MODELS OF LINEAR LOGIC BASED ON THE SCHWARTZ ε-PRODUCT.

YOANN DABROWSKI AND MARIE KERJEAN

ABSTRACT. In this talk we want to present the recent results of [DK]. We construct several smooth classical denotational models of Linear Logic: they are smooth as non-linear proofs are interpreted as infinitely differentiable functions, and they feature an involutive linear negation. The starting point of this work consists in noticing that the multiplicative disjunction corresponds to the well-known Schwartz' epsilon product. Requiring its associativity then asks for a completeness notion, while the linear involutive negation is ensured by considering a good topology (the Arens topology) on the dual, ensuring that the linear involutive negation works as an orthogonality relation.

1. INTRODUCTION

Since the discovery of linear logic by Girard [Gir87], thirty years ago, many attempts have been made to obtain denotational models of linear logic in the context of some classes of vector spaces with linear proofs interpreted as linear maps [Blu96, Ehr02, Gir04, Ehr05, BET]. Models of linear logic are often inspired by coherent spaces, or by the relational model of linear logic. Coherent Banach spaces [Gir99], coherent probabilistic or coherent quantum spaces [Gir04] are Girard's attempts to extend the first model, as finiteness spaces [Ehr05] or Köthe spaces [Ehr02] were designed by Ehrhard as a vectorial version of the relational model. Following the construction of Differential linear logic [ER06], one would want moreover to find natural models of it where non-linear proofs are interpreted by some classes of smooth maps. This requires the use of more general objects of functional analysis which were not directly constructed from coherent spaces. We see this as a strong point, as it paves the way towards new computational interpretations of functional analytic constructions, and a denotational interpretation of continuous or infinite data objects.

Three difficulties appear in this semantical study of linear logic. The equivalence between a formula and its double negation in linear logic asks for the considered vector spaces to be isomorphic to their double duals. This is constraining in infinite dimension. This infinite dimensionality is strongly needed to interpret exponential connectives.Then one needs to find a good category with smooth functions as morphisms, which should give a Cartesian closed category. This is not at all trivial, and was solved by using a quantative setting, i.e. power series as the interpretation for non-linear proofs, in most of the previous works [Gir99, Gir04, Ehr05, Ehr02]. Finally, imposing a reflexivity condition to respect the first requirement usually implies issues of stability by natural tensor products of this condition, needed to model multiplicative connectives. This corresponds to the hard task of finding $*$-autonomous categories [Ba79]. As pointed out in [Ehr16], the only model of differential Linear logic using smooth maps [BET] misses annoyingly the $*$-autonomous property for classical linear logic.

Our paper solves all these issues simultaneously and produces several denotational models of classical linear logic with some classes of smooth maps as morphism in the Kleisli category of the monad.

2. A minimal model of MALL with the ε product

Consider a smooth and classical denotational model of Linear Logic in which formulas are interpreted by locally convex and separated topological $\mathbb{R}$-vector spaces (lcs). The dual of a space E is denoted by E', and the space of smooth functions between E and F is denoted by $\mathscr{C}^\infty(E,F)$. We denote by $\mathcal{L}(E,F)$ the space of linear continuous functions between two lcs. Then the comonad ! must verify, by involutivity of the linear negation:

$$\begin{aligned} !E &\simeq (!E)'' \\ &\simeq \mathcal{L}(!E,\mathbb{R})' \\ &\simeq \mathscr{C}^\infty(E,\mathbb{R})' \end{aligned}$$

Thus the exponential $!E$ identifies to a space of linear scalar maps defined on the space of scalar smooth functions $\mathscr{C}^\infty(E,\mathbb{R})$, that is to a space of distributions.

In historical denotational models of Linear Logic, the linear negation is interpreted as an orhtogonality. For example, in Kothe spaces [Ehr02], the dual $E^\perp$ of a space of sequence $E \subset \mathbb{K}^I$ is

$$E^\perp = \{y \in \mathbb{K}^I | \forall x \in E, \sum_I |x_i y_i| \text{ converges}\}.$$

Defined through a focused orthogonality, we have $E^{\perp\perp\perp} = E^\perp$ and thus one can complete a space by double negation, making it isomorphic to its double dual.

In the theory of topological vector spaces, it is not the case in general that $E''' \simeq E'$. Thus there is no trivial way to make a tensor product of two space isomorphic to its bidual, and to obtain a classical model of MLL. In this paper we consider the *Arens dual* E'_c, that is the vector space E' endowed with the topology of uniform convergence on absolutely convex compact subsets of E. In that particular case, we have always $((E'_c)'_c)'_c \simeq E'_c$. Thus this dual behaves like an orthogonality, and leaves hope for a smooth and classical model of LL.

The interpretation of the multiplicative conjunction $\otimes$ in an algebraic setting is straightforward : it's the tensor product, which enjoys a universal property with respect to bilinear functions. However, with topological vector spaces several notions of bilinear functions exists (continuous, separately continuous, hypocontinuous with respect to some bornology ...), and through it several notions of tensor product. However, the interpretation for the multiplicative disjunction is straightforward.

So as to construct distributions with values in any vector spaces, Schwartz [S] identifies a ε product, such that

$$\mathscr{C}^\infty(\mathbb{R}^n,\mathbb{R})\varepsilon E \simeq \mathscr{C}^\infty(\mathbb{R}^n, E).$$

This product has thus exactly the behaviour wanted for the $\parr$.

Definition 2.1. The ε product is defined as $E\varepsilon F = \mathcal{L}(E'_c, F)$ endowed with the topology of uniform convergence on equicontinuous subsets of E'.

Schwartz shows that this product is associative and commutative on quasi-complete spaces (that is spaces in which every bounded Cauchy net converges). However, reading his proofs, it is enough for spaces to be k-quasi-complete :

Definition 2.2. A lcs is k-quasi-complete when the closed absolutely convex hull of a compact space is still compact.

Thus we have a monoidal category of k-quasi-complete spaces and continuous linear maps, with ε as monoidal operator. We have a functorial idempotent operation $\hat{\_}^k$ of k-quasi-completion making any lcs k-quasi-complete. If E is a k-quasi-complete space, we write:

$$E^{\perp} := (\hat{E}'_c)^k$$

Proposition 2.3. *For a k-quasi-complete space E, $(E^{\perp})'_c$ is again k-quasi-complete space. Thus $E^{\perp\perp\perp} = E^{\perp}$.*

We call k-reflexive spaces the lcs such that $E \simeq E^{\perp\perp}$. We denote by $k-$ **Ref** the category of k-reflexive spaces and continuous linear maps. We define $E \otimes_k F := (E^{\perp}\varepsilon F^{\perp})^{\perp}$, which is k-reflexive by the above proposition. We obtain thus a $*$-autonomous category, whcih is easily enriched with product and co-product, making it a model of MALL.

3. Smooth functions

Kriegl and Michor [KM] define smooth function s as the functions $f : E \to F$ between lcs E and F preserving smooth curves: if $c : \mathbb{R} \to E$ is everywhere infinitely differentiable, then $f \circ c : \mathbb{R} \to F$ must be everywhere infinitely differentiable. This definition has the noticeable advantage to result in a cartesian closed category of Mackey-complete spaces and smooth functions, and was used in [BET] to construct an intuitionnistic model of DiLL.

Definition 3.1. A lcs is Mackey-complete if for every absolutely convex and closed bounded set B, the vector space E_B generated by B is complete for the norm $p_B : x \mapsto \sup\{\lambda \in \mathbb{R} | \lambda x \in B\}$.

Here we take a slighly different notion of smooth functions, similar to the one by Meise [Me], which behave better with respect to the Arens dual.

Definition 3.2. For X, F separated lcs we call $C^{\infty}_{co}(X, F)$ the space of infinitely many times Gâteaux-differentiable functions with derivatives continuous on compacts with value in the space $L^{n+1}_{co}(E, F) = L_{co}(E, L^n_{co}(E, F))$ with at each stage the topology of uniform convergence on compact sets. We put on it the topology of uniform convergence on compact sets of all derivatives in the space $L^n_{co}(E, F)$.

Proposition 3.3. *For any space $X_1, X_2 \in k-$ **Ref** and any Mackey-complete lcs F we have:*

$$C^{\infty}_{co}(X_1 \times X_2, F) \simeq C^{\infty}_{co}(X_1, C^{\infty}_{co}(X_2, F)).$$

4. Smoothness: new models from universal generators

From this preceding notion of smoothness and completeness, one defines (at least) two smooth and classical models of LL. Consider $\mathscr{C} \subset k-$ **Ref** a small category, and define for E and F Mackey-complete spaces:

$$\mathscr{C}^{\infty}_{\mathscr{C}}(E, F) := \{f : E \to F | \forall X \in \mathscr{C}, \forall c \in \mathscr{C}^{\infty}_{co}(X, E), f \circ c \in \mathscr{C}^{\infty}_{co}(X, F)\}$$

This space is endowed with the inductive topology induced by the family of $\mathscr{C}^{\infty}_{co}(X, F)$, for all $X \in \mathscr{C}$ and all $c \in \mathscr{C}^{\infty}_{co}(X, E)$. We show that linear functions are in particular of that type of smooth functions, and thus the inclusion $E' \subset \mathscr{C}^{\infty}_{\mathscr{C}}(E, \mathbb{R})$ induces a topology on E', which we denote by $E'_{\mathscr{C}}$. This new definition for smooth functions defines also a new topology on E which we describe now.

We first consider $\mathscr{C} \subset k-$ **Ref** a full Cartesian subcategory.

Let $\mathscr{C}^\infty$ be the smallest class of locally convex spaces containing $C^\infty_{co}(X,\mathbb{K})$ for $X \in \mathscr{C}$ ($X = \{0\}$ included) and stable by products and subspaces. Consider $\mathscr{S}_\mathscr{C}$ the functor on **LCS** of associated topology in this class described by [Ju, 2.6.4]. This functor maps a lcs E to the vector space E endowed with the finest topology, coarser than the original one on E, such that $\mathscr{S}_\mathscr{S}(E) \in \mathscr{C}^\infty$.

Then if E is Mackey complete we have $E'_\mathscr{C} \simeq \mathscr{S}_\mathscr{C}(E'_c)$. In the article, we construct moreover an inductive Mackey-completion procedure $\hat{\_}^M$ which is functorial with respect to continuous linear maps. We define then for E Mackey-complete:

$$E^{\perp_\mathscr{C}} := \widehat{\mathscr{S}_\mathscr{C}(E'_c)}^M$$

We say that a space is $\mathscr{C}$-reflexive if $E^{\perp_\mathscr{C}\perp_\mathscr{C}} \simeq E$. We have in particular that $E^{\perp_\mathscr{C}}$ is always $\mathscr{C}$-reflexive.

Theorem 4.1. *Consider $\mathscr{C}$ a small category containing finite dimensional vector spaces and contained in Banach spaces. Then the category of $\mathscr{C}$-reflexive spaces is $*$-autonomous, with tensor product $E\hat{\otimes}_\mathscr{C}F := (E^{\perp_\mathscr{C}}\varepsilon E^{\perp_\mathscr{C}})^{\perp_\mathscr{C}}$. It is a model of LL, with exponential:*

$$!_\mathscr{C}E := \mathscr{C}^\infty_\mathscr{C}(E,\mathbb{R})^{\perp_\mathscr{C}}.$$

We have two concrete examples of models smooth models of LL generated this way:

Example 4.1. If $\mathscr{C}$ consists in the category of all finite dimensional spaces, then $C^\infty_{co}(\mathbb{R}^n,\mathbb{K}) \simeq \mathfrak{s}^{\mathbb{N}}$ [V82, (7) p 383], where $\mathfrak{s}$ denotes the Köthe space of rapidly decreasing sequences. This space is a universal generator for nuclear lcs, meaning that every nuclear lcs is a subset of a product of copies of $\mathfrak{s}$. Thus the associated topology functor is $\mathscr{N}(E) = \mathscr{S}_{Fin}(E)$., and our model consists of Nuclear Mackey-complete spaces which equals their double $\mathscr{C}$-dual.

Example 4.2. If $\mathscr{C}$ is the category of Banach space, then $\mathscr{C}^\infty$ is the category of all Schwartz spaces, and our model consists of all Mackey-complete Schwartz space which equals their double $\mathscr{C}$-dual.

5. Differentiation

Smooth linear maps in the sense of Frölicher are bounded but not necessarily continuous. Taking the differential at 0 of functions in $\mathscr{C}^\infty(E,F)$ thus would not give us a morphisms in $\mathbf{k}-\mathbf{Ref}$, thus we have no interpretation for the codereliction $\overline{\mathsf{d}}$ of DiLL. In the paper, we thus restrict the definition of smooth functions to those which admits continuous differentials.

References

[Ba79] M Barr,:*-autonomous categories LNM 752 Springer-Verlag Berlin 1979.

[BET] R. Blute, T. Ehrhard, and C. Tasson. A convenient differential category. *Cah. Topol. Gom. Diffr. Catg.*, 53(3):211–232, 2012.

[Blu96] R. Blute. Hopf algebras and linear logic. *Mathematical Structures in Computer Science*, 6(2), 1996.

[DK] Y. Dabrowski and M .Kerjean:Models of Linear Logic based on the Schwartz epsilon product arXiv:1712.07344

[Ehr02] T. Ehrhard. On Köthe sequence spaces and linear logic. *Mathematical Structures in Computer Science*, 12(5):579–623, 2002.

[Ehr05] T. Ehrhard. Finiteness spaces. *Mathematical Structures in Computer Science*, 15(4):615–646, 2005.

[Ehr11] T. Ehrhard. A model-oriented introduction to differential linear logic. *preprint*, August 2011.

[Ehr16] T. Ehrhard. An introduction to Differential Linear Logic: proof-nets, models and antiderivatives *Math. Structures in Computer Science*, 1-66 (2017), published online.

[ER06] T. Ehrhard and L. Regnier. Differential Interaction nets. *Theoretical Computer Science*, 364(2): 166-195, 2006.

[Gir87] J.-Y. Girard. Linear logic. *Theoretical Computer Science*, 50(1):101, 1987.

[Gir99] J.-Y. Girard. Coherent Banach spaces: a continuous denotational semantics. *Theoretical Computer Science*, 227(1-2):275–297, 1999. Linear logic, I (Tokyo, 1996).

[Gir04] J.-Y. Girard. Between logic and quantic: a tract. In *Linear logic in computer science*, volume 316 of *London Math. Soc. Lecture Note Ser.*, pages 346–381. Cambridge Univ. Press, Cambridge, 2004.

[Ju] H. Junek: *Locally Convex Spaces and Operator Ideals*. Leipzig: B. G. Teubner (1983)

[Ker] M. Kerjean Weak topologies for Linear Logic *Logical Method in Computer Science, Volume 12, Issue 1, Paper 3, 2016* .

[KT] M. Kerjean and C. Tasson *Mackey-complete spaces and power series - A topological model of Differential Linear Logic* Mathematical Structures in Computer Science, 2016, 1–36. .

[KM] A. Kriegl, and P.W. Michor: *The Convenient Setting of Global Analysis*. Providence: American Mathematical Society (1997)

[Me] R. Meise Spaces of differentiable functions and the approximation property. *in Approximation Theory and functional Analysis.* J.B. Prolla (ed.) North-Holland 1979, 263–307.

[PAM] P.-A. Melliès. Categorical semantics of linear logic. In *Interactive models of computation and program behavior*, volume 27 of *Panorama Synthèses*, pages 1–196. Soc. Math. France, Paris, 2009.

[V82] M. Valdivia *Topics in Locally convex spaces.* North-Holland 1982.

[S] L. Schwartz Théorie des distributions valeurs vectorielles. *AIHP.* 7(1957), 1–141.

Université de Lyon, Université Lyon 1, Institut Camille Jordan UMR 5208, 43 blvd. du 11 novembre 1918, F-69622 Villeurbanne cedex, France

E-mail address: `dabrowski@math.univ-lyon1.fr`

Univ Paris Diderot, Sorbonne Paris Cit, IRIF, UMR 7126, CNRS, F-75205 Paris, France

E-mail address: `kerjean@irif.fr`