dualized logics. In [4], Wansing introduces a logic **2Int** that is a conservative extension of intuitionistic logic (with respect to validity) and dual-intuitionistic logic (with respect to dual validity), and also has both canonicity and duality. However, the proof system for **2Int** is not a sequent calculus. Additionally, the semantics of **2Int** allows models in which an atomic formula is true and false at the same time.

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[1] This derivation is constructed using the inference rules of Pinto and Uustalu's presentation of a multiple conclusion "Dragalin style" **BiInt** sequent calculus from [3]. This sequent calculus is not complete without cut, but they go on to provide a complete cut-free system in which $A \vee (\top \prec A)$ is also a theorem.

[2] The formulas $-A$ and $\sim A$ are both common shorthand notations for the formula $\top \prec A$.

[3] Though the meaning of the subtraction connective is somewhat subtle in **BiInt**, the equivalence in classical logic between $A \prec B$ and $A \wedge \neg B$ provides a helpful hint.

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$$\dfrac{\dfrac{}{\vdash \top, A}\ \top\mathrm{R} \quad \dfrac{}{A \vdash A, \top \prec A}\ \mathrm{hyp}}{\dfrac{\vdash \top \prec A, A}{\vdash A \vee (\top \prec A)}\ \vee\mathrm{R}}\ \prec\mathrm{R}^{4}$$

Figure 1: Derivation of $A \vee (\top \prec A)$ in Dragalin-style sequent calculus

$$\begin{array}{lcl} \text{Polarities } p & ::= & + \mid - \\ \text{Formulas } F & ::= & \sigma \mid \top_p \mid F_1 \wedge_p F_2 \mid F_1 \rightarrow_p F_2 \end{array}$$

Figure 2: Syntax of polarities and $\mathbf{2Int}^x$ formulas

In this paper we introduce a sequent calculus proof system–called $\mathbf{2Int}^x$–that derives proofs of bi-intuitionistic propositional formulas and consists of dualized inference rules (Section 2). In Section 3 we define a Kripke semantics for $\mathbf{2Int}^x$ that is distinct from the semantics given to **2Int**, in that atomic formulas cannot be both true and false in a particular world. We also prove soundness of $\mathbf{2Int}^x$, and examine some of its properties. We are optimistic that $\mathbf{2Int}^x$ has the property of canonicity because the formula $A \vee (\top \prec A)$ is not valid with respect to the given semantics.

2 Proof System

The $\mathbf{2Int}^x$ proof system is a single conclusion propositional sequent calculus that incorporates a notion of polarity into both formulas and connectives in order to derive proofs of bi-intuitionistic propositional formulas. The proof rules for introducing logical connectives are dual by definition, such that a sequent that concludes a formula that introduces a connective is derived using the same rule as a sequent that concludes the dual of that formula. Logical connectives are annotated with positive and negative polarities, such that each polarity is dual to the other (see Figure 3). Additionally, sequents also annotate the formulas in both the conclusion position as well as the assumption positions. However, these annotations use a related but different form of polarity that allows for the possibility of "uncertainty".

2.1 Polarities and syntax of formulas

Figure 2 shows the syntax of polarities and formulas. A polarity p appears in a logical connective as a syntactic parameter that determines between one of two duals. A connective with a positive polarity functions in the typical way, and a connective with a negative polarity functions as its dual. For example, the dual logical connective of $\wedge$ is $\vee$, so a $\wedge_-$ (resp. $\wedge_+$) that appears in a $\mathbf{2Int}^x$ formula will function as $\vee$ (resp. $\wedge$). Figure 4 provides a recursive definition of a function that translates formulas from **BiInt** to $\mathbf{2Int}^x$ syntax. This mapping determines the dual for each logical connective of typical bi-intuitionistic propositional logic.

Figure 3 defines inversion on polarities.

[4] The persistence of the $\top \prec A$ conclusion in the right premise is purely a technicality because Pinto and Uustalu define this particular rule to also serve as a contraction rule for subtraction conclusions.

$$\begin{array}{lcl} \overline{+} & = & - \\ \overline{-} & = & + \end{array}$$

Figure 3: Inverse of a polarity

$$\begin{array}{lcl} \ulcorner\sigma\urcorner & = & \sigma \\ \ulcorner\top\urcorner & = & \top_+ \\ \ulcorner\bot\urcorner & = & \top_- \\ \ulcorner F_1 \wedge F_2\urcorner & = & \ulcorner F_1\urcorner \wedge_+ \ulcorner F_2\urcorner \\ \ulcorner F_1 \vee F_2\urcorner & = & \ulcorner F_1\urcorner \wedge_- \ulcorner F_2\urcorner \\ \ulcorner F_1 \rightarrow F_2\urcorner & = & \ulcorner F_1\urcorner \rightarrow_+ \ulcorner F_2\urcorner \\ \ulcorner F_2 \prec F_1\urcorner & = & \ulcorner F_1\urcorner \rightarrow_- \ulcorner F_2\urcorner \end{array}$$

Figure 4: Translation from standard syntax for **BiInt** to $\mathbf{2Int}^x$ syntax

2.2 Uncertain Polarities

A u-polarity (see Figure 5) appears only as an annotation on formulas in sequents of $\mathbf{2Int}^x$ proof derivations, and determines a kind of knowledge status/sense regarding its associated formula. The u-polarity attached to a formula indicates two orthogonal aspects of the formula: whether it is “positive” or “negative”, and also whether the formula is “certain” or “uncertain”. The descriptions below convey intuitions about the meaning of each of the four unique combinations of u-polarity aspects. The meaning of “forward” and “backward persistent” assumptions will become clear in section 2.3 where we discuss abstract Kripke structures. That section will explain that every formula in a sequent’s context represents an assumption (resp. counter-assumption) that only pertains to a specific “state” of knowledge, along with every other “state” of knowledge that is “forwardly” (resp. “backwardly”) “connected” to it.

- “negative” and “certain”: A formula annotated with a u-polarity of $-$ denotes a certainly false (or *falsified*) formula. A derivation ending with an empty context sequent that concludes with a formula at this u-polarity *disproves* that formula, and a sequent that has a formula at this u-polarity in its context has that formula as a backward persistent *counter-assumption*.
- “positive” and “certain”: A formula annotated with a u-polarity of $+$ denotes a certainly true (or *verified*) formula. A derivation ending with an empty context sequent that concludes with a formula at this u-polarity *proves* just that formula, and a sequent with a formula at this u-polarity in its context has that formula as a forward persistent *assumption*.
- “negative” and “uncertain”: A formula annotated with a u-polarity of $-?$ denotes an uncertainly false (or *not verified*) formula. A derivation ending with an empty context sequent that concludes with a formula at this u-polarity proves the *non-existence* of a *proof* of that formula, and a sequent that has a formula at this u-polarity in its context has the *intuitionistic negation* of that formula as

$$\text{U-polarities } u \quad ::= \quad + \mid - \mid +? \mid -?$$

Figure 5: Syntax U-polarities (polarities with uncertainty)

$$\begin{array}{lcl} \lfloor + \rfloor & = & + \\ \lfloor - \rfloor & = & - \\ \lfloor +? \rfloor & = & + \\ \lfloor -? \rfloor & = & - \end{array}$$

Figure 6: Truncation of a u-polarity

$$\begin{array}{lcl} \widetilde{+} & = & -? \\ \widetilde{-} & = & +? \\ \widetilde{+?} & = & - \\ \widetilde{-?} & = & + \end{array}$$

Figure 7: Inverse of a u-polarity

a backward persistent *assumption*.[5].

- "positive" and "uncertain": A formula annotated with a u-polarity of $+?$ denotes an uncertainly true (or *not falsified*) formula. A derivation ending with an empty context sequent that concludes with a formula at this u-polarity proves the *non-existence* of a *disproof* of that formula, and a sequent that has a formula at this u-polarity in its context has the *dual-intuitionistic negation* of that formula as a forward persistent *counter-assumption*.

The latter two descriptions are justified in Section 2.5. Also note that a context that contains a formula at a "negative" and "uncertain" u-polarity is meaningfully distinct from a context that contains the intuitionistic negation of that formula at a "positive" and "certain" polarity precisely because of the difference in the persistence direction associated to each of the formulas by their respective u-polarities.

Figure 6 defines a truncation function that maps a u-polarity to a polarity by eliminating uncertainty. Figure 7 defines an involution on u-polarities that maps each u-polarity to its dual by inverting both its certainty and its polarity. This mapping corresponds with the interpretation of u-polarities given above in a natural way. For example, $\widetilde{+} = -?$ because the *dual* of a verified (or certainly true) formula is a not verified (or certainly not true, or *weakly false*) formula, which is equivalent to a formula that is either falsified or has a completely unknown truth value.

2.3 Contexts and Abstract Kripke Structures

Every sequent of a $\mathbf{2Int}^x$ proof derivation is a triple of a finite binary relation R over a set of symbols, a finite set of assumed formulas Γ, and a conclusion formula F. We call the relation R an *abstract Kripke structure*, and we call the symbols related by R *abstract worlds*. Every formula in the sequent is annotated with a u-polarity as well as an abstract world, and we denote the formula F at u-polarity u and abstract world n by $u F \mathbin{@} n$.

[5] This is distinct from counter-assuming the formula because we are in an intuitionistic setting. Verifying the intuitionistic negation of a formula establishes only that the formula is not verified, and does not establish that the formula is falsified (a strictly stronger notion). In other words, verifying the intuitionistic negation of a formula eliminates the possibility of verifying that formula (because that would form a contradiction) and is, of course, consistent with falsifying that formula, but it does not necessarily imply the falsification of that formula.

$$\begin{array}{lcl} n \prec_+ n' & = & (n,n') \\ n \prec_- n' & = & (n',n) \\ R \vdash e & \Leftrightarrow & e \in R^* \end{array}$$

Figure 8: Polarized edges and accessibility in abstract Kripke structures

The $\mathbf{2Int}^x$ inference rules contain certain side conditions that correspond to the structure of a Kripke semantics, and serve the purpose of preventing the $\mathbf{2Int}^x$ system from being able to prove non-intuitionistic formulas. These side conditions enforce relations between the abstract Kripke structure of a rule's conclusion sequent, and the abstract Kripke structures of its premise sequents. The abstract Kripke structures of these sequents determine a directed graph that describes a kind of "connectedness" relationship on the set of abstract worlds. Similar to the notion of a u-polarity annotation (introduced in section 2.2), an abstract world that annotates a formula enhances the meaning of the formula by refining its knowledge status/sense. Broadly speaking, each inference rule relates its conclusion formula at abstract world n to only those assumed formulas at abstract worlds that can *reach* n (we will define *reach* precisely below). This is similar to the systems found in [3] and [1]. However, the $\mathbf{2Int}^x$ inference rules use abstract Kripke structures differently than those systems because the precise meaning of an abstract world being able to *reach* another abstract world depends on the polarity of the conclusion formula. For example, suppose that the conclusion has polarity + or +? and is annotated by abstract world n'. In this case an abstract world n *reaches* n' iff (n,n') is in the reflexive-transitive closure of the abstract Kripke structure. Otherwise, when the u-polarity is negative, n *reaches* n' iff (n',n) is in the reflexive-transitive closure.

2.4 Proof rules

The inference rules for $\mathbf{2Int}^x$ are defined in Figure 9. In addition to depending on proof derivations, some inference rules of $\mathbf{2Int}^x$ also depend on the satisfaction of side conditions that are denoted by a premise enclosed by square brackets ($[\cdot]$). The following describes the precise meaning of these side conditions with respect to any rule in which they might appear. A $\mathbf{2Int}^x$ rule can require a relation between the abstract world of its conclusion and the abstract worlds of its assumptions via a side condition of the form $R \vdash n \prec_p n'$. An inference rule that depends on this form of side condition requires that an abstract world n' is reachable from another abstract world n via edges (of the reachability relation R) polarized at polarity p (see Figure 8). An inference rule that requires that a polarity p' is equal to a truncated uncertain polarity u will depend on the condition $p' = \lfloor u \rfloor$. If it requires that p' is equal to the inverse of truncated u, then it will depend on the condition $p' = \overline{\lfloor u \rfloor}$. For an inference rule that requires that the u-polarity of a formula be equivalent to polarity p', $p'\,F\,@\,n$ will denote the formula at u-polarity u such that $u \in \{+,-\}$ and $p' = \lfloor u \rfloor$. Finally, an inference rule that requires that an abstract world n is not a member of a set A of abstract worlds will depend on the condition $n \notin A$. In particular, a rule that requires that an abstract world is "fresh" will depend on a side condition of this form with occurrences of the sets $\mathit{field}(R)$ and $\mathit{worlds}(\Gamma)$: $\mathit{field}(R)$ will denote the set of abstract worlds that occur in either the domain or the range of the abstract Kripke structure R; and $\mathit{worlds}(\Gamma)$ will denote the set of abstract worlds n such that $u\,F\,@\,n$ is in the context Γ, for some u-polarity u and formula F.

The use of explicit polarities allows us to cut the number of rules that would otherwise be required at least in half, although it does result in more dense rules. Here are some example deductions, using the usual non-polarized syntax (Figure 4). First, let Γ be the context containing just $+(\top \to \top) \to r\,@\,n$. Then

$$\frac{[R \vdash n \prec_{p'} n'] \quad [p' = \lfloor u \rfloor]}{R;\Gamma, p'\,F \,@\, n \vdash u\,F \,@\, n'}\ \text{Assume}^{\prec} \qquad \frac{}{R;\Gamma, u\,F \,@\, n \vdash u\,F \,@\, n}\ \text{Assume}$$

$$\frac{R;\Gamma \vdash u\,F \,@\, n}{R;\Gamma, u'\,F' \,@\, n' \vdash u\,F \,@\, n}\ \text{Weaken} \qquad \frac{[p' = \lfloor u \rfloor]}{R;\Gamma \vdash u\,\top_{p'} \,@\, n}\ \text{Unit}$$

$$\frac{R;\Gamma \vdash u\,F_1 \,@\, n \quad R;\Gamma \vdash u\,F_2 \,@\, n \quad [p' = \lfloor u \rfloor]}{R;\Gamma \vdash u\,F_1 \wedge_{p'} F_2 \,@\, n}\ \text{And}$$

$$\frac{R;\Gamma \vdash u\,F_i \,@\, n \quad [p' = \overline{\lfloor u \rfloor}] \quad [i \in \{1,2\}]}{R;\Gamma \vdash u\,F_1 \wedge_{p'} F_2 \,@\, n}\ \overline{\text{And}}$$

$$\frac{R \cup (n \prec_{p'} n');\Gamma, p'\,F_1 \,@\, n' \vdash u\,F_2 \,@\, n' \quad [p' = \lfloor u \rfloor] \quad [n' \notin \{n\} \cup \mathit{field}(R) \cup \mathit{worlds}(\Gamma)]}{R;\Gamma \vdash u\,F_1 \rightarrow_{p'} F_2 \,@\, n}\ \text{Implies}$$

$$\frac{R;\Gamma \vdash p'\,F_1 \,@\, n' \quad R;\Gamma \vdash u\,F_2 \,@\, n' \quad [p' = \overline{\lfloor u \rfloor}] \quad [R \vdash n \prec_{p'} n']}{R;\Gamma \vdash u\,F_1 \rightarrow_{p'} F_2 \,@\, n}\ \overline{\text{Implies}}$$

$$\frac{R;\Gamma, \widetilde{u}F_1 \,@\, n \vdash p'\,F_2 \,@\, n' \quad R;\Gamma, \widetilde{u}F_1 \,@\, n \vdash \widetilde{p'}\,F_2 \,@\, n' \quad [p' = \lfloor u \rfloor]}{R;\Gamma \vdash u\,F_1 \,@\, n}\ \text{Cut}$$

Figure 9: Inference Rules for **2Int**^{x}

we have

$$\dfrac{\dfrac{\overline{\Gamma \vdash +(\top \to \top) \to r @ n}}{\varnothing;\Gamma,-?r @ n \vdash +(\top \to \top) \to r @ n} \qquad \dfrac{\dfrac{\overline{\varnothing,(n,n');\Gamma,-?r @ n,+\top @ n' \vdash +\top @ n'}}{\varnothing;\Gamma,-?r @ n \vdash +\top \to \top @ n} \qquad \overline{\varnothing;\Gamma,-?r @ n \vdash -?r @ n}}{\varnothing;\Gamma,-?r @ n \vdash -?(\top \to \top) \to r @ n}}{\varnothing;\Gamma \vdash +r @ n}\ \text{Cut}$$

An example of $\phi \to (\psi \to \phi)$ (eliding side conditions):

$$\dfrac{\dfrac{\dfrac{\varnothing,(n_0,n_1),(n_1,n_2) \vdash n_1 \prec_+ n_2 \qquad + = \lfloor + \rfloor}{\varnothing,(n_0,n_1),(n_1,n_2);\varnothing,+\phi @ n_1,+\psi @ n_2 \vdash +\phi @ n_2}\ \text{Assume}^{\prec} \qquad \cdots}{\varnothing,(n_0,n_1);\varnothing,+\phi @ n_1 \vdash +\psi \to \phi @ n_1}\ \text{Implies} \qquad \cdots}{\varnothing;\varnothing \vdash +\phi \to (\psi \to \phi) @ n_0}\ \text{Implies}$$

Let $R = \{(n_0,n_1),(n_1,n_2)\}$ for some abstract worlds n_0,n_1,n_2, and let $\Gamma = \{+\phi @ n_1, +\neg\phi @ n_2\}$ for some formula ϕ (some side conditions are omitted):

$$\dfrac{\dfrac{\dfrac{\overline{R;\Gamma,-?\bot @ n_2 \vdash +\neg\phi @ n_2} \qquad \dfrac{\overline{R;\Gamma,-?\bot @ n_2 \vdash +\phi @ n_2}\ \text{As.}^{\prec} \qquad \overline{R;\Gamma,-?\bot @ n_2 \vdash -?\bot @ n_2}\ \text{Assume}}{R;\Gamma,-?\bot @ n_2 \vdash -?\neg\phi @ n_2}\ \text{Implies}}{R;\Gamma \vdash +\bot @ n_2}\ \text{Cut}}{\varnothing,(n_0,n_1);\varnothing,+\phi @ n_1 \vdash +\neg\neg\phi @ n_1}\ \text{Implies}}{\varnothing;\varnothing \vdash +\phi \to \neg\neg\phi @ n_0}\ \text{Implies}$$

2.5 Relationship Between $\tilde{u}F @ n$ and $uF @ n$

In Section 2.2, we described an empty context proof derivation with conclusion formula at a negative (resp. positive) uncertain u-polarity as a proof of the non-existence of a proof (resp. disproof) of that formula. This description is justified because a proof of a formula at polarity u can be combined with a proof of that formula at polarity $\tilde{u}$ to derive a contradiction. For example suppose that there exists a proof derivation $\mathcal{D}_1$ ending with the sequent $R;\Gamma \vdash -?\phi @ n$. The existence of another derivation $\mathcal{D}_2$ ending with $R;\Gamma \vdash +\phi @ n$ would permit the following derivation of $R;\Gamma \vdash +\bot @ n$ [6]:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_2}{R;\Gamma \vdash +\phi @ n}}{R;\Gamma,-?\bot @ n \vdash +\phi @ n}\ \text{Wkn.} \qquad \dfrac{\dfrac{\mathcal{D}_1}{R;\Gamma \vdash -?\phi @ n}}{R;\Gamma,-?\bot @ n \vdash -?\phi @ n}\ \text{Wkn.} \qquad + = \lfloor + \rfloor}{R;\Gamma \vdash +\bot @ n}\ \text{Cut}$$

By the soundness theorem (stated in Section 3.3), the existence of derivation $\mathcal{D}_1$ either implies the non-existence of derivation $\mathcal{D}_2$, or implies that the context $R;\Gamma$ is inconsistent. The empty context cannot be inconsistent, so therefore a proof derivation of $\cdot;\cdot \vdash uF @ n$ precludes the existence of a derivation of $\cdot;\cdot \vdash \tilde{u}F @ n$.

[6] It is easy to check that a proof derivation ending with $R;\Gamma \vdash +?\phi @ n$ causes a similar dual result, corresponding naturally with the description given in Section 2.2 of an "uncertain" and "positive" polarity.

3 The Semantics of $\mathbf{2Int}^x$

In the following sections we define Kripke models for $\mathbf{2Int}^x$, claim soundness of $\mathbf{2Int}^x$ with respect to the Kripke models, and then observe some properties of $\mathbf{2Int}^x$.

3.1 Kripke Models

The definitions below describe a polarized semantic interpretation of any $\mathbf{2Int}^x$ formula as a truth state of "unknown", "verified", or "falsified". This interpretation indicates established knowledge of a formula by the "verified" state, or by its dual, the "falsified" state. By proving a soundness theorem for $\mathbf{2Int}^x$, we will see that the u-polarities that annotate the conclusions of sequents formally correspond with the semantic polarity in the expected way. For example, an empty context proof of a formula at u-polarity $+$, $-$, $+?$, or $-?$, should imply that the formula is valid, dually valid, unfalsifiable, or unverifiable (respectively).

Like the models defined by Wansing for **2Int**, the interpretation utilizes Kripke models endowed with two valuation functions instead of just one. A "verifier" function determines the worlds at which an atomic formula is verified, and a "falsifier" function determines the worlds at which an atomic formula is falsified. However, properties 2, 3, and 4 of Definition 1 distinguish the Kripke models of $\mathbf{2Int}^x$ from those of **2Int**: property 2 persists backwards instead of forwards, and the other two together prohibit models with inconsistent verifier and falsifier functions.

Definition 1. *Let Σ be an enumerable set of atomic formulas. We define a structure M as a tuple $\langle I, \leq, v^+, v^- \rangle$, where I is a non-empty set, $\leq$ is a preorder relation on the set I, and v^+, v^- are both functions that map each atomic formula $\sigma \in \Sigma$ to a subset of I. Additionally, v^+, v^- must satisfy:*

1. *for any $w, w' \in I$ with $w \leq w'$: if $w \in v^+(\sigma)$ then $w' \in v^+(\sigma)$*
2. *for any $w, w' \in I$ with $w \geq w'$: if $w \in v^-(\sigma)$, then $w' \in v^-(\sigma)$*
3. *for any $w \in I$: if $w \in v^+(\sigma)$, then $w \notin v^-(\sigma)$*
4. *for any $w \in I$: if $w \in v^-(\sigma)$, then $w \notin v^+(\sigma)$*

Properties 1 and 2 cause the verification (resp. falsification) of a symbol at some world to persist to all reachable (resp. inversely reachable) worlds. Properties 3 and 4 ensure that the function v^+ (representing a verification perspective) and the function v^- (representing a falsification perspective) can never contradict each other. That is, no symbol can ever be verified and falsified at the same world.

The following definition interprets a $\mathbf{2Int}^x$ formula with respect to particular world of a Kripke model.

Definition 2. *For a structure M, world w, and formula F, the relation $M, w \vDash^+ F$ and $M, w \vDash^- F$ is* *inductively defined by:*

$$
\begin{array}{lll}
M,w \vDash^+ \sigma & \textit{iff} & w \in v^+(\sigma) \\
M,w \vDash^- \sigma & \textit{iff} & w \in v^-(\sigma) \\
M,w \vDash^+ \top_+ & & \\
M,w \nvDash^+ \top_- & & \\
M,w \vDash^- \top_- & & \\
M,w \nvDash^- \top_+ & & \\
M,w \vDash^+ F_1 \to_+ F_2 & \textit{iff} & \textit{for every } w' \geq w : M,w' \nvDash^+ F_1 \textit{ or } M,w' \vDash^+ F_2 \\
M,w \vDash^- F_1 \to_+ F_2 & \textit{iff} & \textit{there exists } w' \geq w : M,w' \vDash^+ F_1 \textit{ and } M,w' \vDash^- F_2 \\
M,w \vDash^+ F_1 \to_- F_2 & \textit{iff} & \textit{there exists } w' \leq w : M,w' \vDash^- F_1 \textit{ and } M,w' \vDash^+ F_2 \\
M,w \vDash^- F_1 \to_- F_2 & \textit{iff} & \textit{for every } w' \leq w : M,w' \nvDash^- F_1 \textit{ or } M,w' \vDash^- F_2 \\
M,w \vDash^+ F_1 \wedge_+ F_2 & \textit{iff} & M,w' \vDash^+ F_1 \textit{ and } M,w' \vDash^+ F_2 \\
M,w \vDash^- F_1 \wedge_+ F_2 & \textit{iff} & M,w' \vDash^- F_1 \textit{ or } M,w' \vDash^- F_2 \\
M,w \vDash^+ F_1 \wedge_- F_2 & \textit{iff} & M,w' \vDash^+ F_1 \textit{ or } M,w' \vDash^+ F_2 \\
M,w \vDash^- F_1 \wedge_- F_2 & \textit{iff} & M,w' \vDash^- F_1 \textit{ and } M,w' \vDash^- F_2
\end{array}
$$

A formula F is *valid (*resp. *dually valid) in structure M* iff for all $w \in I$ we have $M,w \vDash^+ F$ (resp. $M,w \vDash^- F$). $M \vDash^+ F$ (resp. $M \vDash^- F$) denotes that formula F is valid (resp. dually valid) in structure M. A formula F is *unverifiable (*resp. *unfalsifiable) in structure M* iff for all $w \in I$ we have $M,w \nvDash^+ F$ (resp. $M,w \nvDash^- F$). $M \nvDash^+ F$ (resp. $M \nvDash^- F$) denotes that formula F is unverifiable (resp. unfalsifiable) in structure M. A formula is *valid* (resp. *dually valid*) in $\mathbf{2Int}^x$ iff it is valid (resp. dually valid) in every structure. $\vDash^+ F$ (resp. $\vDash^- F$) denotes that formula F is valid (resp. dually valid). A formula F is *unverifiable* (resp. *unfalsifiable*) iff it is unverifiable (resp. unfalsifiable) in every structure. $\nvDash^+ F$ (resp. $\nvDash^- F$) denotes that formula F is unverifiable (resp. unfalsifiable).

In the introduction we established that **BiInt** lacks canonicity by observing that the formula $A \vee (\top \prec A)$ is valid in **BiInt**, and is a theorem in some example **BiInt** proof systems. Naturally, now that we have given a semantics for $\mathbf{2Int}^x$ formulas we will verify that $A \vee (\top \prec A)$ is not valid in $\mathbf{2Int}^x$ with the following countermodel: $M = \langle \{w_0\}, \{(w_0,w_0)\}, \sigma \mapsto \varnothing, \sigma \mapsto \varnothing \rangle$ is a structure for $\mathbf{2Int}^x$, and clearly the formula $\ulcorner \sigma \vee (\top \prec \sigma) \urcorner = \sigma \wedge_- (\sigma \to_- \top_+)$ is not verified in this structure because $M \nvDash^+ \sigma$ and $M \nvDash^+ \sigma \to_- \top_+$.

3.2 Semantics of Sequents

We now define semantic relations between structures, annotated formulas, sets of annotated formulas, and sequents.

Definition 3 (Abstract Bridge function)**.** *Let $M = \langle I, \leq, v^+, v^- \rangle$ be a structure, and let R be a finite relation on a set of abstract worlds. An abstract bridge function for structure M is a function $h : R \to I$, such that for all $(n,n') \in R$: $h(n) \leq h(n')$.*

Let u be a u-polarity, F be a formula, and n be an abstract world. The relation $M,h \vDash u F \,@\, n$ is defined as follows:

Definition 4.

$$
\begin{array}{lll}
M,h \vDash +F \mathbin{@} n & \textit{iff} & M,h(n) \vDash^{+} F \\
M,h \vDash -F \mathbin{@} n & \textit{iff} & M,h(n) \vDash^{-} F \\
M,h \vDash +?F \mathbin{@} n & \textit{iff} & M,h(n) \nvDash^{-} F \\
M,h \vDash -?F \mathbin{@} n & \textit{iff} & M,h(n) \nvDash^{+} F
\end{array}
$$

Let Γ be a set of annotated functions, then $M,h \vDash \Gamma$ iff for every annotated formula $uF \mathbin{@} n$ in Γ we have $M,h \vDash uF \mathbin{@} n$.

A structure $M = \langle I, \leq, v^{+}, v^{-} \rangle$ is a model of a sequent $R;\Gamma \vdash uF \mathbin{@} n$ iff for every abstract bridge function $h : R \to I$: if $M,h \vDash \Gamma$, then $M,h \vDash uF \mathbin{@} n$.

A sequent $R;\Gamma \vdash uF \mathbin{@} n$ is valid iff every structure M is a model of the sequent. A valid sequent is denoted $R;\Gamma \Vdash uF \mathbin{@} n$.

3.3 Soundness of 2Int^x

Theorem 5 (Soundness of **2Int**^{x})**.** *If there exists a **2Int**^{x} proof derivation of the sequent $R;\Gamma \vdash uF \mathbin{@} n$, then that sequent is valid.*

Proof: We have constructed a proof of this theorem in the Agda programming language.

3.4 Theorems of 2Int^x and 2Int

We now observe that there exists a theorem of **2Int**^{x} that is not a theorem of **2Int**. The proof derivation below proves that $(\phi \prec \phi) \to \psi$ is a theorem in **2Int**^{x}. Let $R = \{(n_0, n_1)\}$ for any abstract worlds n_0 and n_1, and let $\Gamma = \{+\phi \prec \phi \mathbin{@} n_1, -?\psi \mathbin{@} n_1\}$ for any formulas ϕ and ψ.

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{}{R;\Gamma \vdash +\phi \prec \phi \mathbin{@} n_1}\ \text{As.}
\qquad
\dfrac{\dfrac{R,(n_1,n_2) \vdash (n_2 \prec_{-} n_2) \quad - = \lfloor -? \rfloor}{R,(n_1,n_2);\Gamma, -\phi \mathbin{@} n_2 \vdash -?\phi \mathbin{@} n_2}\ \text{As.}^{\prec} \quad - = \lfloor -? \rfloor}{R;\Gamma \vdash -?\phi \prec \phi \mathbin{@} n_1}\ \text{Im.}
\quad + = \lfloor + \rfloor}{R;\varnothing, +\phi \prec \phi \mathbin{@} n_1 \vdash +\psi \mathbin{@} n_1}\ \text{Cut}}{}}{}
}{\varnothing;\varnothing \vdash +(\phi \prec \phi) \to \psi \mathbin{@} n_0}\ \text{Im.}
$$

However, $(\phi \prec \phi) \to \psi$ is not a theorem of **2Int**. Let $I = \{w_0\}$ and $M = \langle I, \{(w_0, w_0)\}, v^{+}, v^{-} \rangle$ be a structure for **2Int**, where v^{+}, v^{-} are functions from the set of atomic formulas $\{\sigma_0, \sigma_1\}$ to the set of information states I, and are defined by $v^{+} = v^{-} = \{(\sigma_0, I), (\sigma_1, \varnothing)\}$. According to the Kripke semantics of **2Int**, $M, w_0 \vDash^{+} \sigma_0 \prec \sigma_0$ because $M, w_0 \vDash^{+} \sigma_0$ and $M, w_0 \vDash^{-} \sigma_0$. However $M, w_0 \nvDash^{+} (\sigma_0 \prec \sigma_0) \to \sigma_1$ because at world w_0 we have $M, w_0 \vDash^{+} \sigma_0 \prec \sigma_0$ and $M, w_0 \nvDash^{+} \sigma_1$. This implies that $(\phi \prec \phi) \to \psi$ is not a theorem of **2Int**, since M is a countermodel.

4 Future Work

In the future, we intend to continue analysis of **2Int**^{x}, working toward our goal of a dualized logic with canonicity. Specifically, we intend to investigate whether **2Int**^{x} has the property of canonicity, and look into term assignment with respect to the Curry-Howard isomorphism. Additionally, we plan on proving a completeness theorem for $\mathbf{2Int}^x$.

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