Useless Explicit Induction Reasoning

Sorin Stratulat

Université de Lorraine, CNRS, LORIA, F-57000 Metz, France
sorin.stratulat@univ-lorraine.fr

Abstract

In the setting of classical first-order logic with inductive predicates, we give as example a theory for which the conjectures that cannot be proved without induction reasoning are still not provable by adding explicit induction reasoning.

1 Introduction

In [2, 3, 4], Brotherston and Simpson presented different sequent-based inference systems, as LKID, able to perform induction reasoning in the setting of the classical first-order logic with inductive predicates and equality ($\mathrm{FOL}_{\mathrm{ID}}$). By adding induction reasoning, the effort spent for building proofs with these systems may be infinite and the result undecidable. In this context, it would be interesting to find theories where the addition of induction reasoning when proving conjectures in these theories is useless.

On the other hand, it is arguably much more difficult to show that a conjecture cannot be proved by induction. As example, Berardi and Tatsuta [1] disproved the Brotherston-Simpson conjecture (see Conjecture 7.7 in [4]) using arguments based on a model-theoretic approach by showing that there is a conjecture, known as 2-Hydra, which cannot be proved by explicit induction using the LKID system. Stratulat also identified in [5] a set of conjectures which did not succeed to prove using only Peano induction, by experimenting with the Coq proof assistant [6].

Here is presented a theory for which every conjecture that cannot be proved by LKID without induction reasoning also cannot be proved by adding explicit induction reasoning. We denote this theory by $\mathrm{LKID}(\{0, s, N\}, \Phi_N)$, where 0 is a constant symbol, s is a unary function symbol and N is an inductive predicate symbol, defined by the set Φ_N of productions:

$$N(0) \tag{1}$$

$$N(x) \Rightarrow N(s(x)) \tag{2}$$

The full version of the paper is given at https://members.loria.fr/SStratulat/files/PARISlong.pdf. In this paper, we omit the sections that describe the setting of $\mathrm{FOL}_{\mathrm{ID}}$ with equality and the LKID system, as well as the full proofs. Section 2 introduces our approach to analyse, exhaustively and syntactically, the application of the explicit induction rule $(Ind\ N)$

$$\frac{\Gamma \vdash F(0), \Delta \quad \Gamma, F(y) \vdash F(s(y)), \Delta \quad \Gamma, F(t) \vdash \Delta}{\Gamma, N(t) \vdash \Delta}\ (Ind\ N)$$

which is a left introduction of the inductive atom $N(t)$, for some term t, and uses $F(x_1)$ as induction hypothesis formula. The main result shows that there is no LKID proof of a sequent S, defined in the theory $\mathrm{LKID}(\{0, s, N\}, \Phi_N)$, if S cannot be proved by LKID without the use of the induction rule, excepting when it trivially integrates explicit induction principles. The last section outlines future works.

2 Exhaustive syntactic analysis of LKID derivations

Generally speaking, we say that a conjecture is proved by explicit induction reasoning if some of the induction hypotheses, representing 'not-yet proved' formulas, are used to prove induction conclusions, as they are defined by some explicit induction rule. When applied for proving sequents from the LKID($\{0, s, N\}, \Phi_N$) theory, this means that $F(y)$, which plays the role of the explicit induction hypothesis in the definition of $(Ind\ N)$, is used to reason on $F(s(y))$ or formulas deriving from it after the application of $(Ind\ N)$. Notice that not all the proofs including applications of the (Ind)-rule are using induction reasoning.

The sequents that can be proved without induction reasoning are called $(Ind)^{0,1}$-*provable*, i.e., they can be proved by LKID without (Ind)-steps or where the only allowed (Ind)-step is when applied on sequents that capture some induction principle. For the LKID($\{0, s, N\}, \Phi_N$) theory, these sequents may produce, by using (Ind)- and (Cut)-free derivations, sequents having in the antecedent part formulas of the form $\exists x, F(x) \Rightarrow F(s(x))$, where $F(x_1)$ is one of the formulas $(\neg)s^m 0 = s^n x_1$, $(\neg)s^m 0 = s^n x_1$, $(\neg)s^m x_1 = s^n x_1$, $(\neg)s^m x_2 = s^n x_1$, or $(\neg)s^m x_1 = s^n x_2$, for some naturals m and n, and variable x_2. $s^n t$ means the successive application for n times of s on t, for any term t.

We give below an example of $(Ind)^{0,1}$-provable sequents.

Lemma 1. *$\Gamma \vdash N s^n 0, \Delta$ is $(Ind)^{0,1}$-provable, for every $n \geq 0$.*

Our idea for showing that a sequent S is not LKID-provable in a given LKID theory is based on two assertions: Ai) S is not $(Ind)^{0,1}$-provable, and Aii) whenever (Ind) is applied on a sequent that is not $(Ind)^{0,1}$-provable, one of the premises of the (Ind) rule is also not $(Ind)^{0,1}$-provable.

Theorem 1. *For every sequent S that satisfies the assertion Ai) and LKID theory that satisfies the assertion Aii), S is not LKID-provable in that theory.*

Notice that Ai) is a property related to a particular sequent, while Aii) is a property that holds for a particular LKID theory. It is enough to check the two assertions only for (Cut)-free LKID derivations, thanks to Corollary 3.7 from [4].

We give as example the sequents $N(x), s^n x = s^{n+1} x \vdash$, for any $n \geq 0$, and show that they are not $(Ind)^{0,1}$-provable in the theory LKID($\{0, s, N\}, \Phi_N$).

2.1 Checking the assertion Ai)

We firstly prove the following lemma.

Lemma 2. *For every (Ind)-free LKID proof there is a (Cut)-free LKID proof which is also (Ind)-free.*

Then, this lemma is used to validate the assertion Ai) by the next lemma.

Lemma 3. *There is no $(Ind)^{0,1}$-proof of $N(x), s^n x = s^{n+1} x \vdash$, for any $n \geq 0$.*

2.2 Checking the assertion Aii)

We will perform a case analysis on the disjunctive normal form (DNF) of the induction hypothesis formula $F(x_1)$ used by the application of the $(Ind\ N)$ rule. The case analysis should take into account the different 'interpretations' of skolem terms, representing terms including skolem function symbols.

Let us assume that the DNF of $F(x_1)$ is $(L^1_1 \wedge \ldots \wedge L^1_{n_1}) \vee \ldots \vee (L^k_1 \wedge \ldots \wedge L^k_{n_k})$, where $k > 0$, $n_1 > 0$, $n_2, \ldots, n_k$ are naturals, and each L^i_j, with $i \in [1..k]$ and $j \in [1..n_i]$, is a literal that may include skolem function symbols. We denote by $\lfloor L \rfloor$ the interpretation of any literal L with skolem terms which can be either of the form $s^n 0$ or $s^n x$, for some natural $n \geq 0$ and variable x. We are looking for the interpretations for the literals of the DNF of $F(x_1)$ such that all the premises of $(Ind\ N)$ are $(Ind)^{0,1}$-provable. If none can be found, by exhaustive analysis of these interpretations, one can conclude that at least one of the premises is not $(Ind)^{0,1}$-provable.

The minor premise $\Gamma \vdash (L^1_1 \wedge \ldots \wedge L^1_{n_1}) \vee \ldots \vee (L^k_1 \wedge \ldots \wedge L^k_{n_k})[\{x_1 \mapsto 0\}], \Delta$ can be transformed by successively applying $(\vee R)$ for k times into the sequent $\Gamma \vdash (L^1_1 \wedge \ldots \wedge L^1_{n_1})[\{x_1 \mapsto 0\}], \ldots, (L^k_1 \wedge \ldots \wedge L^k_{n_k})[\{x_1 \mapsto 0\}], \Delta$. The premise is not $(Ind)^{0,1}$-provable if there is $j \in [1..k]$ such that $\Gamma \vdash (L^j_1 \wedge \ldots \wedge L^j_{n_j})[\{x_1 \mapsto 0\}], \Delta$ is not $(Ind)^{0,1}$-provable. By successively applying $(\wedge R)$ for n_j-times, we get n_j sequents of the form $\Gamma \vdash L^j_i[\{x_1 \mapsto 0\}], \Delta$, with $i \in [1..n_j]$, which should be $(Ind)^{0,1}$-provable. In the following, we will represent the DNF of $F(x_1)$ as $(L^j_1 \wedge \ldots \wedge L^j_{n_j}) \vee \Phi$, where Φ is the DNF of $F(x_1)$ without the disjunct $(L^j_1 \wedge \ldots \wedge L^j_{n_j})$.

We continue to analyse the minor premise $\Gamma, ((L^j_1 \wedge \ldots \wedge L^j_{n_j}) \vee \Phi)[\{x_1 \mapsto y\}] \vdash ((L^j_1 \wedge \ldots \wedge L^j_{n_j}) \vee \Phi)[\{x_1 \mapsto s(y)\}], \Delta$. By applying $(\vee R)$, we get $\Gamma, ((L^j_1 \wedge \ldots \wedge L^j_{n_j}) \vee \Phi)[\{x_1 \mapsto y\}] \vdash (L^j_1 \wedge \ldots \wedge L^j_{n_j})[\{x_1 \mapsto s(y)\}], \Phi[\{x_1 \mapsto s(y)\}], \Delta$, to which we can apply $(\vee L)$ to get the two sequents $\Gamma, (L^j_1 \wedge \ldots \wedge L^j_{n_j})[\{x_1 \mapsto y\}] \vdash (L^j_1 \wedge \ldots \wedge L^j_{n_j})[\{x_1 \mapsto s(y)\}], \Phi[\{x_1 \mapsto s(y)\}], \Delta$ and $\Gamma, \Phi[\{x_1 \mapsto y\}] \vdash (L^j_1 \wedge \ldots \wedge L^j_{n_j})[\{x_1 \mapsto s(y)\}], \Phi[\{x_1 \mapsto s(y)\}], \Delta$. By successively applying $(\wedge L)$ for n_j times on the first sequent, we get $\Gamma, L^j_1[\{x_1 \mapsto y\}], \ldots, L^j_{n_j}[\{x_1 \mapsto y\}] \vdash (L^j_1 \wedge \ldots \wedge L^j_{n_j})[\{x_1 \mapsto s(y)\}], \Phi[\{x_1 \mapsto s(y)\}], \Delta$. Again, by successively applying $(\wedge R)$ for n_j-times, we get the n_j sequents $\Gamma, L^j_1[\{x_1 \mapsto y\}], \ldots, L^j_{n_j}[\{x_1 \mapsto y\}] \vdash L^j_i[\{x_1 \mapsto s(y)\}], \Phi[\{x_1 \mapsto s(y)\}], \Delta$, for $i \in [1..n_j]$.

Finally, we analyse the major premise $\Gamma, ((L^j_1 \wedge \ldots \wedge L^j_{n_j}) \vee \Phi)[\{x_1 \mapsto t\}] \vdash \Delta$. By applying $(\vee L)$, we get the two sequents $\Gamma, (L^j_1 \wedge \ldots \wedge L^j_{n_j})[\{x_1 \mapsto t\}] \vdash \Delta$ and $\Gamma, \Phi[\{x_1 \mapsto t\}] \vdash \Delta$. Again, by successively applying $(\wedge L)$ for n_j times on the first sequent, we get $\Gamma, L^j_1[\{x_1 \mapsto t\}], \ldots, L^j_{n_j}[\{x_1 \mapsto t\}] \vdash \Delta$.

To sum up, we have to show that there is no interpretation such that the conditions

C-1) the n_j sequents $\Gamma \vdash \lfloor L^j_i \rfloor[\{x_1 \mapsto 0\}], \Delta$, with $i \in [1..n_j]$, are all $(Ind)^{0,1}$-provable;

C-2)
- (a) the n_j sequents $\Gamma, \lfloor L^j_1 \rfloor[\{x_1 \mapsto y\}], \ldots, \lfloor L^j_{n_j} \rfloor[\{x_1 \mapsto y\}] \vdash \lfloor L^j_i \rfloor[\{x_1 \mapsto s(y)\}], \Phi[\{x_1 \mapsto s(y)\}], \Delta$, with $i \in [1..n_j]$ are all $(Ind)^{0,1}$-provable, and
- (b) $\Gamma, \Phi[\{x_1 \mapsto y\}] \vdash (\lfloor L^j_1 \rfloor \wedge \ldots \wedge \lfloor L^j_{n_j} \rfloor)[\{x_1 \mapsto s(y)\}], \Phi[\{x_1 \mapsto s(y)\}], \Delta$ is $(Ind)^{0,1}$-provable;

C-3)
- (a) $\Gamma, \lfloor L^j_1 \rfloor[\{x_1 \mapsto t\}], \ldots, \lfloor L^j_{n_j} \rfloor[\{x_1 \mapsto t\}] \vdash \Delta$ is $(Ind)^{0,1}$-provable, and
- (b) $\Gamma, \Phi[\{x_1 \mapsto t\}] \vdash \Delta$ is $(Ind)^{0,1}$-provable

are *simultaneously* satisfied, meaning that some of the premises of $(Ind\ N)$ are not $(Ind)^{0,1}$-provable, as requested.

In the following, we analyse LKID($\{0, s, N\}, \Phi_N$), by considering the cases when i) $n_j = 1$, and ii) $n_j > 1$.

Lemma 4 (case $n_j = 1$)**.** *Let $\Gamma, N(t) \vdash \Delta$ be a sequent that is not $(Ind)^{0,1}$-provable. For every literal L and DNF Φ, if $(Ind\ N)$ is applied on $\Gamma, N(t) \vdash \Delta$, with $N(t)$ as principal formula and using $L \vee \Phi$ as induction hypothesis formula, then some of the premises of the (Ind)-step are not $(Ind)^{0,1}$-provable.*

Lemma 5 (case $n_j > 1$)**.** *Let $\Gamma, N(t) \vdash \Delta$ be a sequent that is not $(Ind)^{0,1}$-provable. If $(Ind\ N)$ is applied on $\Gamma, N(t) \vdash \Delta$, with $N(t)$ as principal formula and using $(L_1^j \wedge \ldots \wedge L_{n_j}^j) \vee \Phi$ as induction hypothesis formula, with $n_j > 1$, then some of the premises of the (Ind)-step are not $(Ind)^{0,1}$-provable.*

Theorem 2. *Let us assume that S is a sequent that is not $(Ind)^{0,1}$-provable in the theory LKID$(\{0, s, N\}, \Phi_N)$. Then, there is no LKID$(\{0, s, N\}, \Phi_N)$-proof of S.*

By Lemma 3 and Theorem 2, we can conclude that there is no LKID$(\{0, s, N\}, \Phi_N)$-proof of $N(x), s^n x = s^{n+1} x \vdash$, for any $n \geq 0$.

3 Conclusions and future works

We have shown that the conjectures not provable in LKID$(\{0, s, N\}, \Phi_N)$ without the use of induction reasoning are still not provable by adding explicit induction reasoning. Compared to [1], our setting does not need to assume that i) s is injective ($\forall x, y\ s(x) = s(y) \Rightarrow x = y$), and ii) 0 and s are free constructors ($\forall x, \neg 0 = s(x)$). The corresponding sequents, i.e., $N(x), N(y), s(x) = s(y) \vdash x = y$ and $N(x) \vdash \neg 0 = s(x)$, are not $(Ind)^{0,1}$-provable and we can conclude that there is no LKID$(\{0, s, N\}, \Phi_N)$-proof of them.

We think that there are many other similar theories to be discovered. The presented approach, based on the exhaustive syntactic analysis of the LKID rule applications, can be used for this purpose. In this direction, we intend to check the theory LKID$(\{0, s, N, R\}, \Phi_N \cup \Phi_R)$, where R is defined for the set of critical conjectures from [5].

It would be interesting to check our results also for cyclic reasoning, for example in the theory CLKID$^\omega(\{0, s, N\}, \Phi_N)$, by using the CLKID$^\omega$ system [4].

References

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