Riesz Modal Logic
with Threshold Operators

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Abstract

We present a sound and complete axiomatisation of the Riesz modal logic extended with one inductively defined operator which allows the definition of threshold operators. This logic is capable of interpreting the bounded fragment of the logic probabilistic CTL over discrete and continuous Markov chains.

Keywords Probabilistic logics, Real–Valued Semantics, Axiomatisation

1 Introduction

Modal and temporal logics are formalisms designed to express properties of directed graphs representing the behavior of computing systems. Well studied examples include *Computation Tree Logic* (CTL), the *modal μ-calculus*, *Propositional Dynamic Logic* (PDL), among others. Probabilistic temporal logics are variants of temporal logics specifically designed to express properties of probabilistic computing systems, often modelled as Markov chains, Markov decision processes or similar structures. Examples include *probabilistic CTL* (pCTL [1, 3, 8]), the *probabilistic μ-calculus* ([10, 12]), *probabilistic* (pPDL [5]), *etc.*

Among these logic, pCTL is arguably the best known at the present moment because it has a relatively simple definition, it is capable of expressing useful properties and, yet, it poses significant research problems. Indeed, despite more than three decades of research, some basic problems regarding pCTL remain open. These include the problem of axiomatisation and the decidability of the satisfiability problem. A seminal result was obtained by Lehmann and Shelah in [8] (see also [4]). They provided a sound and complete axiomatisation for the *qualitative fragment* of pCTL* (an extension of pCTL) and proved that the satisfiability problem for formulas belonging to this fragment is decidable. As the authors clearly state, however, their methods cannot be easily extended to the analysis of full (*quantitative*) pCTL. In fact, since this seminal paper, no much progress has been made towards an axiomatisation of full pCTL. The recent paper [7] presents an axiomatisation of a logic which can interpret a very limited fragment of quantitative pCTL but, again, it is not clear how to extend the axiomatisation to full pCTL.

In an attempt to make progress, some research has focused on an alternative, *real–valued* approach to probabilistic logics. Following Kozen's seminal work [5], formulas ϕ of these logics are interpreted as real–valued functions $[\![\phi]\!] : X \to \mathbb{R}$ on the state space X of a Markov chain, rather than (characteristic functions of) sets $[\![\phi]\!] : X \to \{0, 1\}$ as in ordinary probabilistic logics such as pCTL.

Following this line of research, the author has focused on studying probabilistic real–valued modal logics for probabilistic systems. One of the first observation made (see [10] and [12]) was that extending a simple basic real–valued modal logic with (co)inductively defined operators (*à la* Kozen μ-calculus) results in a very expressive formalism capable of interpreting the full logic pCTL:

$$\begin{array}{c}\text{Basic real–valued modal logic}\\ +\\ \text{(co)inductive operators}\end{array} \quad \supseteq \quad \text{pCTL}$$

This observation suggested the following research program towards the axiomatisation of pCTL:

1. identify a convenient basic real–valued modal logic $\mathcal{L}$ which, once extended with (co)inductively defined operators (as in [10] and [12]), can interpret pCTL,
2. obtain a sound and complete axiomatisation of $\mathcal{L}$,
3. extend $\mathcal{L}$ with the required (co)inductive operators,
4. try to axiomatise the extended logic using the axiomatisation of $\mathcal{L}$ and the theoretical machinery for reasoning about fixed–points and (co)inductive definitions.

Carrying on this research program, in [11] a basic real–valued modal logic called *Riesz modal logic* ($\mathcal{R}$) was identified (step 1). The semantics of the logic $\mathcal{R}$ is based on the theory of *Riesz spaces*, also known as *vector lattices*. Historically, research on Riesz spaces was pioneered in the 1930's by F. Riesz, L. Kantorovich and H. Freudenthal among others and was motivated by the applications in the study of function spaces ($X \to \mathbb{R}$) in functional analysis. Using some key results of this theory, most notably the Riesz representation theorem and the Yosida representation theorem, a sound and complete axiomatisation of $\mathcal{R}$ was obtained in [11] with respect to a large class of (discrete and continuous) Markov chains (step 2).

1.1 Contribution

This work is a continuation of [11] and deals with steps 3–4 from the research program outlined above. We extend the Riesz modal logic $\mathcal{R}$ with a unary inductively defined predicate P(_) whose semantics is:

$$[\![\mathrm{P}\phi]\!](x) = \begin{cases} 1 & \text{if } [\![\phi]\!](x) > 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus the semantics of $[\![\mathrm{P}\phi]\!]$ is the characteristic function of a set of states. The possibility of turning a real–valued function ϕ into a set, which comes available when the logic $\mathcal{R}$ is extended with P (denoted by $\mathcal{R}_{\{\mathrm{P}\}}$), is a key element for the interpretation of Boolean logics. We indeed show that the logic $\mathcal{R}_{\{\mathrm{P}\}}$ can interpret the *bounded fragment* of pCTL (step 3).

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The main contribution of this work is a sound and complete axiomatisation of the logic $\mathcal{R}_{\{P\}}$ over (discrete and continuous) Markov chains (step 4). This result is obtained once again by applying key results from the theory of Riesz spaces, most notably a theorem of Yudin (about the Dedekind completion of Archimedean Riesz spaces) and a theorem of Kantorovich (about the extension of positive additive operators).

Remarkably, our axiomatisation is purely equational and the only inference rules are those of equational logic. This may be seen as an improvement over axiomatisations of other probabilistic logics (see, e.g., [6, 7]) which rely on the use of infinitary inference rules (i.e., having countable number of premises).

2 Markov Chains and Probabilistic Logics

Definition 2.1. Given a set X we denote with $\mathcal{D}^{\leq 1}(X)$

$$\mathcal{D}^{\leq 1}(X) = \{d : X \to [0,1] \mid \sum_{x} d(x) \leq 1\}$$

the collection of *sub–probability distributions* on X. We denote with $\mathcal{D}^{=1}(X)$ the collection of full probability distributions. With some abuse of notation, given a subset $A \subseteq X$ we denote with $d(A)$ the cumulative probability of A:

$$d(A) = \sum \{d(x) \mid x \in A\}.$$

The value $d(X)$ is called the *mass* of d.

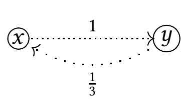
Definition 2.2. Let X be a set. Then for every $d \in \mathcal{D}^{\leq 1}(X)$ and bounded function $f : X \to \mathbb{R}$ we define the *expectation of f relative to the sub–probability distribution d* as:

$$\mathbf{E}(f, d) = \sum_{x \in X} f(x) \cdot d(x)$$

Definition 2.3. A *Markov chain* is a pair (X, τ) where X is the (possibly infinite) set of states and $\tau : X \to \mathcal{D}^{\leq 1}(X)$ it the transition function which maps each state to a sub–probability distribution over states.

The intended interpretation is that, at a state $x \in X$, the computation stops with probability $1 - d(X)$, where $d = \tau(x)$, and continues with probability $d(X)$ following the (normalized) probability distribution d.

Example 2.1. Consider the Markov chain (X, τ) with $X = \{x, y\}$ and $\tau : X \to \mathcal{D}^{\leq 1}(X)$ defined by: $\tau(x) = \{0 : x, 1 : y\}$ and $\tau(y) = \{\frac{1}{3} : x, 0 : y\}$,



From state x the computation progresses to y with probability 1. From state y the computation halts with probability $\frac{2}{3}$ and progresses to x with probability $\frac{1}{3}$.

Remark 2.2. Other similar definitions include Markov chains with full distributions rather than sub–probability distributions, labelled Markov chains $(X, \{\tau_b\}_{b \in L})$ having a set of transition functions indexed by a set L of labels, and many others. Our definition follows the development of [11]. All the results of this work can be adapted to most variants of Markov chains with only minor variations. See discussion in Section 7.

We denote with X^∞ the set of non–empty finite and infinite sequences of states of the Markov chain (X, τ). Given a state $x \in X$ and a finite sequence $\vec{y} = \langle y_0, \ldots, y_n \rangle \in X^\infty$ of states, the Markov chain naturally determines the probability $\overline{\tau_x}(\vec{y})$ of this sequence occurring when starting from the state x as follows:

$$\overline{\tau_x}(\vec{y}) = \begin{cases} 0 & \text{if } y_0 \neq x_0 \\ \lambda_0 \cdot \lambda_1 \cdot \ldots \lambda_{n-1} & \text{otherwise} \end{cases}$$

where $\lambda_i = \big(\tau(y_i)\big)(y_{i+1})$, for all $i \in \{0, \ldots, n-1\}$, and where we adopt the convention $\Pi\emptyset = 1$ so that $\tau_x(\langle x \rangle) = 1$. Using standard methods (see, e.g., [1, §10.1]) it is possible to extend the assignment $\overline{\tau_x}$ to a sub–probability Borel measure μ_x on the whole topological space X^∞ (where X is endowed with the discrete topology). We omit the standard definitions and proceed directly with the following.

Definition 2.4. Given a Markov chain (X, τ) and $x \in X$ we denote with μ_x the sub–probability Borel measure on X^∞ induced by $\overline{\tau_x}$.

We adopt the standard notation of linear temporal logics for describing subsets of X^∞.

Definition 2.5. Given subsets $A, B \subseteq X$ and $L \subseteq X^\infty$ we denote:

- $\{A\} = \{(x_0, x_1, \ldots) \in X^\infty \mid x_0 \in A\}$
- $\circ L = \{(x_0, x_1, \ldots) \in X^\infty \mid (x_1, x_2, \ldots,) \in L\}$
- $A\, \mathcal{U}^{\leq 0}\, B = \{B\}$
- $A\, \mathcal{U}^{\leq n+1} B = \{B\} \cup \Big(\{(A \cap (X \setminus B))\} \cap \circ\big(A\, \mathcal{U}^{\leq n}\, B\big)\Big)$
- $A\, \mathcal{U}\, B = \bigcup_n (A\, \mathcal{U}^{\leq n}\, B)$

Thus $\{A\}$ is the set of sequences whose first state is in A and $\circ L$ is the set of sequences $(x_0, x_1, \ldots)$ whose tail $(x_1, \ldots)$ belongs to L. So for example $\circ\{A\}$ denotes the set of sequences having at least two elements and whose second element is in A. The inductively defined set $A\, \mathcal{U}^{\leq n} B$ is made of sequences $(x_0, \ldots)$ with $x_i \in B$, for some $i \leq n$ and $x_j \in A$ for all $j < i$. Lastly, $A\, \mathcal{U}\, B$ consists of sequences $(x_0, \ldots)$ with $x_i \in B$, for some $i \in \mathbb{N}$ and $x_j \in A$ for all $j < i$.

2.1 Probabilistic CTL and its Bounded Fragment

In this subsection we give the basic definitions of the logic *probabilistic CTL* (or just pCTL) and its interpretation on Markov chains. We refer to [1, §10.2] for a detailed introduction.

Definition 2.6 (Syntax)**.** The set of pCTL formulas is generated by the following grammar $F, G ::=$

$$\top \mid F \wedge G \mid \sim G \mid \mathbb{P}_{*p} \circ F \mid \mathbb{P}_{*p}(F\, \mathcal{U}^{\leq n}\, G) \mid \mathbb{P}_{*p}(F\, \mathcal{U}\, G)$$

where $* \in \{>, \geq\}$ and $p \in [0, 1]$.

The set of *bounded formulas* is given by formulas not containing the $\mathbb{P}_{*p}(F\, \mathcal{U}\, G)$ operator.

Remark 2.3. The version of the logic pCTL presented in [1, §10.1] is interpreted over Markov chains having full probability distributions and not sub–probability distributions. As already mentioned (see also discussion in Section 7) this is a minor difference and all the results of this paper can be reformulated to agree with the definition of [1, §10.1].

Definition 2.7 (Semantics)**.** Given a Markov chain (X, τ), the semantics (or interpretation) of a pCTL formula F is the set $(\!| F |\!)_\tau \subseteq X$ defined by induction on F as follows:

$$(\!| \top |\!)_\tau = X \qquad (\!| \sim F |\!)_\tau = X \setminus (\!| F |\!)_\tau \qquad (\!| F \vee G |\!)_\tau = (\!| F |\!)_\tau \cup (\!| G |\!)_\tau$$

$$(\!|\mathbb{P}_{\star p}(\circ F)|\!) = \{x \in X \mid \mu_x(\circ\{A\}) \star p\}$$
$$(\!|\mathbb{P}_{\star p}(F\,\mathcal{U}^{\leq n}\,G)|\!) = \{x \in X \mid \mu_x(A\,\mathcal{U}^{\leq n}\,B) \star p\}$$
$$(\!|\mathbb{P}_{\star p}(F\,\mathcal{U}\,G)|\!) = \{x \in X \mid \mu_x(A\,\mathcal{U}\,B) \star p\}$$

with $\star p \in \{> p, \geq p\}$ and where $A, B \subseteq X$ are defined as $A = (\!|F|\!)_\tau$ and $B = (\!|G|\!)_\tau$.

Example 2.4. Consider the Markov chain of Example 2.1 and the bounded pCTL formula $F = \mathbb{P}_{>\frac{1}{2}}(\circ\top)$. Then $(\!|F|\!)_\tau = \{x\}$ as the formula holds on those states that can progress to any other state with probability strictly greater than $\frac{1}{2}$. Consider now the formula $G = \mathbb{P}_{\geq 1}(\top\,\mathcal{U}^{\leq 7}\,\sim F)$. The formula G holds on those state that reach a state satisfying $\sim F$, in at most 7 steps, with probability 1. Clearly both y and x satisfy this property since y is already in $\sim F$ and x reaches y with probability 1 in one step.

2.2 Riesz Modal Logic

We now recall the definition of the Riesz modal logic $\mathcal{R}$ from [11].

Definition 2.8 (Syntax)**.** The set of formulas of $\mathcal{R}$ is generated by the following grammar:

$$\phi, \psi ::= 0 \mid 1 \mid r\phi \mid \phi + \psi \mid \phi \sqcup \psi \mid \phi \sqcap \psi \mid \Diamond\phi \qquad \text{where } r \in \mathbb{R}.$$

The quantitative semantics of formulas is given as follows.

Definition 2.9 (Semantics)**.** Given a Markov chain (X, τ), the semantics (or interpretation) of a formula ϕ is the real–valued function $[\![\phi]\!]_\tau : X \to \mathbb{R}$ defined by induction on ϕ as follows:

$$[\![0]\!]_\tau(x) = 0 \qquad [\![1]\!]_\tau(x) = 1$$
$$[\![r\phi]\!]_\tau(x) = r \cdot ([\![\phi]\!]_\tau(x)) \qquad [\![\phi + \psi]\!]_\tau(x) = [\![\phi]\!]_\tau(x) + [\![\psi]\!]_\tau(x)$$
$$[\![\phi \sqcup \psi]\!]_\tau(x) = \max\{[\![\phi]\!]_\tau(x), [\![\psi]\!]_\tau(x)\}$$
$$[\![\phi \sqcap \psi]\!]_\tau(x) = \min\{[\![\phi]\!]_\tau(x), [\![\psi]\!]_\tau(x)\}$$
$$[\![\Diamond\phi]\!]_\tau(x) = \mathbb{E}([\![\phi]\!]_\tau, \tau(x)) = \sum_{y \in X} [\![\phi]\!]_\tau(y) \cdot \tau(x)(y)$$

Beside the arithmetic operators having the expected meaning, the formula $\Diamond\phi$ is the function that assigns to the state x the expectation of the (interpretation of) the formula ϕ relative to to the sub–probability distribution $\tau(x)$.

Remark 2.5. It is immediate to verify that the semantics of every formula ϕ is a bounded function and therefore the semantics of the $\Diamond$ connective, which uses the expectation functional, is well defined.

Example 2.6. Let us consider the Markov chain of Example 2.1 and consider the Riesz modal logic formula $\Diamond 1$. The formula $\Diamond 1$ can be understood as mapping each state $x \in X$ to the total mass of the sub–probability distribution $\tau(x)$. Thus we have $[\![\Diamond 1]\!]_\tau(x) = 1$ and $[\![\Diamond 1]\!]_\tau(y) = \frac{1}{3}$.

When the Markov chain (X, τ) is clear from the context, and no confusion arises, we will just write $\phi(x)$ in place of $[\![\phi]\!]_\tau(x)$. We will often make use of the following useful abbreviations: $-\phi = (-1)\phi$, $\phi^+ = \phi \sqcup 0$ (the non–negative part), $|\phi| = \phi^+ + (-\phi)^+$ (the absolute value) and $[\phi] = \phi^+ \sqcap 1$ (the restriction to $[0, 1]$).

2.3 Riesz modal logic with threshold operators

We now extended the Riesz modal logic with one additional unary operator denoted by P.

Definition 2.10. The unary operator P is defined as follows:

$$[\![\mathrm{P}\phi]\!]_\tau(x) = \begin{cases} 1 & \text{if } [\![\phi]\!]_\tau(x) > 0 \\ 0 & \text{otherwise} \end{cases}$$

We denote from now on with $\mathcal{R}_{\{\mathrm{P}\}}$ the Riesz modal logic extended with P.

The interpretation of the formula $\mathrm{P}\phi$ is always a $\{0, 1\}$-valued function on X and it is the characteristic function of the set of states getting strictly positive values by ϕ.

We will refer to $\{0, 1\}$-valued functions as *Boolean* functions, since they corresponds to (characteristic functions of) subsets of X. Note that a function f is Boolean if and only if $f = \mathrm{P}f$ and that $\mathrm{PP}f = \mathrm{P}f$.

Remark 2.7. The usual operations of union ($\cup$), intersection and complementation ($\neg$) can be easily defined on Boolean functions f, g as follows:

$$f \cup g = f \sqcup g \quad \neg f = 1 - f \quad f \cap g = f \sqcap g.$$

The P operator is useful because it allows the definition of other *threshold operators.* For example the useful operators $\mathbb{T}_{\star p}$, with $p \in [0, 1]$ and $\star \in \{>, \geq\}$ defined as:

$$[\![\mathbb{T}_{\star p}\phi]\!]_\tau(x) = \begin{cases} 1 & \text{if } [\![\phi]\!]_\tau(x) \star p \\ 0 & \text{otherwise} \end{cases}$$

can be encoded using the P operator. First, note that for $p = 0$

$$\mathbb{T}_{>0}\phi = \mathrm{P}\phi$$
$$\mathbb{T}_{\geq 0}\phi = \neg\mathbb{T}_{>0}(-\phi)$$

Then, for the general case $p \in [0, 1]$, we have that

$$\mathbb{T}_{>p}\phi = \mathbb{T}_{>0}(\phi - p1)$$
$$\mathbb{T}_{\geq p}\phi = \mathbb{T}_{\geq 0}(\phi - p1).$$

Example 2.8. Consider the formula $\psi = \mathbb{T}_{>\frac{1}{2}}(\Diamond\top)$. The semantics of the formula ψ is the (characteristic function of the) set of states in the Markov chain having outgoing probability strictly greater than $\frac{1}{2}$. The same property is expressible by the pCTL formula $\mathbb{P}_{>\frac{1}{2}}(\circ\top)$. Considering the Markov chain (X, τ) of Example 2.1 we have that $[\![\psi]\!]_\tau(x) = 1$ and $[\![\psi]\!]_\tau(y) = 0$

Example 2.9. Let $\gamma = \neg\psi$, where ψ is defined as above. Thus γ expresses the same property of the pCTL formula $\sim F$ of Example 2.4. Consider the $\mathcal{R}$ formula $\mathbb{T}_{\geq 1}(\gamma \sqcup \Diamond\gamma)$. It is an useful exercise to verify that this formula expresses the same property as the pCTL formula $\mathbb{P}_{\geq 1}(\top\,\mathcal{U}^{\leq 1}\,\sim F)$.

We are now ready to state the following expressivity results.

Proposition 2.11. *The bounded fragment of pCTL can be interpreted in the logic $\mathcal{R}_{\{\mathrm{P}\}}$ by the following inductively defined translation t from pCTL formulas to $\mathcal{R}_{\{\mathrm{P}\}}$ formulas:*

pCTL formula F		$\mathcal{R}_{\{\mathrm{P}\}}$ *formula* $t(F)$
$t(\top)$	=	1
$t(\sim F)$	=	$1 - t(F)$
$t(F \vee G)$	=	$t(F) \sqcup t(G)$
$t(\mathbb{P}_{\star p}(\circ F))$	=	$\mathbb{T}_{\star p}(\Diamond t(F))$
$t(\mathbb{P}_{\star p}(F\,\mathcal{U}^{\leq n}\,G))$	=	$\mathbb{T}_{\star p}(\theta^n_{F,G})$

where the formula $\theta^n(F,G)$ is defined inductively as follows:

$$\begin{array}{rcl}\theta^0_{F,G} & = & t(G)\\ \theta^{n+1}_{F,G} & = & t(G) \sqcup \big(t(F) \sqcap \Diamond\theta^n_{F,G}\big)\end{array}$$

Proof. We need to show that for all bounded pCTL formulas F it holds that $(\!| F |\!)_\tau = [\![t(F)]\!]_\tau$ (the set $(\!| F |\!)_\tau \subseteq X$ is identified as its corresponding Boolean function).

The proof, by structural induction on F, is straightforward for the cases regarding the Boolean connectives and the pCTL operator $\mathbb{P}_{\star p}(\circ\_)$ given the definition of $\mathbb{T}_{\star p}$. For the case regarding the pCTL operator $\mathbb{P}_{\star p}(F\,\mathcal{U}^{\leq n}\,G)$ it is sufficient to verify, by induction on n, that $[\![\theta^n_{F,G}]\!](x) = \mu_x((\!| F |\!)_\tau\,\mathcal{U}^{\leq n}\,(\!| G |\!)_\tau)$, for all $x\in X$.

Case $n=0$. By Definition 2.5 we have

$$\mu_x((\!| F |\!)_\tau\,\mathcal{U}^{\leq 0}\,(\!| G |\!)_\tau) = \mu_x(\{(\!| G |\!)_\tau\})$$

and $\mu_x(\{(\!| G |\!)_\tau\})(y)=1$ if and only if $y \in (\!| G |\!)_\tau$. This, by induction hypothesis on G, coincides with the semantics of $[\![\theta^0_{F,G}]\!] = [\![t(G)]\!]_\tau$ as desired.

Case $n+1$. By definition of $\theta^n_{F,G}$ we get

$$[\![\theta^{n+1}_{F,G}]\!]_\tau(x) = \begin{cases} 1 & \text{if } x \in [\![t(G)]\!]_\tau\\ 0 & \text{if } x \notin [\![t(G)]\!]_\tau \cap [\![t(F)]\!]_\tau\\ [\![\Diamond\theta^n_{F,G}]\!]_\tau(x) & \text{otherwise}\end{cases}$$

From the definition of the set $L = ((\!| F |\!)_\tau\,\mathcal{U}^{\leq n}\,(\!| G |\!)_\tau) \subseteq X^\infty$ we get:

$$\mu_x(L) = \begin{cases} 1 & \text{if } x \in (\!| G |\!)_\tau\\ 0 & \text{if } x \notin (\!| G |\!)_\tau \cap (\!| F |\!)_\tau\\ \mu_x(\circ((\!| F |\!)_\tau\,\mathcal{U}^{\leq n}\,(\!| G |\!)_\tau)) & \text{otherwise}\end{cases}$$

It remains to verify that $[\![\Diamond\theta^n_{F,G}]\!]_\tau(x)$ and $\mu_x(\circ((\!| F |\!)_\tau\,\mathcal{U}^{\leq n}\,(\!| G |\!)))$ coincide and this follows from the inductive hypothesis and the definition of $\circ(\_)$ and $[\![\Diamond\_]\!]_\tau$. □

3 Inductive characterization of P

The goal of this section is to express the semantics of the operators P, introduced in the previous section, by means of an inductive definition.

We first recall the fundamental Knaster–Tarski fixed–point theorem.

Definition 3.1. Given a lattice L and a monotone function $f : L \to L$, a point $a \in L$ is called a *prefixed–point* (of f) if $a \geq f(a)$ and it is called a *fixed–point* if $a = f(a)$.

Theorem 3.2 (Knaster–Tarski (A)). *Let L be a lattice and $f : L \to L$ be monotone function. If $a \in L$ is the least prefixed–point of f then it is also the least fixed–point.*

Theorem 3.3 (Knaster–Tarski (B)). *Let L be a complete lattice and $f : L \to L$ be monotone function. Then the least fixed–point of f exists and is definable as $\bigsqcup_\alpha f^\alpha$ where*

$$f^0 = \bot \quad f^{\alpha+1} = f(f^\alpha) \quad f^\beta = \bigsqcup_{\alpha<\beta} f^\alpha$$

where α ranges over ordinals and β over limit ordinals. Furthermore the least fixed–point is the least prefixed–point.

Recall, from the previous section, that the meaning of the abbreviation $[\phi]$ is

$$[\![[\phi]]\!]_Y(x) = \begin{cases} 1 & \text{if } [\![\phi]\!]_\tau(x) > 1\\ [\![\phi]\!]_Y(x) & \text{if } [\![\phi]\!]_\tau(x) \in [0,1]\\ 0 & \text{if } [\![\phi]\!]_\tau(x) < 0\end{cases}$$

Proposition 3.4. *Let (X,τ) be a Markov chain. Then for every formula ϕ, the semantics $[\![\mathrm{P}\phi]\!]_Y$ is the least solution (in the lattice of functions $f : X \to \mathbb{R}$ ordered pointwise) satisfying the equation $f = [\phi + f]$, or just written as:*

$$\mathrm{P}\phi \overset{\mu}{=} [\phi + \mathrm{P}\phi].$$

Proof. By Lemma 3.5 and Lemma 3.6 below, the least prefixed–point of $f \mapsto [\phi + f]$ is the indicator function of the smallest set which contains $\{x \mid \phi(x) > 0\}$ and this is the semantics of $\mathrm{P}\phi$. □

Lemma 3.5. *A function $f : X \to \mathbb{R}$ is a prefixed–point of $f \mapsto [\phi + f]$ if and only if it has the following properties:*

1. *for all $x \in X$ it holds that $f(x) \geq 0$, and*
2. *for all $x \in X$, if $\phi(x) > 0$ then $f(x) \geq 1$,*

Proof. First we show that any prefixed–point has these two properties. The first property follows directly from the definition of the operation $[\_]$. For the second point, assume towards a contradiction that $\phi(x) = a > 0$ and $f(x) = b < 1$, for some $x \in X$. Since f is a prefixed–point we get $b \geq [a + b]$ and this is a contradiction because $b < [a + b]$.

Now, take any function f with the two properties above, we need to show that f is prefixed–point, i.e., $f(x) \geq [\phi + f(x)]$ for all $x \in X$. There are two cases: if $\phi(x) \leq 0$ then $[\phi(x) + f(x)] \leq [f(x)]$ and, since $f(x) \geq 0$ we have that $[f(x)] \leq f(x)$ and thus the desired inequality holds; if $\phi(x) > 0$ then, by the second property $f(x) \geq 1$, and since $[\phi(x)+f(x)] \leq 1$ the desired equality holds. □

Lemma 3.6. *The least (pre)fixed–point $f \overset{\mu}{=} [\phi + f]$ is a Boolean function.*

Proof. First observe that $f \leq 1$. Indeed, towards a contradiction, if $f \not\leq 1$ then $f \sqcap 1 \lneq f$ and since $f \sqcap 1$ satisfies the properties of Lemma 3.5 it is a prefixed–point smaller than f. Hence we have established that $f : X \to [0,1]$.

To show that f is a Boolean function, assume towards a contradiction that $f : X \to [0,1]$ is the least (pre)fixed–point and, for some $x \in X$ it holds that $0 < f(x) < 1$. But then the function $g(x) = f(x) \cdot f(x)$ has the properties of Lemma 3.5 and thus is a prefixed–point and clearly $g \lneq f$. A contradiction. □

4 The Axiomatisation

The goal of this section is to present the axiomatisation of the logic $\mathcal{R}_{\{\mathrm{P}\}}$ which will later be proved complete with respect to the appropriate class of models.

The axiomatisation is simply obtained by putting together axioms expressing:

1. the axioms of the Riesz modal logic $\mathcal{R}$ from [11], shown in Figure 1, and
2. some axioms expressing the inductive definition

$$\mathrm{P}(\phi) \overset{\mu}{=} [\phi + \mathrm{P}\phi]$$

shown in Figure 2.

1a) axioms of $\mathbb{R}$-vector spaces: $f + (g + h) = (f + g) + h$, $f + g = g + f$, $f + 0 = f$, $f + (-f) = 0$, $1f = f$, $r(r'f) = (r \cdot r')f$, $r(f + g) = rf + rg$, $(r + r')f = rf + r'f$, where $-f$ is the abbreviation of $(-1)f$.
1b) axioms of lattices: $f \sqcup (g \sqcup h) = (f \sqcup g) \sqcup h$, $f \sqcup g = g \sqcup f$, $f \sqcup (f \sqcap g) = f$, $f \sqcap (g \sqcap h) = (f \sqcap g) \sqcap h$, $f \sqcap g = g \sqcap f$, $f \sqcap (f \sqcup g) = f$,
1c) compatibility axioms: $0 \leq 1$, $(f \sqcap g) + h \leq g + h$ and $0 \leq r(f \sqcup 0)$, for all $r \in \mathbb{R}_{\geq 0}$, where $f \leq g$ is the abbreviation of $f \sqcap g = f$,
1d) modal axioms:
(Linearity) $\diamond(f+g) = \diamond(f)+\diamond(g)$ and $\diamond(rf) = r(\diamond f)$, for all $r \in \mathbb{R}$,
(Positivity) $\diamond(f \sqcup 0) \geq 0$,
(1-decreasing) $\diamond(1) \leq 1$.

Figure 1. Axioms of the Riesz Modal Logic $\mathcal{R}$.

Note that the axiomatisation of the Riesz modal logic $\mathcal{R}$ from [11], shown in Figure 1, is a purely equational axiomatisation and the only deduction rules are those of equational logic (i.e., substitution of equals for equals).

2a) $\mathrm{P}(g) \leq \mathrm{P}(g \sqcup g')$
2b) $\mathrm{P}g = [g + \mathrm{P}g]$
2c) $g - [g] \leq g - \mathrm{P}(g - [g])$

Figure 2. Axioms of the P operator.

An axiomatisation of the inductively defined operator P could be obtained using the well–known Park induction rules [14] expressing that $\mathrm{P}\phi$ is the least prefixed–point of $[f + \phi]$:

$$[f + \mathrm{P}f] \leq \mathrm{P}f \qquad [f + g] \leq g \Rightarrow \mathrm{P}f \leq g$$

The second of these rules is, however, an equational implication and not an equation. The methods we will use to investigate the axiomatisation and prove the completeness theorem are algebraic in nature. For this reason it is useful to work with a purely equational axiomatisation such as the one for $\mathcal{R}$ of Figure 1. Fortunately, by invoking a theorem of Santocanale [15], it is possible to express the inductive definition of the operator P by means of equalities. The price to pay is a less transparent axiomatisation.

The rest of this section is devoted to the application of Santocanale's theorem from [15] to derive the equations expressing the inductive definition of P shown in Figure 2. The full axiomatisation of the logic $\mathcal{R}_{\{\mathrm{P}\}}$ is obtained by merging the equations of Figure 1 and Figure 2.

Remark 4.1. We will never make use of the concrete equational axioms of Figure 2 and instead just use the knowledge that they correctly express the least prefixed–point definition of P. Hence the reader can safely skip Subsection 4.1.

4.1 Application of Santocanale's Theorems

The techniques of [15] for deriving equational axiomatisations of operators defined by least fixed–point equations are applicable to all equational theories extending the theory of *lattices with a closed structure*. The theory of $\mathcal{R}$ satisfies this requirement as it extends the theory of lattice–ordered abelian groups and it is therefore an ordered structure (the lattice order $f \leq g \Leftrightarrow f = f \sqcap g$) where the closed structure is given by the operations $x \otimes y = x + y$ and $x \multimap y = -x + y$. We refer to [15] for further details.

We now state Proposition 2.11 from [15] as a Theorem by adapting the notation and terminology to the present setting.

Theorem 4.1. *Let $\mathcal{T}$ be an equational theory extending the equational theory of $\mathcal{R}$. Let $F_1(f)$ and $F_2(f)$ be terms with variable f and assume they are both monotone in f, i.e., that the equalities $F_1(f) \leq F_1(f \sqcup g)$ and $F_2(f) \leq F_2(f \sqcup g)$ both belong to the theory $\mathcal{T}$. Let $F(f, g) = F_1(F_2(f) + g)$. Lastly, let $G(g)$ be any term such that the following equations belong to $\mathcal{T}$:*

1. *$G(g) \leq G(g \sqcup g')$ (G is monotone in g),*
2. *$F(G(g), g) = G(g)$*
3. *$g - F^*(g) \leq g - F_2\big(G(g - F^*(g))\big)$*

where $F^*(g) = F_2(F_1(g))$.

Then the term $G(g)$ is the unique, up to equality in the theory $\mathcal{T}$, least (pre)fixed-point of $F(f, g)$, written $G(g) \overset{\mu}{=} F(G(g), g)$.

The above result can be used to axiomatise the operator P by instantiating the statement of Theorem 4.1 as follows:

$$F_1(f) = [f] \quad F_2(f) = f \quad G(g) = \mathrm{P}g \quad F(f, g) = [f + g]$$

The axioms for the operator P are listed in Figure 2. Note that Theorem 4.1 is indeed applicable because F_1 and F_2 are clearly monotone.

5 Algebraic Semantics and Completeness

The goal of this section is to investigate the axiomatisation of the $\mathcal{R}_{\{\mathrm{P}\}}$ logic with algebraic methods. Since all the axioms (Figures 1 and 2) are equational, the axiomatisation defines a variety, in the usual sense of universal algebra, of structures over the appropriate signature.

We remark here that formulas of the $\mathcal{R}_{\{\mathrm{P}\}}$ logic coincide with closed terms (no variables) in the language $\{0, 1, +, r(\_), \sqcup, \sqcap, \diamond, \mathrm{P}\}$. For this reason, in what follows, the letters ϕ and ψ will range over closed terms.

5.1 Riesz spaces

The axioms 1a, 1b and 1c of Figure 1 define the variety of algebras over the signature $\{0, 1, +, r(\_), \sqcap, \sqcup\}$ called *Riesz spaces*, also known as *vector lattices*. We refer to the textbooks [16] and [9] for an introduction to the theory of Riesz spaces. The recent conference paper [11] also provides a succinct introduction to some of the basic required concepts.

Among the class of Riesz spaces we will be mostly interested in the Archimedean and unital spaces.

Definition 5.1. Let R be a Riesz space. An element $a \neq 0$ of R is *infinitely small* if there exists some $b \in R$ such that $n|a| \leq |b|$, for all $n \in \mathbb{N}$. A Riesz space is *Archimedean* if it does not have any infinitely small element.

Definition 5.2. We say that a Riesz space is *unital* if for every $a \in R$ there exists some $n \in \mathbb{N}$ such that $|a| \leq n1$.

Example 5.1. For any given set X, the Riesz space $(X \to \mathbb{R})$ with operations defined pointwise is Archimedean. However, if X is infinite, it is not unital. Indeed consider $X = \mathbb{N}$ and the identity function $f(x) = x$. Then, for every $n \in \mathbb{N}$, it is not true that $f \leq n1$.

The subspace $(X \to \mathbb{R})^b$ of all *bounded functions* is Archimedean and unital.

We denote with **AURiesz** the class of Archimedean and unital Riesz spaces. The importance of this class comes from the Yosida representation theorem (see, e.g., [9, §45] and [11, §E]) which states that each $R \in$ **AURiesz** is isomorphic to a subalgebra of $C(X)$, the Riesz space of real–valued continuous functions with operations defined pointwise on a (essentially unique) compact Hausdorff space. Hence, elements of a **AURiesz** space can always be identified with real–valued bounded functions. For this reason we often use the letters f, g, h to range over elements of **AURiesz** spaces.

The following notion is also very important in the theory of Riesz spaces.

Definition 5.3. A Riesz space R is *Dedekind complete* if every bounded subset $A \subseteq R$ has a least upper bound ($\bigsqcup A$) and a greatest lower bound ($\bigsqcap A$).

Example 5.2. For example, the Riesz space $\mathbb{R}^2$ is Dedekind complete. More generally, for any set X, the Riesz space of function $(X \to \mathbb{R})$ is Dedekind complete. The subspace $(X \to \mathbb{R})^b$ is Archimedean unital and Dedekind complete. The space $C([0, 1])$ is Archimedean and unital but not Dedekind complete. Indeed one can verify (see, e.g., [2, 9E.4]) that the set $F = \{f \geq 0 \mid f(x) \geq 1 \text{ for all } x \in [\frac{1}{2}, 1]\}$ has no infimum in $C([0, 1])$. Note that the pointwise infimum of F, which is the indicator function of $[\frac{1}{2}, 1]$, is not continuous.

We write **DAURiesz** for the class of *Dedekind complete* Archimedean unital spaces.

The following result, due to Yudin (see, e.g., [16, §IV.11]) is a central theorem in the Theory of Riesz spaces.

Theorem 5.4 (Dedekind completion)**.** *For every Archimedean unital Riesz space R there exists a Dedekind complete Archimedean and unital space $\overline{R}$, called the Dedekind completion of R, such that:*

1. *R embeds in $\overline{R}$, so we can just write $R \subseteq \overline{R}$,*
2. *existing sups and infs in R are preserved in $\overline{R}$. This means that for every $A \subseteq R$ and $f = \bigsqcup A$ (sup taken in R) then $f = \bigsqcup A$ in $\overline{R}$ too.*
3. *$\overline{R}$ is the smallest Dedekind complete space satisfying the two properties above.*

Note that the above theorem implies that the equational theories of **DAURiesz** and **AURiesz** coincide.

5.2 modal Riesz spaces

In [11] we investigated the theory of *modal Riesz spaces* which are Riesz spaces endowed with a unary operation $\diamond$ satisfying the axioms of Figure 1(1d). The terminology introduced before extends naturally to the modal extension. For example we say that the modal Riesz space $(R, \diamond)$ is Archimedean if R is Archimedean. We denote with $\mathbf{Riesz}_\diamond$ the class of all modal Riesz spaces.

The following result (see also Corollary 5.7 below) is the first technical contribution of this work and follows directly from a theorem of Kantorovich about the extension of positive linear operators on Riesz spaces.

Theorem 5.5 (Dedekind extension of modal Riesz spaces)**.** *Let $(R, \diamond)$ be an Archimedean and unital modal Riesz space. Then there exists a Dedekind complete Archimedean and unital modal Riesz space $(\overline{R}, \overline{\diamond})$ such that:*

1. *$\overline{R}$ is the Dedekind completion of R, so we view $R \subseteq \overline{R}$,*
2. *$\overline{\diamond}$ extends $\diamond$, i.e., $\diamond(f) = \overline{\diamond}(f)$ for all $f \in R$.*

Proof. We know from the axioms of Figure 1 that $\diamond : R \to R$ is positive, linear and 1-decreasing. Kantorovich's theorem (see, e.g., Theorem X.3.1 and subsequent discussion in §X.4.1 in [16]) states that any function $F : R \to R$ which is positive ($F(0) \geq 0$) and linear ($F(f + g) = F(f) + F(g)$ and $F(rf) = rF(f)$) can be extended to a positive and linear operator $\overline{F} : \overline{R} \to \overline{R}$ on the Dedekind completion of R. Thus we just need to verify that the resulting $\overline{\diamond}$ is also 1-decreasing ($\overline{\diamond}(1) \leq 1$) and this is clear since $1 \in R$ and therefore $\overline{\diamond}(1) = \diamond(1)$ and $\diamond(1) \leq 1$ because $\diamond$ is 1-decreasing. □

Remark 5.3. The choice of $\overline{\diamond}$ is, in general, not unique.

In other words $(R, \diamond)$ embeds (preserving the modal operation) in $(\overline{R}, \overline{\diamond})$ and existing suprema and infima are preserved. Once again, the result of Theorem 5.5 implies that the equational theories of $\mathbf{DAURiesz}_\diamond$ spaces and $\mathbf{AURiesz}_\diamond$ spaces coincide.

As already observed, closed terms (i.e., without variables) in the language of modal Riesz spaces are exactly formulas ϕ of the Riesz modal logic (see Definition 2.8). The following result from [11] states that, on closed terms (i.e., formulas), the theories of $\mathbf{AURiesz}_\diamond$ and $\mathbf{Riesz}_\diamond$ spaces coincide.

Proposition 5.6. *The equality $\phi = \psi$ holds in the variety $\mathbf{Riesz}_\diamond$ if and only if it holds in the class of $\mathbf{AURiesz}_\diamond$. Therefore $\phi = \psi$ is derivable from the axioms of Figure 1 if and only if it holds in all $\mathbf{AURiesz}_\diamond$ spaces.*

Proof. This is proven in [11] showing that the initial $\mathbf{Riesz}_\diamond$ algebra (free algebra with no generators) is Archimedean ([11, VI.2]) and unital ([11, VI.3]) and, therefore, it is also the initial algebra in $\mathbf{AURiesz}_\diamond$. It is a standard result from universal algebra that an equation $f = g$ between terms with variables $V = \{x_1, \dots, x_n\}$ holds in the free algebra generated by V if and only if it is derivable from the axioms. □

Corollary 5.7. *The equality $\phi = \psi$ holds in the variety $\mathbf{Riesz}_\diamond$ if and only if it holds in the class of $\mathbf{DAURiesz}_\diamond$.*

5.3 Modal Riesz spaces with the P operator

In this subsection we study the vareity of algebras over the signature $\{0, 1, +, r(\_), \sqcup, \sqcap, \diamond, \mathrm{P}\}$ axiomatised as in Figure 1 and Figure 2. We denote this variety as $\mathbf{Riesz}^{\mathrm{P}}_\diamond$.

The first observation is that, since the equations of Figure 2 define the operators P as a least prefixed–point, any modal Riesz space $(R, \diamond)$ in $\mathbf{Riesz}_\diamond$ can either:

1. be extended with some P to $(R, \diamond, \mathrm{P})$ in a unique way, or
2. not admit such extension,

since least prefixed-points, if they exist, are unique.

Example 5.4. Let $R = (C[0, 1], \diamond)$ be the (Archimedean and unital) modal Riesz space with $\diamond(f) = f$ (identity function). Then R cannot be endowed with a P operation to a $\mathbf{Riesz}^{\mathrm{P}}_\diamond$ space. Indeed let $f : [0, 1] \to \mathbb{R}$ be defined as $f(x) = x - \frac{1}{2}$. Then $\mathrm{P}(f)$ must satisfies the properties of Lemma 3.5 and Lemma 3.6 and must therefore be $\{0, 1\}$-valued. But the only $\{0, 1\}$-valued continuous functions on $[0, 1]$ are the characteristic functions of the clopen sets $\emptyset$ ($x \mapsto 0$) and $[0, 1]$ ($x \mapsto 1$). And $\emptyset$ is not a prefixed–point and $[0, 1]$ is not the least prefixed–point.

However, by the Knaster–Tarski fixed–point theorem, fixed–points of monotone bounded functions always exists, if the space is Dedekind complete.

Proposition 5.8. *Every* $(R, \Diamond) \in \mathbf{DAURiesz}_\Diamond$ *can be extended to a* $\mathbf{DAURiesz}^{\mathrm{P}}_\Diamond$ *space* $(R, \Diamond, \mathrm{P})$ *in a unique way.*

Example 5.5. Let (X, τ) be a Markov chain. Then $(X \to \mathbb{R})^b$ is a Dedekind complete Riesz space and the interpretations of the Riesz modal logic operations, from Section 2.2, make the structure $((X \to \mathbb{R})^b, [\![\Diamond]\!]_\tau, [\![\mathrm{P}]\!]_\tau)$ a $\mathbf{DAURiesz}^{\mathrm{P}}_\Diamond$ space.

The following theorem is the most important contribution of this paper and states that every $\mathbf{AURiesz}^{\mathrm{P}}_\Diamond$ space can be embedded in a $\mathbf{DAURiesz}^{\mathrm{P}}_\Diamond$ space.

Theorem 5.9 (Main Extension Theorem)**.** *Let* $R = (R, \Diamond, \mathrm{P})$ *be in* $\mathbf{AURiesz}^{\mathrm{P}}_\Diamond$*. Then* R *can be embedded into a Dedekind complete* $\mathbf{DAURiesz}^{\mathrm{P}}_\Diamond$ *space* $\overline{R} = (\overline{R}, \overline{\Diamond}, \overline{\mathrm{P}})$ *such that:*

1. $R \subseteq \overline{R}$ *and* $\overline{R}$ *is the Dedekind completion of* R*,*
2. *The operators* $\overline{\Diamond}$ *and* $\overline{\mathrm{P}}$ *extends* P *and* $\Diamond$*, respectively.*

Proof. We already know, from Theorem 5.5, that $(R, \Diamond)$ can be extended to $(\overline{R}, \overline{\Diamond})$. By proposition 5.8, the $\mathbf{DAURiesz}_\Diamond$ space $(\overline{R}, \overline{\Diamond})$ can be extended uniquely to a $\mathbf{DAURiesz}^{B}_\Diamond$ space $(\overline{R}, \overline{\Diamond}, \overline{\mathrm{P}})$. It remains to show that for all $f \in R$ it holds that $\mathrm{P}(f) = \overline{\mathrm{P}}(f)$.

Let us fix an arbitrary $f \in R$. By assumption, both $\mathrm{P}(f)$ and $\overline{\mathrm{P}}(f)$ are prefixed–points of the mapping $g \mapsto [f + g]$, with $\mathrm{P}(f)$ being the least prefixed–point among those in $R \subseteq \overline{R}$ and $\overline{\mathrm{P}}(f)$ the least among all prefixed–points in $\overline{R}$. Thus, clearly $\overline{\mathrm{P}}(f) \leq \mathrm{P}(f)$.

Assume, towards a contradiction, that $\overline{\mathrm{P}}(f) \lneq \mathrm{P}(f)$. Let us fix an intermediary function g such that $\overline{\mathrm{P}}(f) \lneq g \lneq \mathrm{P}(f)$ (e.g., $g = \frac{1}{2}\overline{\mathrm{P}}(f) + \frac{1}{2}\mathrm{P}(f)$). In fact, since R is dense in $\overline{R}$ (due to the properties of the Dedekind completion) we can pick $g \in R$. We now claim that $g \geq [f + g]$, i.e., g is prefixed–point. Since $g \in R$ and $g \lneq \mathrm{P}(f)$ this is a contradiction with the assumption that $\mathrm{P}f$ is the least prefixed–point in R.

Proof of claim: $g \geq [f + g]$. It is useful to apply Yosida theorem and identify $R = C(X)$ for some compact Hausdorff space X. Thus we need to show that $g(x) \geq [f(x) + g(x)]$ for all $x \in X$. It is enough to show that g possesses the two properties of Lemma 3.5:

- $g(x) \geq 0$, and
- if $f(x) > 0$ then $g(x) \geq 1$.

Both properties follow immediately from the assumption that $\overline{\mathrm{P}}(f) \lneq g$ since $\overline{\mathrm{P}}(f)$ is itself a prefixed–point and thus, by Lemma 3.5, satisfies the two properties. □

We thus get the following corollary.

Corollary 5.10. *The equational theories of the classes* $\mathbf{AURiesz}^{\mathrm{P}}_\Diamond$ *and* $\mathbf{DAURiesz}^{\mathrm{P}}_\Diamond$ *coincide.*

By adopting the same proof method used in [11] to prove Theorem 5.6, we can now show that on closed terms (i.e., formulas of the $\mathcal{R}_{\{\mathrm{P}\}}$ logic), the equational theory of $\mathbf{AURiesz}^{\mathrm{P}}_\Diamond$ coincides with that of the full variety of $\mathbf{Riesz}^{\mathrm{P}}_\Diamond$ algebras. This in turn proves that an equation between closed terms holds in (D)$\mathbf{AURiesz}^{\mathrm{P}}_\Diamond$ if and only if it is derivable by the axioms of Figure 1 and Figure 2.

Theorem 5.11 (Algebraic Completeness)**.** *Let* ϕ *and* ψ *be closed terms over the signature of* $\mathbf{Riesz}^{\mathrm{P}}_\Diamond$ *algebras. The following are equivalent:*

- $\phi = \psi$ *holds in all models of the variety* $\mathbf{Riesz}^{\mathrm{P}}_\Diamond$*,*
- $\vdash \phi = \psi$*, i.e., the equality is derivable (by the rules of equational logic) from the axioms of Figure 1 and 2,*
- $\phi = \psi$ *holds in all* $\mathbf{AURiesz}^{\mathrm{P}}_\Diamond$ *spaces,*
- $\phi = \psi$ *holds in all* $\mathbf{DAURiesz}^{\mathrm{P}}_\Diamond$ *spaces.*

Proof. It is a standard result of universal algebra that $\vdash \phi = \psi$ if and only if $\psi = \psi$ holds in the free algebra on the empty set of generators, which following the notation [11, VI.A] we denote by $\mathbb{I}$. This algebra is obtained by quotienting the set of closed terms by the equivalence relation $\phi \sim \psi \iff \vdash \phi = \psi$ and defining the operations on equivalence classes in the expected way (see, e.g., Section VI.A in [11]).

We then show that $\mathbb{I}$ is Archimedean and unital. The proof of latter property is identical to the proof of Theorem VI.3 [11] since $\mathrm{P}\phi \leq 1$. To show that $\mathbb{I}$ has the Archimedean property we adapt the proof of Theorem VI.4 in [11] by extending the function g : Form $\to \mathbb{R}$ as:

$$g(\mathrm{P}\phi) = \begin{cases} 1 & \text{if } g(\phi) > 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus $\mathbb{I}$ is both Archimedean and unital, i.e., $\mathbb{I} \in \mathbf{AURiesz}^{\mathrm{P}}_\Diamond$. Hence if $\nvdash \phi = \psi$ then the equality fails in some model of $\mathbf{AURiesz}^{\mathrm{P}}_\Diamond$ and this proves the third point. Lastly, if an equation fails in some $\mathbf{AURiesz}^{\mathrm{P}}_\Diamond$ space then, by Theorem 5.9, it fails in some $\mathbf{DAURiesz}^{\mathrm{P}}_\Diamond$ space and this proves the fourth point. □

6 Model Semantics and Completeness

The completeness result (Theorem 5.11) of the previous section is with respect to a class of algebras. In this section we make use of the duality theory developed in [11] and reformulate it as a completeness result with respect to a class of Markov processes, i.e., topological Markov chains. We first recall the definition of Markov process from [11, §II].

Definition 6.1. A *Markov process* is a pair (X, τ) where X is a compact Hausdorff space and $\tau : X \to \mathcal{R}^{\leq 1}(X)$ is a continuous map from X to the space of Radon sub–probability measures on X.

Hence Markov processes are a topological generalization of Markov chains with a continuous transition function (i.e., Markov kernel). The Riesz modal logic can be interpreted over Markov process in a natural way by generalizing the semantics of the $\Diamond$ modality from Section 2.2 using the Lebesgue integral:

$$[\![\Diamond\phi]\!]_\tau(x) = \int_X [\![\phi]\!]_\tau \, \mathrm{d}\tau(x).$$

We refer to [11, §III] for a detailed presentation of this topological semantics. Under this interpretation, any formula ϕ of the Riesz modal logic is interpreted as a real–valued continuous bounded function $f \in C(X)$.

It turns out that, for every Markov process (X, τ), the structure $(C(X), [\![\Diamond]\!]_\tau)$ is a $\mathbf{AURiesz}_\Diamond$ space which is universally complete (see Definition [11, II.17]). One of the key results of [11] states that, in fact, all universally–complete $\mathbf{AURiesz}_\Diamond$ spaces are of the form above, for some (essentially) unique Markov process (X, τ). This correspondence between objects extends also to morphisms in the form of a duality between the categories $\mathbf{CAURiesz}_\Diamond$ (of universally complete $\mathbf{AURiesz}_\Diamond$ spaces with structure preserving homomorphisms) and the category **Markov** (of Markov processes with coalgebra morphisms). We refer to [11] for a detailed exposition of this duality.

We now specialize this duality to the subcategory of Markov processes whose underlying state space X is compact Hausdorff and *extremally disconnected*. These spaces are well known since they are exactly the Stone spaces of *complete* Boolean algebras.

Definition 6.2. A topological space is *extremally disconnected* if the closure of every open set is open. We say that a Markov process (X, τ) is extremally disconnected if X is extremally disconnected.

For the following theorem see, e.g., [2, 9E.7].

Theorem 6.3. *For a compact Hausdorff space X, the following are equivalent:*

1. *X is extremally disconnected,*
2. *$C(X)$ is Dedekind complete, i.e., $C(X) \in$* **DAURiesz**.

This implies that the duality between **CAURiesz**$_\Diamond$ and **Markov** from [11] restricts to a duality between **DAURiesz**$_\Diamond$ and **EMarkov**, the subcategory of extremally disconnected Markov processes. As a consequence, we can formulate the following corollary.

Corollary 6.4. *Given two formulas of the Riesz modal logic, the following assertions are equivalent:*

- *$\phi = \psi$ holds in all* **DAURiesz**$_\Diamond$ *spaces,*
- *$[\![\phi]\!]_\tau = [\![\psi]\!]_\tau$ holds, for all extremally disconnected Markov processes (X, τ).*

Working with extremally disconnected Markov processes is useful because they allow the interpretation of the P operator of the logic $\mathcal{R}_{\{P\}}$. As already observed in Proposition 5.8, such interpretation is uniquely determined.

Proposition 6.5. *Let X be a compact Hausdorff extremally disconnected and let $R = C(X)$ be the corresponding Dedekind complete Riesz space. Then, for every $f \in R$, the least prefixed–point of the mapping $g \mapsto [g + f]$ is the indicator function of the closure of the set $\{x \mid f(x) > 0\}$.*

Proof. Let $u : X \to \{0, 1\}$ be the least–prefixed point. By the properties listed in Lemma 3.5 and 3.6, the function u is the characteristic function of a set $K \supseteq \{x \mid f(x) > 0\}$ and, since u is continuous, the set K is clopen. The closure of $\{x \mid f(x) > 0\}$, which is clopen since X is extremally disconnected, is the smallest such set K. □

Hence, by defining the interpretation of the P operator on Markov processes as in Definition 6.6 below, we obtain the completeness Theorem 6.7 as a corollary of Theorem 5.11.

Definition 6.6. Let (X, τ) be a Markov process. The semantics $[\![\phi]\!]_\tau$ of a $\mathcal{R}_{\{P\}}$ formula ϕ is defined by extension of the topological semantics of the Riesz modal logic $\mathcal{R}$ with the following definition:

$$[\![P\phi]\!]_\tau(x) = \begin{cases} 1 & \text{if } x \in \overline{\{y \mid [\![\phi]\!]_\tau(y) > 0\}} \\ 0 & \text{otherwise} \end{cases}.$$

Theorem 6.7 (Model Completeness)**.** *Given two formulas ϕ and ψ of the $\mathcal{R}_{\{P\}}$ logic, the following assertions are equivalent:*

- *$\phi = \psi$ is derivable by the axioms of Figure 1 and 2,*
- *$[\![\phi]\!]_\tau = [\![\psi]\!]_\tau$ holds, for all extremally disconnected Markov processes (X, τ).*

6.1 Extending the Completeness Theorem to other models

The completeness result of Theorem 6.7 may be considered, at a first glance, as slightly artificial. This is because, in computer science, we are most often interested in interpreting probabilistic logics such as pCTL or $\mathcal{R}_{\{P\}}$ on (discrete or continuous) Markov chains whose state space is not a compact Hausdorff space, let alone extremally disconnected.

Example 6.1. Let (X, τ) be a Markov chain in the sense of Definition 2.3 with X infinite. Then X (when viewed as a topological space with the discrete topology) is not a compact space. We have already discussed in Section 2.3 how to interpret the logic $\mathcal{R}_{\{P\}}$ over discrete Markov chains and, under this semantics, the formula ϕ is interpreted as a *bounded* function $[\![\phi]\!]_\tau : X \to \mathbb{R}$.

Example 6.2. Let $\tau : X \to \mathcal{R}^{\leq 1}(X)$ be a Borel measurable Markov kernel on a standard Borel space (e.g., the Euclidean space $X = \mathbb{R}^2$). This is a very natural model to consider in many applications. Yet, it does not fit the definition of Markov process of Definition 6.1 for two reasons: $\mathbb{R}^2$ is not compact and τ is not continuous. Yet we can naturally interpret the logic $\mathcal{R}_{\{P\}}$ over these models with:

$$[\![\Diamond\phi]\!]_\tau(x) = \int_X [\![\phi]\!]_\tau \, \mathrm{d}\tau(x) \quad [\![P\phi]\!]_\tau(x) = \begin{cases} 1 & \text{if } [\![\phi]\!]_\tau(x) > 0 \\ 0 & \text{otherwise} \end{cases}$$

With this interpretation, it is possible to show by induction on the structure of ϕ that the semantics $[\![\phi]\!]_\tau$ of very formula ϕ is a Borel measurable *bounded* function $[\![\phi]\!]_\tau : X \to \mathbb{R}$.

Several other examples can be found in the literature: for example Markov processes defined on analytic spaces [13] or even measurable spaces [6].

However, as we now explain, the completeness result still holds if all the models of the examples above (and arguably most other models in the literature) were considered in addition to extremally disconnected Markov processes. To be more precise, by denoting with C_1 and C_2 the classes of models of the two examples above, we can prove the following theorem:

Theorem 6.8 (Extended Model Completeness)**.** *Given two formulas ϕ and ψ of the $\mathcal{R}_{\{P\}}$ logic, the following assertions are equivalent:*

- *$\phi = \psi$ is derivable by the axioms of Figure 1 and 2,*
- *$[\![\phi]\!]_\tau = [\![\psi]\!]_\tau$ holds in all models in* **EMarkov** $\cup\, C_1 \cup C_2$.

Proof. We need to show that if $\phi \neq \psi$ in some model $(X, \tau) \in C_1 \cup C_2$ then there is some $(Y, \sigma) \in$ **EMarkov** such that $[\![\phi]\!]_\sigma \neq [\![\psi]\!]_\tau$.

Assume first that (X, τ) is in C_1. Then we already observed in Example 5.5 that $((X \to \mathbb{R})^b, [\![\Diamond]\!]_\tau, [\![P]\!]_\tau)$ is a Dedekind complete space. Hence, by the duality between **EMarkov** and **DAURiesz**$_\Diamond$, it is isomorphic to $(C(Y), [\![\Diamond]\!]_\sigma, [\![P]\!]_\sigma)$ for some $(Y, \sigma) \in$ **EMarkov**. If fact one can show that $Y = \mathrm{SC}(X)$ where $\mathrm{SC}(X)$ is the Stone–Cech compactification of X (see, e.g., [2, 9E.8]).

Suppose now that (X, τ) is in C_2. Let us denote with $\mathcal{B}(X, \mathbb{R})$ the collection of bounded Borel measurable functions of type $X \to \mathbb{R}$. The space $(\mathcal{B}(X, \mathbb{R}), [\![\Diamond]\!]_\tau, [\![P]\!]_\tau)$ is a **AURiesz**$^{P}_{\Diamond}$ space but, in general, it is not Dedekind complete. However, using the extension Theorem 5.9, we can embed this space into a Dedekind complete space $(A, \Diamond, P)$ and thus $\phi \neq \psi$ in $(A, \Diamond, P)$. By duality, this space is isomorphic to $(C(Y), [\![\Diamond]\!]_\sigma, [\![P]\!]_\sigma)$ for some $(Y, \sigma) \in$ **EMarkov**, and this concludes the proof. □

The proof of the above theorem shows that, as long as we deal with reasonable models of Markov chains (X, τ), the denotation of $\mathcal{R}_{\{P\}}$ formulas belong to some $\mathbf{AURiesz}^{P}_{\Diamond}$ subspace of $(X \to \mathbb{R})^b$ and thus, using the extension Theorem 5.9 and duality between **EMarkov** and $\mathbf{DAURiesz}^{P}_{\Diamond}$, it can be also be equally interpreted in some **EMarkov** process in the sense of Definition 6.1.

7 Possible variants of this work

In this work we have modeled discrete (continuous) probabilistic systems with Markov chains (Markov processes) based on sub–probability distributions (sub–probability measures). This choice was made, in agreement with the work in [11], for mathematical convenience: the axiomatization of the $\Diamond$ operator of the Riesz modal logic is simple and intelligible.

In this section we explain how several variants can be considered.

7.1 Labeled Markov chains

Labeled Markov chains, still based on sub–probability distributions, can be defined as follows:

Definition 7.1. Let L be a set of labels. A *labeled Markov chain* is a pair $(X, \{\tau_l\}_{l\in L})$ where X is the (possibly infinite) set of states and $\tau_l : X \to \mathcal{D}^{\leq 1}(X)$ is the l-labeled transition function which maps each state to a subprobability distribution over states.

The Riesz modal logic $\mathcal{R}$ and its extension $\mathcal{R}_{\{P\}}$ can naturally be adapted to be interpreted over labeled Markov chains by replacing the single modality $\Diamond$ with a L-indexed family of modalities $\langle l\rangle$ whose interpretation is:

$$[\![\langle l\rangle\phi]\!](x) = \mathbb{E}([\![\phi]\!], \tau_l(x))$$

And similarly for labeled Markov processes.

The multimodal variant of the Riesz modal logic and of its extension $\mathcal{R}_{\{P\}}$ can be axiomatized by duplicating the axioms of $\Diamond$ for each $\langle l\rangle$ and $l \in L$. For example, if $L = \{a, b\}$, the axiomatization is obtained by replacing the equations $1d$ from Figure 1 with the equations $1da$ and $1db$ of Figure 3.

1da) modal axioms for $\langle a\rangle$
- (Linearity) $\langle a\rangle(f+g) = \langle a\rangle(f) + \langle a\rangle(g)$ and $\langle a\rangle(rf) = r(\langle a\rangle f)$, for all $r \in \mathbb{R}$,
- (Positivity) $\langle a\rangle(f \sqcup 0) \geq 0$,
- (1-decreasing) $\langle a\rangle(1) \leq 1$.

1db) modal axioms for $\langle b\rangle$
- (Linearity) $\langle b\rangle(f+g) = \langle b\rangle(f) + \langle b\rangle(g)$ and $\langle b\rangle(rf) = r(\langle b\rangle f)$, for all $r \in \mathbb{R}$,
- (Positivity) $\langle b\rangle(f \sqcup 0) \geq 0$,
- (1-decreasing) $\langle b\rangle(1) \leq 1$.

Figure 3. Axioms of the multimodal Riesz Modal Logic.

7.2 Full Markov Chains

One of the most widespread variants of Markov chains is based on full probability distributions and on predicates defined on the set of states. This is also the definition mostly used for interpreting the logic pCTL (see, e.g., [1]).

Definition 7.2. A *full Markov chain* is a tuple $(X, \tau, Q_0, \dots, Q_n)$ where X is the (possibly infinite) set of states, $\tau : X \to \mathcal{D}^{=1}(X)$ is the transition function mapping each state to a full probability distribution, and $Q_i \subseteq X$ for all $0 \leq i \leq n$.

Let us fix the alphabet $L = \{a, q_0, \dots, q_n\}$. We now show how to view any labeled Markov chain $(X, \{\tau_l\}_{l\in L})$ such that $\tau_a \in \mathcal{D}^{=1}$ (i.e., $\tau_a(x)$ is a full probability distribution for all x) as a *full Markov chain* $(X, \tau, Q_0, \dots, Q_n)$.

- the set X of states of the full Markov chain coincides with the set of states of the labeled Markov chain,
- $\tau = \tau_a$, i.e., the a-transition function of the labeled Markov chain is the transition function of the full Markov chain,
- $x \in Q_i(x) \Leftrightarrow \tau_{q_i}(x)$ has positive mass: $\sum_{y\in X} d(y) > 0$ where $d = \tau_{q_i}(x)$.

Clearly any full Markov chain is represented by some labelled Markov chain.

Now note that that the requirement "τ_a is a full probability distribution" can be expressed in the logic $\mathcal{R}_{\{P\}}$ by the axiom:

$$\langle a\rangle 1 = 1$$

Hence the axiomatisation of the multimodal logic $\mathcal{R}_{\{P\}}$ interpreted over labeled Markov chain $(X, \{\tau_l\}_{l\in L})$ such that τ_a is full, can be obtained by adding the axiom $\langle a\rangle 1 = 1$ to the axiomatisation described in the previous subsection.

Finally, note that the (characteristic function of the) set of states satisfying the predicate Q_i is definable in the $\mathcal{R}_{\{P\}}$ logic by the formula: $\mathrm{P}(\langle b_i\rangle 1)$. This gives an axiomatisation of a variant of the $\mathcal{R}_{\{P\}}$ logic, and thus of bounded pCTL (see Theorem 2.11), interpreted over full Markov chains.

7.3 Conclusions and Future Work

We have introduced the $\mathcal{R}_{\{P\}}$ logic, an extension of the Riesz modal logic which can be interpreted over discrete Markov chains as well as on a wide range of continuous Markov processes. The logic $\mathcal{R}_{\{P\}}$ is sufficiently expressive to interpret the bounded fragment of pCTL. Our main result is Theorem 6.7 (see also Theorem 6.8) which states that the set of equations of Figure 1 and Figure 2 gives a sound and complete axiomatisation of $\mathcal{R}_{\{P\}}$.

The research program outlined in the introduction is not yet completed since the logic $\mathcal{R}_{\{P\}}$, while quite expressive, cannot interpret the full logic pCTL and, more specifically, cannot interpret its *unbounded until operator* $\mathbb{P}_{\geq p}(F\ \mathcal{U}\ G)$ (see Section 2.1 for details). One natural idea is to consider the following inductively defined binary operator

$$\mathrm{U}(\phi, \psi) \stackrel{\mu}{=} \mathrm{P}(\psi) \sqcup (\mathrm{P}(\phi) \sqcap \Diamond(\mathrm{U}(\phi, \psi)))$$

which intuitively denotes the least upper bound of the formulas $\theta^n_{F,G}$ defined in the statement of Proposition 2.11. It is not difficult to prove the analogous of Proposition 2.11 and show that $\mathcal{R}_{\{P\}}$+U can interpret full pCTL on Markov chains. However it does not seem straightforward to adapt the techniques of the present work to prove the analogous of Theorem 5.9, which is the key technical result to obtain a complete axiomatisation. We consider the study the U operator and its axiomatisation as an interesting next step in this line of research.

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