Towards intuitive reasoning in axiomatic geometry

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Abstract

Proving lemmas in synthetic geometry is an often time-consuming endeavour – many intermediate lemmas need to be proven before interesting results can be obtained. Automated theorem provers (ATP) made much progress in the recent years and can prove many of these intermediate lemmas automatically. The interactive theorem prover Elfe accepts mathematical texts written in fair English and verifies the text with the help of ATP. Geometrical texts can thereby easily be formalized in Elfe, leaving only the cornerstones of a proof to be derived by the user. This allows for teaching axiomatic geometry to students without prior experience in formal mathematics.

1 Introduction

Elfe is an interactive system for teaching basic proof methods in mathematics. Its goal is to provide users with a system that gives feedback on proofs entered in a fairly natural Mathematical language. Thereby the users are detached from the technicalities of automated theorem provers (ATP). Giving students immediate feedback on their proofs would greatly increase the learning curve – it is often difficult to see when a proof is complete or what steps are missing. Advanced interactive theorem provers Isabelle and Coq could provide such feedback. However, formal reasoning in axiomatic geometry is especially complex.

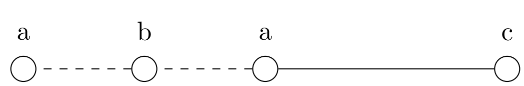
The goal of this work is to provide users with an environment where proofs about geometry can be done closely to how one would write it on a sheet of paper. This allows for letting students in the begin of their studies already use the advances in automated reasoning to dive into mathematical reasoning.

```
Include geometry.
Lemma BetwEquality: for all a,b,c. a-b-c and b-a-c implies a = b.
Proof:
   Assume a-b-c and b-a-c.
   Then a-b-a by BetweennessIdentity, Pasch.
   Hence a = b.
qed.
```

Figure 1: Excerpt of Elfe text about geometry

Consider the exemplary proof in Figure 1 which is in fact a valid Elfe text. First, a background library for geometry is included. This library contains Tarski's axiomatisation [3] of Euclidean geometry, written itself in Elfe. The notation a-b-c means that points a, b and c are collinear and b lies between a and c. The other relation used in Tarski's axiomatisation is congruence of two lines; this is depicted with a-b $\equiv$ c-d in Elfe.

The lemma states that if a lies between b and c and b lies between a and c, then a and b must be the same points. Intuitively, we can make that clear with the following diagram:



The points a and b collapse to the same point since b is enclosed by a on both sides. This exact reasoning is reflected in the proof text: We first assume the left hand side of the lemma and conclude a-b-a from it. That this derivation holds is checked by ATP in the background. Note that we stated explicitly which of the axioms are needed to derive this step, namely the identity criterion for the betweenness relation and the Pasch axiom. This specification can be omitted, then all of the axioms are given to the background provers.

The cooperation with ATP allows for using ELFE in a didactic environment. It is currently being tested by 80 students at the El Bosque University Colombia, who have to complete several tasks about mathematical relations with it[1]. ELFE has previously only been used for proving lemmas with sets, relations and functions. We are currently working on porting more proofs of the GEOCOQ[2] library to ELFE. GEOCOQ has been used in teaching high school students, we hope by comparing the different formalizations we will gain insights how synthetic geometry can best be taught to mathematical beginners.

Reasoning formally in synthetic geometry is often a laborious endeavour. A proof of the above statement in COQ as given in [2] requires to derive many intermediate lemmas before the seemingly simple lemma can be proved. By employing ATP in the background, tedious tasks can immediately be checked and do not require human work.

The ELFE system is implemented in HASKELL and can be accessed through a web interface or a command-line interface (CLI) as shown in Figure 2. After the text is entered via one of its interfaces, it will be transformed into a representation in first-order logic. Keywords like Then and Hence have special meanings in an ELFE proof and are used to structure a mathematical proof. This structure is captured in an intermediate proof representation called statement sequences. The internal representation implies certain proof obligations which are checked by ATPs. A more detailed explanation of the inner workings can be found in [1]; an instance of it can be accessed online[3].

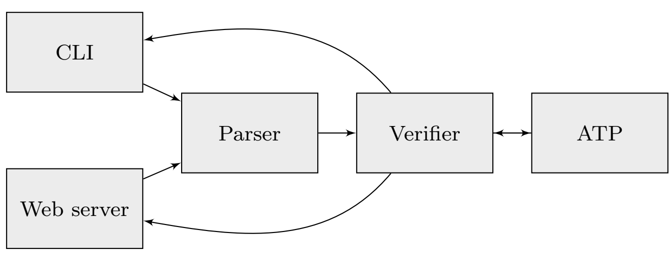


Figure 2: Architecture of the ELFE system

References

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[2] Julien Narboux. Mechanical theorem proving in tarski's geometry. In Francisco Botana and Tomas Recio, editors, *Automated Deduction in Geometry*, pages 139–156, Berlin, Heidelberg, 2007. Springer Berlin Heidelberg.

[3] Alfred Tarski and Steven Givant. Tarski's system of geometry. *Bulletin of Symbolic Logic*, 5:175–214, 1999.

[1] https://elfe-prover.org/tutorial
[2] http://geocoq.github.io/GeoCoq/
[3] https://elfe-prover.org