

Around Classical and Intuitionistic Linear Logics

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Abstract

We revisit many aspects of the syntactic relations between (variants of) classical linear logic (LL) and (variants of) intuitionistic linear logic (ILL) in the propositional setting.

On the one hand, we study different (parametric) “negative” translations from LL to ILL: their expressiveness, the relations with extensions of LL and their use in the proof theory of LL (cut elimination and focusing). In particular, this bridges the intuitionistic restriction on sequents (at most one conclusion) and the focusing property of linear logic. On the other hand, we generalise the known partial results about conservativity of LL over ILL, leading for example to a conservativity proof for LL over tensor logic (TL).

CCS Concepts • Theory of computation → Proof theory; Linear logic;

Keywords Linear logic, Intuitionistic linear logic, Tensor logic, Negative translations, Double negation, Focusing, Conservativity

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1 Introduction

Linear logic (LL) [9] has become a key element of the toolbox in different areas of computer science in particular in the theory of programming languages (type systems, denotational semantics, quantum computing, concurrency theory, implicit complexity, higher-order model checking, etc).

A key property of linear logic, stressed when it was introduced, is the ability of conciliating an involutive negation (as in classical logic) with constructivity (in terms of confluent normalization, or more generally denotational semantics, but also through disjunction and existence properties). However soon after, an intuitionistic variant ILL was also presented [11]. It is defined in sequent calculus as the restriction of LL under the *intuitionistic constraint*: “exactly one formula on the right-hand side of sequents”. It relies on a restricted subset of connectives (in particular the involutive negation is lost) with a focus on linear implication \multimap . ILL appears to be more natural than its classical version in various contexts such as typing systems and the analysis of the λ -calculus, categorical semantics, or for game interpretations [1].

The goal of this paper is to revisit various ways of relating LL and ILL to get a better understanding of the specificities of each of these logics. The central question is then to understand what makes these two systems different (or not).

Conservativity. In studying the relations between LL and ILL, a first natural question is the conservativity property: since any ILL formula can be seen as an LL formula (by unfolding $A \multimap B$ into $A^\perp \wp B$), can we compare provability in ILL and provability in LL? A first direction is easy since any ILL sequent can be turned into an LL sequent (by translating $\Gamma \vdash^i A$ into $\vdash \Gamma^\perp, A$) and any ILL associated proof into an LL proof. The key question is the converse: can we find ILL formulas which are provable in LL but not in ILL?

Let us first mention that, if this question looks similar to the more standard one of comparing provability in classical logic (in the LK sequent calculus for example) and in intuitionistic logic (in the LJ sequent calculus for example), it is rather different. Indeed, even with implication as the only connective, conservativity of LK over LJ fails, as shown by Peirce’s law $((A \rightarrow B) \rightarrow A) \rightarrow A$. This is because the intuitionistic restriction (“at most one formula on the right-hand side of sequents”) of LJ has a direct impact on the use of structural rules (contraction and weakening). However in a linear setting, these rules are already controlled: that is the purpose of the exponential connectives $!$ and $?$. This makes LL and ILL much closer to each other than LK and LJ. The main (positive and negative) results on conservativity of LL over ILL have been obtained by H. Schellinx [22]:

- in the presence of both \multimap and 0 , conservativity may fail as shown by the counterexample $(X \multimap (0 \multimap Y) \multimap Y') \multimap ((X \multimap X') \multimap 0) \multimap Y'$;
- for formulas without \multimap or without 0 , conservativity holds;
- for formulas in the image of Girard’s translation of LJ into LL $(A \rightarrow B \mapsto !A \multimap B)$ [9], conservativity holds.

Parametric Negative Translation. While conservativity is about understanding the trivial embedding of ILL into LL, one can also study translations of LL into ILL.

Double-negation translations or negative translations or continuation-passing style (CPS) translations are well known tools to map classical logic into intuitionistic logic which led to computational understandings of classical logic [12, 20]. This logical analysis of control operators has early been related with linear logic [10]. Moreover it has been stressed that, while implication is usually considered as the central connective of intuitionistic logic (in relation with the λ -calculus in particular), in the context of intuitionistic logic used as a target of translations from classical logic, the key connective is negation [14, 24].

These translations have inspired works on translating LL into ILL [3, 8, 15, 18, 25] since they give ways to enforce the intuitionistic constraint on sequents. Once such a translation is settled, one can wonder what it says about the starting system, and what is its expressiveness. This last point can be stated through faithfulness analysis: are the sequents/formulas provable in the image of the

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translation, the images of provable sequents of the source system? or can we express more through the translation than in the original system? This question has shown to be very fruitful in the context of CPS-translations leading to the introduction of delimited control operators [7].

We follow Chang-Chaudhuri-Pfenning [3] which propose to use a parametric negative translation of LL into ILL and to analyse, for different values of the parameter, the extensions of linear logic represented by the translation (*i.e.* for which provability is exactly the provability in the image). While they use a specific extension JILL of ILL, we work here with standard ILL. The translation $(_)^\bullet$ we use is close to the one of [4] (parametrized by an arbitrary ILL formula R) with the idea that it uses the minimal number of negations required to get a decoration of proofs of LL into proofs of ILL: during the translation, each rule is macro-expanded into a corresponding rule of ILL together with some negation rules (but no introduction of fresh cut for example is required, and in particular cut-free proofs are turned into cut-free proofs). We make a detailed analysis of the relations between LL and ILL through the translation $(_)^\bullet$ (in particular regarding the expressiveness of the image). The extension of LL with the equivalence $R \dashv \vdash \perp$ happens to play a key role.

Focusing. The negations used in the translation $(_)^\bullet$ are those required to get a decoration of LL proofs as ILL proofs. However fewer negations are required if we just want to preserve provability. This comes from a precise analysis of polarities of formulas as defined by the theory of focusing [2]. Indeed the work on focusing and polarization [2, 10] has put forward a partition of connectives into two classes: positive and negative ones (a.k.a. synchronous and asynchronous). Each class happens to group connectives which share common proof-theoretical properties and which can cluster into macro-connectives. Concretely it led to the definition of focused systems for LL which structurally impose that consecutive connectives of a given polarity are used as a cluster. The focusing property then states the completeness of these focused systems with respect to LL provability: any LL proof can be turned into a focused one with the same conclusion.

We present the optimised translation with tensor logic (TL) [18] as target. Tensor logic is the fragment of intuitionistic linear logic which focuses on positive connectives (\otimes , \oplus and $!$) together with negation (as a restriction of \multimap). The idea of tight links between polarization, focusing and double-negation translations is not new [10, 18, 19]. It has been developed in both classical and linear settings. We prove the stronger result that the focusing property of linear logic can be directly deduced from the optimised negative translation from LL to TL. This shows how the “at most one positive formula in the focus” constraint of focused linear systems is a particular case of the “at most one formula on the right” of intuitionistic ones.

CPS-transformations are used in particular in compiling because of the strong structural properties they provide on the generated terms [21]. Similarly focused proofs are providing structural constraints on proofs (important in their use in proof search for example). We thus show these constraints (focusing and CPS) to be of the same nature since focusing happens to be obtained from CPS.

Contributions. Starting from the definition of the parametric translation $(_)^\bullet$ from LL to ILL, Section 2 studies the expressiveness of $(_)^\bullet$ while making the parameter vary. We introduce and discuss

the parametric logic RLL(\mathcal{R}) (extending LL with $\mathcal{R} \dashv \vdash \perp$) which provides an upper bound on the expressiveness of $(_)^\bullet$. We then give a sufficient condition on the parameter for this bound to be lower as well. Finally we consider some important possible values for the parameter leading to equivalences between: provability in the image of $(_)^\bullet$, provability in RLL, provability in LL with some additional rules, and provability in LL of enriched sequents. A typical example being: $\Gamma^\bullet \vdash^i !\Phi \multimap \Phi$ in ILL (Φ atomic) $\iff \vdash_{?1}^r \Gamma$ in RLL($?1$) $\iff \vdash^0 \Gamma$ in LL with (mix_0) $\iff \vdash \Gamma, ?1$ in LL. On the way, we prove some proof-theoretical properties of LL and ILL: cut-elimination of LL can be simply deduced from cut-elimination in ILL (by using $(_)^\bullet$), and the connectives \perp and $?$ cannot be defined in propositional ILL.

In Section 3, we generalise the work of H. Schellinx on conservativity of LL over ILL [22]. First we give a new counterexample which has implicative order 2 (the minimal possible value): $((X \otimes \top) \& (Y \otimes \top)) \multimap 0 \multimap ((X \multimap X') \oplus (Y \multimap Y'))$. On the positive side, we extend the conservativity result for the image of Girard’s translation of intuitionistic logic: we prove that it comes from the fact that \multimap is only used in the shape $!_ \multimap _$, and we generalise this $!_$ pattern to a more general set of formulas. Concerning the conservativity for formulas without \multimap or 0 , we extend it to formulas without $_ \multimap 0$ (up to linear equivalence). This gives us the conservativity of LL over TL (with no restriction).

By choosing the parameter value for $(_)^\bullet$ to be a fixed propositional variable, its target becomes tensor logic. In Section 4, we define the polarized optimisations $(_)^-$ and $(_)^+$ of $(_)^\bullet$. By analysing the image of an LL proof through the induced translation in TL, we show the obtained proof is exactly a proof of the starting sequent in the focused system LL_{foc} , thus proving the focusing property for linear logic.

Most of the presented results are formalised in the Coq proof assistant with the help of the Yalla library [17]:

<https://perso.ens-lyon.fr/olivier.laurent/yalla/acaiil/>

2 From LL to ILL

In this section, we study a parametric negation-based translation from LL to ILL. Formulas of LL [9] are denoted F, G, H , etc:

$$F ::= X \mid X^\perp \mid 1 \mid \perp \mid F \otimes F \mid F \wp F \mid 0 \mid \top \mid F \oplus F \mid F \& F \mid !F \mid ?F .$$

Formulas of ILL [11] are denoted I, J, K , etc:

$$I ::= X \mid 1 \mid I \otimes I \mid I \multimap I \mid 0 \mid \top \mid I \oplus I \mid I \& I \mid !I .$$

Sequents are denoted $\vdash \Gamma$ for LL and $\Gamma \vdash^i I$ for ILL. The logical systems we consider come with exchange rules which allow us to permute formulas in sequents. To make derivations shorter, we omit exchange rules in proof trees. They are very easy to reconstruct if needed.

2.1 Negative Translation

Let us fix an arbitrary formula R of ILL which will be used as a parameter for translating LL into ILL. We use $_ \multimap R$ as a *defined* negation connective in ILL denoted \neg_R . The following two rules are derivable:

$$\frac{\Gamma, I \vdash^i R}{\Gamma \vdash^i \neg_R I} \neg_R R \qquad \frac{\Gamma \vdash^i I}{\Gamma, \neg_R I \vdash^i R} \neg_R L$$

Lemma 2.1. *In ILL, for all formulas I, J, I' and J' , we have:*

- (i) $I, \neg_R I' \vdash^i R$ and $J, \neg_R J' \vdash^i R$ implies $I \otimes J, \neg_R (I' \otimes J') \vdash^i R$

(ii) $I, \neg_{\mathbb{R}} I' \vdash^i \mathbb{R}$ and $J, \neg_{\mathbb{R}} J' \vdash^i \mathbb{R}$ implies $I \oplus J, \neg_{\mathbb{R}} (I' \oplus J') \vdash^i \mathbb{R}$

Definition 2.2 (Translation of Formulas). The translation F^\bullet of a formula F of LL is a formula of ILL:

$$\begin{array}{ll} X^\bullet & = \neg_{\mathbb{R}} X & (X^\perp)^\bullet & = X \\ 1^\bullet & = \neg_{\mathbb{R}} 1 & \perp^\bullet & = 1 \\ 0^\bullet & = \neg_{\mathbb{R}} 0 & \top^\bullet & = 0 \\ (F \otimes G)^\bullet & = \neg_{\mathbb{R}} (\neg_{\mathbb{R}} F^\bullet \otimes \neg_{\mathbb{R}} G^\bullet) & (F \wp G)^\bullet & = F^\bullet \otimes G^\bullet \\ (F \oplus G)^\bullet & = \neg_{\mathbb{R}} (\neg_{\mathbb{R}} F^\bullet \oplus \neg_{\mathbb{R}} G^\bullet) & (F \& G)^\bullet & = F^\bullet \oplus G^\bullet \\ (!F)^\bullet & = \neg_{\mathbb{R}} !\neg_{\mathbb{R}} F^\bullet & (?F)^\bullet & = !\neg_{\mathbb{R}} \neg_{\mathbb{R}} F^\bullet \end{array}$$

When the value of \mathbb{R} needs to be explicitly mentioned, we use the notation $(_)^\bullet[\mathbb{R}]$. Almost the same translation is considered in [4, 18]. The main difference is in $(?F)^\bullet$ (to be discussed later).

Lemma 2.3 (Translation and Dual).

For all F , $\neg_{\mathbb{R}} F^\bullet, \neg_{\mathbb{R}} (F^\perp)^\bullet \vdash^i \mathbb{R}$ is derivable in ILL.

Proposition 2.4 (Translation of Proofs).

If $\vdash \Gamma$ is provable in LL then $\Gamma^\bullet \vdash^i \mathbb{R}$ is provable in ILL.

The resulting proof is obtained by “decorating” (in the spirit of [23]) the original one. This means that the structure of the proof is preserved: each LL rule is macro-expanded into a corresponding rule in ILL together with some additional (\neg -R) and (\neg -L) rules. Only the case of the (*cut*) rule is more involved since a call to Lemma 2.3 is required as a medium between the two cut proofs (but still a cut in the source induces a cut in the target). Concerning the additive connectives, it is also possible to keep a decoration while decreasing the number of negations by defining $(F \oplus G)^\bullet = F^\bullet \& G^\bullet$. However this would break a polarity policy which is central in Section 4 for removing more negations. Concerning the exponentials, for any I in ILL, we have $!I \vdash^i !\neg_{\mathbb{R}} \neg_{\mathbb{R}} I$. As a consequence, one can optimise $(?F)^\bullet$ into $(?F)^\bullet = !F^\bullet$ as it is defined in [4, 18]. This would preserve the validity of Proposition 2.4 (since $!F^\bullet \vdash^i !\neg_{\mathbb{R}} \neg_{\mathbb{R}} F^\bullet$), except that we would have to eliminate cuts in LL before translating. Indeed this optimisation breaks Lemma 2.3. Lemma 2.3 is also necessary in the presence of quantifiers for Lemma 5.1 to hold.

Following Proposition 2.4, a natural question is then to understand if more than LL can be encoded into ILL through this translation. We study this now while making the parameter \mathbb{R} vary.

We start with a result about the (*mix_n*) rules ($n \geq 0$):

$$\frac{\vdash \Gamma_1 \quad \cdots \quad \vdash \Gamma_n}{\vdash \Gamma_1, \dots, \Gamma_n} \text{mix}_n \quad (\text{in particular } \frac{}{\vdash} \text{mix}_0)$$

We use the notation $\vdash^n \Gamma$ for sequents in LL extended with (*mix_n*) (and $\vdash^{02} \Gamma$ for LL with both (*mix₀*) and (*mix₂*)). If F is a formula, $\otimes_n F$ is defined by: $\otimes_0 F = 1$ and $\otimes_{n+1} F = F \otimes \otimes_n F$.

Lemma 2.5. Let \mathbb{R} be an intuitionistic linear formula, for all $n \geq 0$, if $\otimes_n \mathbb{R} \vdash^i \mathbb{R}$ is provable in ILL then LL with (*mix_n*) can be translated by $(_)^\bullet$ into ILL.

2.2 Response Linear Logic

We introduce an extension of linear logic which will happen to be directly related with the image of the translation $(_)^\bullet$. Given a formula \mathcal{R} of LL, *response linear logic* $\text{RLL}(\mathcal{R})$ is the extension of LL incorporating the two axioms:

$$\frac{}{\vdash_{\mathcal{R}} \mathcal{R}^\perp} \perp_{\mathcal{R}} \quad \frac{}{\vdash_{\mathcal{R}} \mathcal{R}, 1} 1_{\mathcal{R}}$$

This corresponds to adding the equivalence $\mathcal{R} \dashv \perp$ to LL. Be aware that, because of the introduction of non-trivial axioms in $\text{RLL}(\mathcal{R})$,

cut elimination does not hold in general. Sequents of $\text{RLL}(\mathcal{R})$ are denoted $\vdash_{\mathcal{R}}^r \Gamma$.

Lemma 2.6 (Substitution). If $\vdash_{\mathcal{R}}^r \Gamma$ is provable in $\text{RLL}(\mathcal{R})$ then, for all F and X , $\vdash_{\mathcal{R}[F/X]}^r \Gamma[F/X]$ is provable in $\text{RLL}(\mathcal{R}[F/X])$.

2.2.1 Alternative Presentations

If the definition of $\text{RLL}(\mathcal{R})$ above does not satisfy cut elimination, it is possible to give alternative presentations with equivalent provability power but better proof-theoretical properties.

Lemma 2.7. $\vdash_{\mathcal{R}}^r \Gamma$ in $\text{RLL}(\mathcal{R})$ if and only if $\vdash \Gamma, ?\mathcal{R}, ?(\mathcal{R}^\perp \otimes \perp)$ in LL.

This presentation has the advantage of having an admissible cut rule since cut is admissible in LL and we can build:

$$\frac{\frac{\vdash \Gamma, \Delta, ?\mathcal{R}, ?(\mathcal{R}^\perp \otimes \perp) \quad \vdash \Delta, A^\perp, ?\mathcal{R}, ?(\mathcal{R}^\perp \otimes \perp)}{\vdash \Gamma, \Delta, ?\mathcal{R}, ?\mathcal{R}, ?(\mathcal{R}^\perp \otimes \perp), ?(\mathcal{R}^\perp \otimes \perp)} \text{cut}}{\vdash \Gamma, \Delta, ?\mathcal{R}, ?(\mathcal{R}^\perp \otimes \perp)} ?c$$

We now consider some more specific values of \mathcal{R} .

Lemma 2.8. $\vdash_{?R}^r \Gamma$ in $\text{RLL}(?R)$ if and only if $\vdash \Gamma, ?R$ in LL.

2.2.2 The Particular Case $\mathcal{R} = \underline{\mathbb{R}}$

We denote by $\underline{\quad}$ the canonical embedding of the ILL formula I into LL based on $\underline{J} \multimap K = (J)^\perp \wp K$. Given a formula \mathbb{R} of ILL, it is possible to relate $\text{RLL}(\underline{\mathbb{R}})$ and the image of the translation $(_)^\bullet$.

Lemma 2.9. Given a formula F of LL, $\vdash_{\underline{\mathbb{R}}}^r \underline{F}^\bullet, F$ is provable in $\text{RLL}(\underline{\mathbb{R}})$.

Proposition 2.10. If $\Gamma^\bullet \vdash^i \mathbb{R}$ is provable in ILL then $\vdash_{\underline{\mathbb{R}}}^r \Gamma$ is provable in $\text{RLL}(\underline{\mathbb{R}})$.

Proof. We have $\vdash (\underline{\quad})^\perp, \underline{\quad}$ provable in LL thus in $\text{RLL}(\underline{\mathbb{R}})$. Then if $\Gamma = F_1, \dots, F_k$, we introduce a cut with $\vdash_{\underline{\mathbb{R}}}^r \underline{\mathbb{R}}^\perp$ and k cuts with proofs from Lemma 2.9. \square

We have presented a general pattern:

$$\vdash \Gamma \text{ in LL} \implies \Gamma^\bullet \vdash^i \mathbb{R} \text{ in ILL} \implies \vdash_{\underline{\mathbb{R}}}^r \Gamma \text{ in RLL}(\underline{\mathbb{R}})$$

valid for any \mathbb{R} . We thus have general lower and upper bounds on the expressiveness of $(_)^\bullet$.

2.3 From $\text{RLL}(\underline{\mathbb{R}})$ to ILL

We have seen $\Gamma^\bullet \vdash^i \mathbb{R}$ in ILL $\implies \vdash_{\underline{\mathbb{R}}}^r \Gamma$ in $\text{RLL}(\underline{\mathbb{R}})$. We can try to find values \mathbb{R} such that the converse holds. In this case, the upper bound on the expressiveness of $(_)^\bullet$ will happen to be an exact characterisation. In order to translate $\vdash_{\underline{\mathbb{R}}}^r \Gamma$ into ILL with $(_)^\bullet$, we have to extend Proposition 2.4 with the translations through $(_)^\bullet$ of the two additional axioms. This requires the provability of:

$$(\underline{\mathbb{R}}^\perp)^\bullet \vdash^i \mathbb{R} \quad \text{and} \quad (\underline{\mathbb{R}})^\bullet, \mathbb{R} \vdash^i \mathbb{R}.$$

We do not currently know which are precisely the formulas \mathbb{R} making these two sequents provable, but here are some partial positive and negative results.

On the negative side, one can see that counterexamples exist: $X \multimap X, X \multimap Y, X \multimap 1, X \& Y, (X \& X') \multimap Y$ or $!X$ for example.

On the positive side, we introduce a notion of *purely positive affine formula* (rejecting $!, \multimap$ and $\&$):

$$E ::= X \mid 1 \mid E \otimes E \mid 0 \mid E \oplus E \mid \top.$$

Note that $\underline{E} = E$ since it does not contain the connective \neg .

The properties $(E^\perp)^\bullet[E] \vdash^i E$ and $E^\bullet[E], E \vdash^i E$ are rather badly behaved for a proof by induction on E . This is why we are moving to more general statements first.

Lemma 2.11. *If E is a purely positive affine formula and R is an intuitionistic linear formula, $(E^\perp)^\bullet[R] \vdash^i E$ and $E \vdash^i \neg_R E^\bullet[R]$ are provable in ILL.*

Proposition 2.12. *Let E be a purely positive affine formula, $\Gamma^\bullet[E] \vdash^i E$ in ILL if and only if $\vdash_E^r \Gamma$ in RLL(E).*

Let us now consider some of the formulas E , and try to see whether we can go further than Section 2.2.1 in understanding RLL(E) and $(_)^\bullet[E]$.

2.3.1 $R = 1$

We are going to use the (mix_0) and (mix_2) rules in LL to characterise RLL(1) as suggested by Lemma 2.5.

Lemma 2.13. *If $\vdash^{02} \Gamma$ is provable in LL with (mix_0) and (mix_2) then $\vdash_R \Gamma^\bullet[1R] \vdash^i \Gamma$ is provable in ILL for any R .*

Proposition 2.14. *The following statements are equivalent:*

- (i) $\vdash^{02} \Gamma$ in LL with (mix_0) and (mix_2)
- (ii) $\Gamma^\bullet[1] \vdash^i 1$ in ILL
- (iii) $\vdash_1^r \Gamma$ in RLL(1)

Proof. We prove the following three implications:

- $(i) \Rightarrow (ii)$: we apply Lemma 2.5 or we can use Lemma 2.13 with $R = \top$ (since $1 \dashv\vdash \top$ in ILL).
- $(ii) \Rightarrow (iii)$: this is Proposition 2.10.
- $(iii) \Rightarrow (i)$: the two additional rules of RLL(1) are derivable in LL extended with (mix_0) and (mix_2) . \square

Using linear equivalence with 1, we immediately get the same results for $R = \top, R = !, R = 1 \multimap 1$, etc.

2.3.2 $R = \top$

RLL(\top) is uninteresting since provability is trivial: any sequent becomes provable.

$$\frac{\frac{\vdash_{\top}^r 0 \quad \perp \mathcal{R}}{\vdash_{\top}^r \Gamma} \quad \frac{\vdash_{\top}^r \top, \Gamma}{\vdash_{\top}^r \Gamma} \top}{\vdash_{\top}^r \Gamma} cut$$

2.3.3 $R = 0$

Affine logic AL is LL extended with the rule: $\frac{\vdash^a \Gamma}{\vdash^a \Gamma, F} wk$.

Proposition 2.15. *The following statements are equivalent:*

- (i) $\vdash^a \Gamma$ in AL
- (ii) $\Gamma^\bullet[0] \vdash^i 0$ in ILL
- (iii) $\vdash_0^r \Gamma$ in RLL(0)

Proof. We prove the following three implications:

- $(i) \Rightarrow (ii)$: this is Proposition 2.4 with $R = 0$ for the standard LL rules. We then have to consider:

$$\frac{\text{IH} \quad \frac{\Gamma^\bullet[0] \vdash^i 0 \quad \frac{\vdash^a \Gamma}{\vdash^a \Gamma, F} 0L}{\Gamma^\bullet[0], F^\bullet[0] \vdash^i 0} 0L}{\Gamma^\bullet[0], F^\bullet[0] \vdash^i 0} cut$$

- $(ii) \Rightarrow (iii)$: this is Proposition 2.10.
- $(iii) \Rightarrow (i)$: the two additional rules of RLL(0) are derivable in LL extended with (wk) . \square

2.3.4 $R = \Phi$

We consider Φ to be a(n intuitionistic) propositional variable.

Proposition 2.16. *The following statements are equivalent:*

- (i) $\vdash \Gamma$ in LL
- (ii) for all $R, \Gamma^\bullet[R] \vdash^i R$ in ILL
- (iii) there exists $\Phi \notin \Gamma$ such that $\Gamma^\bullet[\Phi] \vdash^i \Phi$ in ILL
- (iv) there exists $\Phi \notin \Gamma$ such that $\vdash_{\Phi}^r \Gamma$ in RLL(Φ)
- (v) $\vdash_{\perp}^r \Gamma$ in RLL(\perp)

Proof. We prove the following five implications:

- $(i) \Rightarrow (ii)$: this is Proposition 2.4.
- $(ii) \Rightarrow (iii)$: let Φ be a propositional variable not free in Γ , we simply instantiate R with Φ .
- $(iii) \Rightarrow (iv)$: this is Proposition 2.10 with $\underline{\Phi} = \Phi$.
- $(iv) \Rightarrow (v)$: by Lemma 2.6, $\vdash_{\perp}^r \Gamma$ in RLL(\perp) since $\Phi \notin \Gamma$.
- $(v) \Rightarrow (i)$: the two additional rules of RLL(\perp) are derivable in LL. \square

Using linear equivalence, we immediately get the same results for $1 \multimap \Phi$, etc.

We can also apply this result to prove cut elimination in LL from the corresponding result for ILL:

Corollary 2.17 (Cut Elimination). *Cut elimination in ILL entails cut elimination in LL.*

Proof. If $\vdash \Gamma, F$ and $\vdash \Delta, F^\perp$ are provable in LL, by Proposition 2.4, we have $\Gamma^\bullet, F^\bullet \vdash^i R$ and $\Delta^\bullet, (F^\perp)^\bullet \vdash^i R$ and thus $\Gamma^\bullet, \Delta^\bullet \vdash^i R$ in ILL by using Lemma 2.3. By cut elimination in ILL, one can build a cut-free proof of $\Gamma^\bullet, \Delta^\bullet \vdash^i R$, and thus a cut-free proof of $\vdash \Gamma^\bullet, \Delta^\bullet, R$ in LL. We choose R to be a fresh propositional variable Φ and, by substitution, we have $\vdash \Gamma^\bullet, \Delta^\bullet, \perp$.

We can check by induction on F that F^\bullet, \perp is obtained from F by adding some $_ \wp \perp$ and $_ \otimes 1$ in it. We conclude by induction on the proof of $\vdash \Gamma^\bullet, \Delta^\bullet, \perp$ that $\vdash \Gamma, \Delta$ in LL. \square

2.4 Relating RLL(\mathcal{R}) and ILL

We have seen in Section 2.3.4 that, in the study of $(_)^\bullet$, not only RLL(R) naturally appears, but also RLL(\mathcal{R}) with \mathcal{R} not an intuitionistic formula (\perp for example in Proposition 2.16). This really adds something since, for example, \perp could not be replaced directly by an intuitionistic formula: there is no R in ILL such that $\underline{R} \dashv\vdash \perp$ (Lemma 2.19).

Lemma 2.18. *It is not possible to have both $\vdash^0 (I)^\perp$ and $\vdash^0 !F, I$ provable in LL with (mix_0) , for any I of ILL and any F of LL.*

Proof. By induction on I . The key cases are:

- If both $\vdash^0 !F, \underline{J} \otimes \underline{K}$ and $\vdash^0 (J)^\perp \wp (\underline{K})^\perp$ are provable in LL with (mix_0) then $\vdash^0 (J)^\perp, (\underline{K})^\perp$ as well. Moreover either $\vdash^0 !F, \underline{J}$ and $\vdash^0 \underline{K}$, or $\vdash^0 !F, \underline{K}$ and $\vdash^0 \underline{J}$ are provable. In the first case for example, we can build:

$$\frac{\vdash^0 (J)^\perp, (\underline{K})^\perp \quad \vdash^0 \underline{K}}{\vdash^0 (J)^\perp} cut$$

and we apply the induction hypothesis on J .

- If both $\vdash^0 !F, (J)^\perp \wp \underline{K}$ and $\vdash^0 \underline{J} \otimes (\underline{K})^\perp$ are provable in LL with (mix_0) then $\vdash^0 !F, (J)^\perp, \underline{K}$, $\vdash^0 \underline{J}$, and $\vdash^0 (\underline{K})^\perp$ as well. So that we can build:

$$\frac{\frac{\vdash^0 !F, (J)^\perp, K}{\vdash^0 !F, K} \quad \vdash^0 J}{\vdash^0 !F, K} \text{ cut}$$

and we apply the induction hypothesis on K . \square

Lemma 2.19. *It is not possible to have both $\vdash (I)^\perp$ and $\vdash 1, I$ provable in LL, for any I of ILL.*

As a consequence, there is also no formula W in ILL such that for any I , $W \stackrel{I}{/} X \dashv\vdash ?I$ in LL (otherwise one would have $W \stackrel{0}{/} X \dashv\vdash ?0 \dashv\vdash \perp$ in LL): this means the connective $?$ is not definable in ILL.

Let us now consider other values of \mathcal{R} which are also not purely positive affine formulas, or even not equivalent to any ILL formula.

2.4.1 $\mathcal{R} = ?1$ and $R = !\Phi \multimap \Phi$

As we have seen for \perp , $?1$ is also a formula which is “out of the scope” of ILL: there is no R in ILL such that $R \dashv\vdash ?1$ in LL (Lemma 2.20).

Lemma 2.20. *It is not possible to have both $\vdash (I)^\perp, ?1$ and $\vdash !\perp, I$ provable in LL, for any I of ILL.*

We can however relate $(_)^\bullet$ and $\text{RLL}(?1)$:

Proposition 2.21. *The following statements are equivalent:*

- (i) $\vdash^0 \Gamma$ in LL with (mix_0)
- (ii) for all R provable in ILL, $\Gamma^{\bullet[R]} \vdash^i R$ in ILL
- (iii) there exists $\Phi \notin \Gamma$ such that $\Gamma^{\bullet[!\Phi \multimap \Phi]} \vdash^i !\Phi \multimap \Phi$ in ILL
- (iv) there exists $\Phi \notin \Gamma$ such that $\vdash_{\mathcal{R}\Phi \dashv\vdash \mathcal{R}\Phi}^r \Gamma$ in $\text{RLL}(?\Phi^\perp \mathcal{R} \Phi)$
- (v) $\vdash_{?1}^r \Gamma$ in $\text{RLL}(?1)$
- (vi) $\vdash \Gamma, ?1$ in LL

2.4.2 $\mathcal{R} = !\perp$ and $R = !\Phi$

First note there is no R in ILL such that $R \dashv\vdash !\perp$ in LL (Lemma 2.23).

Lemma 2.22. *It is not possible to have both $\vdash (I)^\perp$ provable in LL and $\vdash^0 I$ provable in LL with (mix_0) for each $I \in \Gamma$, for any Γ of ILL.*

Proof. By induction on the proof of $\vdash (I)^\perp$ in LL, we look at the last rule. The key case is the (\otimes) rule. We have $\Gamma = \Gamma', \Gamma''$ with $\vdash (\Gamma')^\perp, J$ and $\vdash (\Gamma'')^\perp, (K)^\perp$, and $\vdash^0 (J)^\perp, K$, as well as $\vdash^0 I$ for any I in Γ', Γ'' thus, using cuts, we can build a proof of $\vdash^0 J$ and then a proof of $\vdash^0 K$. We conclude with the induction hypothesis applied to the proof of $\vdash (\Gamma'')^\perp, (K)^\perp$. \square

Lemma 2.23. *It is not possible to have both $\vdash (I)^\perp, !\perp$ and $\vdash ?1, I$ provable in LL, for any I of ILL.*

It is however possible to relate $(_)^\bullet$ and $\text{RLL}(!\perp)$, and to give an alternative characterisation of $\text{RLL}(!\perp)$. The system $\text{LL}^{!\perp}$ is the extension of LL obtained by adding to the LL rules, the following

$$\text{one: } \frac{\vdash^{!\perp} \Gamma \quad \vdash^{!02} \Delta}{\vdash^{!\perp} \Gamma, \Delta} \text{ mix}^{!\perp}.$$

Lemma 2.24. *If $\vdash^{!\perp} \Gamma$ is provable in $\text{LL}^{!\perp}$ then $\vdash^{!02} \Gamma$ is provable in LL with both (mix_0) and (mix_2) .*

$\text{LL}^{!\perp}$ is thus intermediate between LL with (mix_2) and LL with both (mix_0) and (mix_2) , since (mix_2) is derivable but (mix_0) is constrained to occur only above a $(mix^{!\perp})$ rule. Indeed, we have:

$$\frac{\vdash^{!\perp} \Gamma \quad \frac{\vdash^{!\perp} \Delta}{\vdash^{!02} \Delta} \text{ Lemma 2.24}}{\vdash^{!\perp} \Gamma, \Delta} \text{ mix}^{!\perp}$$

but $\vdash^{!\perp}$ is not provable in $\text{LL}^{!\perp}$.

Proposition 2.25 (Cut elimination in $\text{LL}^{!\perp}$). *$\text{LL}^{!\perp}$ has the cut-elimination property.*

Proposition 2.26. *The following statements are equivalent:*

- (i) $\vdash^{!\perp} \Gamma$ in $\text{LL}^{!\perp}$
- (ii) for all R , $\Gamma^{\bullet[!R]} \vdash^i !R$ in ILL
- (iii) there exists $\Phi \notin \Gamma$ such that $\Gamma^{\bullet[!\Phi]} \vdash^i !\Phi$ in ILL
- (iv) there exists $\Phi \notin \Gamma$ such that $\vdash_{!\Phi}^r \Gamma$ in $\text{RLL}(!\Phi)$
- (v) $\vdash_{!\perp}^r \Gamma$ in $\text{RLL}(!\perp)$

Proof. The main implications are:

- (i) \Rightarrow (ii): this is Proposition 2.4 for the standard LL rules. We then have to consider:

$$\frac{\text{IH} \quad \Gamma^{\bullet[!R]} \vdash^i !R \quad \text{Lemma 2.13} \quad !R, \Delta^{\bullet[!R]} \vdash^i !R}{\Gamma^{\bullet[!R]}, \Delta^{\bullet[!R]} \vdash^i !R} \text{ cut}$$

- (v) \Rightarrow (i): the additional rules of $\text{RLL}(!\perp)$ are derivable in $\text{LL}^{!\perp}$:

$$\frac{\frac{\vdash^{!\perp} 1}{\vdash^{!\perp} ?1} 1}{\vdash^{!\perp} ?1} \quad \frac{\frac{\frac{\vdash^{!02} \text{mix}_0}{\vdash^{!02} \perp} \perp}{\vdash^{!02} !} !}{\vdash^{!\perp} 1, !\perp} \text{ mix}^{!\perp}$$

\square

In the derivations just above, the fact that we allow not only (mix_0) but also (mix_2) in the right premise of the $(mix^{!\perp})$ rule does not seem to be required. However it happens to be necessary for the cut-elimination property to hold in $\text{LL}^{!\perp}$ (Proposition 2.25).

In [3], the cases $\mathcal{R} = 1$, $\mathcal{R} = \perp$, $\mathcal{R} = \top$, $\mathcal{R} = 0$, $\mathcal{R} = ?1$ and $\mathcal{R} = !\perp$ are considered in the slightly different context of the logic JILL. While the authors give the same characterisation as here for $\text{RLL}(\mathcal{R})$ in terms of extensions of LL in the first five cases, they left open the case of $\text{RLL}(!\perp)$.

2.4.3 $\mathcal{R} = ? \otimes_n \perp$ and $R = !((\otimes_n \Phi) \multimap \Phi) \multimap \Phi$

We have already seen characterisations of LL extended with (mix_0) in Section 2.4.1 and of LL extended with both (mix_0) and (mix_2) in Section 2.3.1. We turn our attention to LL extended with (mix_n) .

Proposition 2.27. *For any $n \geq 0$, the following statements are equivalent:*

- (i) $\vdash^n \Gamma$ in LL with (mix_n)
- (ii) for all R such that $\otimes_n R \vdash^i R$ is provable in ILL, $\Gamma^{\bullet[R]} \vdash^i R$ in ILL
- (iii) there exists $\Phi \notin \Gamma$ such that $\Gamma^{\bullet[!(\otimes_n \Phi) \multimap \Phi]} \vdash^i !((\otimes_n \Phi) \multimap \Phi) \multimap \Phi$ in ILL
- (iv) there exists $\Phi \notin \Gamma$ such that $\vdash_{?((\otimes_n \Phi) \otimes \Phi^\perp) \mathcal{R} \Phi}^r \Gamma$ is provable in $\text{RLL}(?((\otimes_n \Phi) \otimes \Phi^\perp) \mathcal{R} \Phi)$
- (v) $\vdash_{? \otimes_n \perp}^r \Gamma$ in $\text{RLL}(? \otimes_n \perp)$
- (vi) $\vdash \Gamma, ? \otimes_n \perp$ in LL

The key point is to use $(?((\otimes_n \Phi) \otimes \Phi^\perp) \mathcal{R} \Phi)[^\perp / \Phi] \dashv\vdash ? \otimes_n \perp$ in (iv) \Rightarrow (v).

Let us consider a few particular cases. For $n = 0$, $!(1 \multimap \Phi) \multimap \Phi \dashv\vdash !\Phi \multimap \Phi$ and we find back the results of Section 2.4.1. For $n = 1$, $!((\Phi \otimes 1) \multimap \Phi) \multimap \Phi \dashv\vdash \Phi \dashv\vdash \Phi$ and we find back the results of Section 2.3.4. For $n = 2$, $\mathcal{R} = ?(\perp \otimes \perp)$ and $R = !((\Phi \otimes \Phi) \multimap \Phi) \multimap \Phi$

give characterisations of LL with (mix_2) alone (in contrast with Section 2.3.1).

2.4.4 $\mathcal{R} = 1$, and $R = \Phi \multimap \Phi$ or $R = !\Phi \multimap 1$ or
 $R = !((\Phi \multimap \Phi) \multimap ((\Phi \multimap \Phi) \multimap \Phi) \multimap \Phi)$

We can revisit Section 2.3.1 with some variations on R .

Proposition 2.28. *The following statements are equivalent:*

- (i) $\vdash^{02} \Gamma$ in LL with (mix_0) and (mix_2)
- (ii) $\Gamma \bullet^{[\Phi \multimap \Phi]} \vdash^i \Phi \multimap \Phi$ in ILL with $\Phi \notin \Gamma$
- (iii) $\Gamma \bullet^{[!\Phi \multimap 1]} \vdash^i !\Phi \multimap 1$ in ILL with $\Phi \notin \Gamma$
- (iv) $\Gamma \bullet^{[!((\Phi \multimap \Phi) \multimap ((\Phi \multimap \Phi) \multimap \Phi) \multimap \Phi)]} \vdash^i !((\Phi \multimap \Phi) \multimap ((\Phi \multimap \Phi) \multimap \Phi) \multimap \Phi)$ in ILL with $\Phi \notin \Gamma$
- (v) $\vdash \Gamma, ?(?1 \otimes ?1)$ in LL

3 Back to LL: Conservativity

Through the translation $I \mapsto \underline{I}$, ILL can be seen as a subsystem of LL (simply replace each $I \multimap J$ by $I^\perp \wp J$). We often identify I and \underline{I} in the present section. Whether LL is a conservative extension of ILL (i.e. formulas or sequents from ILL provable in LL are the same as in ILL) has been studied in [22]. The property does not hold in full generality as shown by H. Schellinx's counterexample [22]:

$$(X \multimap (0 \multimap Y) \multimap Y') \multimap ((X \multimap X') \multimap 0) \multimap Y'.$$

One can give more compact variants of this example by using \otimes and \top : $((X \multimap X') \multimap 0) \multimap (X \otimes (0 \multimap Y))$ and $((X \multimap X') \multimap 0) \multimap (X \otimes \top)$ [13] (possibly the shortest existing counterexample). If one considers the *implicative order* of formulas (0 for formulas without \multimap , and add 1 each time you go to the left of a \multimap in the formula tree), these formulas have order 3. One can also give a new counterexample of order 2 (as we will see below, this is minimal since there is no counterexample of order 1):

$$(((X \otimes \top) \& (Y \otimes \top)) \multimap 0) \multimap ((X \multimap X') \oplus (Y \multimap Y')).$$

In [22], two constraints are given on formulas in order to ensure conservativity. We are going to generalise both. Another result is proposed in [25, Exercise 2 page 39]: for all Γ in ILL, $\vdash \Gamma^\perp, 0$ in LL if and only if $\Gamma \vdash^i 0$ in ILL. However if I is a formula such that $\vdash \underline{I}$ in LL and $\not\vdash^i I$ in ILL (such as H. Schellinx's counterexample), then $I \multimap 0 \vdash^i 0$ is not provable in ILL while $\vdash \underline{I} \otimes \top, 0$ is provable in LL.

3.1 Looking to the Left of \multimap

In [22], it is proved that conservativity holds for the image of Girard's translation of intuitionistic logic into linear logic. That is for formulas in the following grammar:

$$\mathcal{G} ::= X \mid !\mathcal{G} \multimap \mathcal{G} \mid 0 \mid \mathcal{G} \& \mathcal{G} \mid !\mathcal{G} \oplus !\mathcal{G}.$$

Proposition 3.1 ([22, Corollary 3.7]). *A formula in the grammar \mathcal{G} is provable in LL if and only if it is provable in ILL.*

We are going to show that the key point here is the constraint induced on the left-hand side of \multimap . Moreover we generalise the shape $!_ \multimap _$ into $O \multimap _$ where O is what we call a *!-like formula*:

$$O ::= X \mid 1 \mid O \otimes O \mid 0 \mid O \oplus O \mid O \& I \mid I \& O \mid !I.$$

Lemma 3.2. *If $\vdash \underline{\Omega}, \underline{I}$ is provable in LL (where Ω contains !-like formulas only, and all sub-formulas of Γ and I of the shape $J \multimap K$ are such that J is !-like), then $\Gamma \vdash^i I$ is provable in ILL, and if Ω is not empty then $\Gamma, \Sigma \vdash^i I$ in ILL for any Σ .*

Proof. By induction on a cut-free proof of $\vdash \underline{\Omega}, \underline{I}$ with atomic axioms. The key cases are:

- (\wp) rule:
 - If $\underline{I} = F \wp G$, then $I = J \multimap K$ with $\underline{J} = F^\perp$ and $\underline{K} = G$. We have:

$$\frac{\text{IH} \quad \Gamma, J, (\Sigma) \vdash^i K}{\Gamma, (\Sigma) \vdash^i J \multimap K} \multimap R$$

- If $\underline{\Omega}^\perp$ contains $F \wp G$ then $\Gamma = \Gamma', J \otimes K$ with $\underline{J} = F^\perp$ and $\underline{K} = G^\perp$. We have:

$$\frac{\text{IH} \quad \Gamma', J, K, (\Sigma) \vdash^i I}{\Gamma', J \otimes K, (\Sigma) \vdash^i I} \otimes L$$

- Finally, by definition of !-like, $\underline{\Omega}$ cannot contain a formula of the shape $F \wp G$.

- (\otimes) rule:
 - If $\underline{I} = F \otimes G$, then $I = J \otimes K$ with $\underline{J} = F$ and $\underline{K} = G$. In the contexts of the premises, Γ is split into Δ' and Δ'' and Ω is split into Ω' and Ω'' . We have:

$$\frac{\text{IH} \quad \Delta' \vdash^i J \quad \text{IH} \quad \Delta'' \vdash^i K}{\Delta', \Delta'' \vdash^i J \otimes K} \otimes R$$

If Ω is not empty, then at least one of Ω' and Ω'' as well. If, for example, $\Omega' \neq \emptyset$ then we have, for all Σ :

$$\frac{\text{IH} \quad \Delta', \Sigma \vdash^i J \quad \text{IH} \quad \Delta'' \vdash^i K}{\Delta', \Delta'', \Sigma \vdash^i J \otimes K} \otimes R$$

- If $\underline{\Omega}^\perp$ contains $F \otimes G$ then $\Gamma = \Gamma', J \multimap K$ with $\underline{J} = F$ and $\underline{K} = G^\perp$. If \underline{I} belongs to the same premise as F , we have $\Gamma = \Delta', \Delta'', J \multimap K$ with $\vdash \underline{\Delta'}^\perp, \underline{J}, \underline{I}$ and, since J is !-like, for all Σ :

$$\text{IH} \quad \Delta', \Delta'', J \multimap K, \Sigma \vdash^i I$$

If \underline{I} belongs to the same premise as G , we have $\Gamma = \Delta', \Delta'', J \multimap K$ and:

$$\frac{\text{IH} \quad \Delta' \vdash^i J \quad \text{IH} \quad \Delta'', K \vdash^i I}{\Delta', \Delta'', J \multimap K \vdash^i I} \multimap L$$

If Ω is not empty and splits into Ω' and Ω'' in the premises, then at least one of Ω' and Ω'' is not empty. If, for example, $\Omega' \neq \emptyset$ then we have, for all Σ :

$$\frac{\text{IH} \quad \Delta', \Sigma \vdash^i J \quad \text{IH} \quad \Delta'', K \vdash^i I}{\Delta', \Delta'', J \multimap K, \Sigma \vdash^i I} \multimap L$$

- If $\underline{\Omega}$ contains $F \otimes G$ then $\Omega = \Omega', J \otimes K$ with $\underline{J} = F$ and $\underline{K} = G$. In the contexts of the premises, Γ is split into Δ' and Δ'' and \underline{I} must be in the same premise as F or G and we have, for all Σ :

$$\text{IH} \quad \Delta', \Delta'', \Sigma \vdash^i I$$

□

Theorem 3.3 (!-Like Conservativity). *If I is a formula of ILL such that any formula on the left-hand side of a \multimap in I is !-like, then $\vdash \underline{I}$ is provable in LL if and only if $\vdash^i I$ is provable in ILL.*

As a consequence, we get the conservativity of LL over ILL for the fragment corresponding to formulas obtained by Girard's translation from intuitionistic logic into LL (as already shown in [22]) but also for the image of the so-called *call-by-value Girard's translation* ($A \rightarrow B \mapsto !(A \multimap B)$) [9], and for other variants as soon as they only use the connective \multimap in the $O \multimap _$ shape. As a corollary, this gives us the faithfulness of these translations: provability of the image implies provability of the source (move from LL to ILL by conservativity, and then to intuitionistic logic using the notion of skeleton [6]).

3.2 Looking to the Right of \multimap

The other conservativity result from [22] relies on the study of the interaction between \multimap and 0:

Proposition 3.4 ([22, Proposition 3.8]). *If I is a formula of ILL which does not contain \multimap or does not contain 0, then $\vdash \underline{I}$ is provable in LL if and only if $\vdash^i I$ is provable in ILL.*

We are interested in refining this result since, for example, the translation $(_)^\bullet$ uses both 0 and \multimap , thus Proposition 3.4 does not allow us to get a conservativity result for the image of this translation.

Definition 3.5 (Almost Zero). A formula of ILL is *almost 0* if it belongs to the following grammar:

$$Z ::= 0 \mid Z \otimes I \mid I \otimes Z \mid Z \& I \mid I \& Z \mid Z \oplus Z \mid !Z .$$

Lemma 3.6. *Let Z be an almost 0 formula, we have $Z \dashv\vdash 0$ in ILL.*

Definition 3.7 (Zero Clean). A formula of ILL is *0-clean* if it does not contain any subformula of the shape $_ \multimap Z$ with Z an almost 0 formula.

Lemma 3.8. *If Γ contains 0-clean formulas only, and if $\vdash \underline{\Gamma}^\perp$ is provable in LL, then Γ contains an almost 0 formula.*

Proof. By induction on a cut-free proof with atomic axioms. The key case is the (\otimes) rule: if $\underline{I}^\perp = F \otimes G$ then $I = J \multimap K$ with $\underline{J} = F$ and $\underline{K} = G^\perp$. By induction hypothesis applied to the premise $\vdash \underline{\Delta}^\perp, \underline{K}^\perp$, we have an almost 0 formula in $\Delta \subseteq \Gamma$ since I is 0-clean. \square

Theorem 3.9 (Zero-Clean Conservativity). *Let I be a 0-clean formula, $\vdash \underline{I}$ is provable in LL if and only if $\vdash^i I$ is provable in ILL.*

Proof. The interesting direction is the left-to-right one. We prove by induction on a cut-free proof that, if $\vdash \underline{\Gamma}^\perp, \underline{I}$ containing 0-clean formulas only is provable in LL, then $\Gamma \vdash^i I$ is provable in ILL. The key case is the (\otimes) rule.

If $\underline{I} = F \otimes G$, then $I = J \otimes K$ with $\underline{J} = F$ and $\underline{K} = G$. The contexts of the premises are $\underline{\Delta}^\perp$ and $\underline{\Sigma}^\perp$ with $\Gamma = \Delta, \Sigma$. We have:

$$\frac{\frac{\text{IH}}{\Delta \vdash^i J} \quad \frac{\text{IH}}{\Sigma \vdash^i K}}{\Delta, \Sigma \vdash^i J \otimes K} \otimes R$$

If $\underline{\Gamma}^\perp$ contains $F \otimes G$ then $\Gamma = \Gamma', J \multimap K$ with $\underline{J} = F$ and $\underline{K} = G^\perp$. If \underline{I} belongs to the same premise as F , the other premise is of the shape $\vdash \underline{\Sigma}^\perp, G$ with 0-clean formulas only. By Lemma 3.8, Σ, K contains an almost 0 formula Z . We cannot have $K = Z$ otherwise $J \multimap K$ is not 0-clean, thus Z belongs to $\Sigma = \Sigma', Z$. We have $\Gamma = \Delta, \Sigma', Z, J \multimap K$ and:

$$\text{Lemma 3.6} \quad \frac{\frac{Z \vdash^i 0 \quad 0, \Delta, \Sigma', J \multimap K \vdash^i I}{Z, \Delta, \Sigma', J \multimap K \vdash^i I} \text{cut}}{0L}$$

If \underline{I} belongs to the same premise as G , we have $\Gamma = \Delta, \Sigma, J \multimap K$ and:

$$\frac{\frac{\text{IH}}{\Sigma \vdash^i J} \quad \frac{\text{IH}}{\Delta, K \vdash^i I}}{\Delta, \Sigma, J \multimap K \vdash^i I} \multimap L$$

\square

Lemma 3.10. *If $\vdash \underline{\Gamma}^\perp, \underline{\Delta}$ is provable in LL, with Δ containing at least two formulas, then $\underline{\Gamma}^\perp, \underline{\Delta}$ contains \top .*

By looking at the key case of Theorem 3.9 (and similarly for Lemma 3.8):

$$\frac{\frac{\vdash \underline{\Delta}^\perp, \underline{J}, \underline{I} \quad \vdash \underline{\Sigma}^\perp, \underline{K}^\perp}{\vdash \underline{\Delta}^\perp, \underline{\Sigma}^\perp, \underline{J} \otimes \underline{K}^\perp, \underline{I}} \otimes}{\vdash \underline{\Delta}^\perp, \underline{\Sigma}^\perp, \underline{J} \otimes \underline{K}^\perp, \underline{I}} \otimes$$

we can see that we can refine these results: conservativity can only fail if $_ \multimap Z$ appears in negative position (an odd number of times on the left of a \multimap connective in the formula tree), and if the formula contains (outside this Z) a \top in positive position or a 0 in negative position (thanks to Lemma 3.10 applied to the premise $\vdash \underline{\Delta}^\perp, \underline{J}, \underline{I}$).

Similarly, Lemma 3.2 and Theorem 3.3 can be strengthened by only constraining \multimap to be of the shape $O \multimap _$ when it occurs in negative position.

To sum up Sections 3.1 and 3.2, a counterexample to the conservativity of LL over ILL must contain:

- a negative sub-formula $I \multimap J$ with I not !-like (which implies that the implicative order of the global formula is at least 2);
- a negative sub-formula $I \multimap Z$ (this also entails implicative order at least 2);
- a positive sub-formula \top or a negative sub-formula 0 outside this Z .

Moreover, for each such counterexample, thanks to Lemma 3.6, one can obtain a possibly shorter one by replacing Z with 0. One can check these three constraints are true for the formulas mentioned in the beginning of Section 3.

3.3 Tensor Logic

Tensor logic (TL) [18] is a variant of ILL based on a *primitive* negation connective. *Tensor formulas* are generated by:

$$U ::= X \mid 1 \mid U \otimes U \mid 0 \mid U \oplus U \mid !U \mid \neg U .$$

Sequents are $\Gamma \vdash^t \Pi$ where Π is either empty or a single formula U . Rules are given on Figure 1.

Since if we add a primitive negation \neg to ILL, it is possible to prove that \neg and \neg_Φ are linearly equivalent (for a propositional variable Φ), and since the rules of TL for the other connectives are the same as in ILL, we can work with TL as being the fragment of ILL restricted to tensor formulas (with \neg interpreted as \neg_Φ and $\Gamma \vdash^t \Phi$ as $\Gamma \vdash^t \Phi$). Theorem 3.9 is strong enough to prove a conservativity result of LL over TL.

Theorem 3.11 (Conservativity of LL over TL). *Let U be a formula of TL, $\vdash \underline{U}$ is provable in LL if and only if $\vdash^t U$ is provable in TL.*

$$\begin{array}{c}
\frac{}{U \vdash^t U} \text{ax} \qquad \frac{\Gamma \vdash^t U \quad \Delta, U \vdash^t \Pi}{\Gamma, \Delta \vdash^t \Pi} \text{cut} \\
\frac{\Gamma \vdash^t U}{\Gamma, \neg U \vdash^t} \neg L \qquad \frac{\Gamma, U \vdash^t}{\Gamma \vdash^t \neg U} \neg R \\
\frac{\Gamma, U, V \vdash^t \Pi}{\Gamma, U \otimes V \vdash^t \Pi} \otimes L \qquad \frac{\Gamma \vdash^t U \quad \Delta \vdash^t V}{\Gamma, \Delta \vdash^t U \otimes V} \otimes R \\
\frac{\Gamma \vdash^t \Pi}{\Gamma, 1 \vdash^t \Pi} 1L \qquad \frac{}{\vdash^t 1} 1R \\
\frac{\Gamma, U \vdash^t \Pi \quad \Gamma, V \vdash^t \Pi}{\Gamma, U \oplus V \vdash^t \Pi} \oplus L \qquad \frac{}{\Gamma, 0 \vdash^t \Pi} 0L \\
\frac{\Gamma \vdash^t U}{\Gamma \vdash^t U \oplus V} \oplus_1 R \qquad \frac{\Gamma \vdash^t V}{\Gamma \vdash^t U \oplus V} \oplus_2 R \\
\frac{\Gamma, U \vdash^t \Pi}{\Gamma, !U \vdash^t \Pi} !L \qquad \frac{! \Gamma \vdash^t U}{! \Gamma \vdash^t !U} !R \\
\frac{\Gamma \vdash^t \Pi}{\Gamma, !U \vdash^t \Pi} !wL \qquad \frac{\Gamma, !U, !U \vdash^t \Pi}{\Gamma, !U \vdash^t \Pi} !cL
\end{array}$$

Figure 1. Tensor logic (TL)

Proof. This relies on the fact that the only use of the \multimap connective in TL seen as a fragment of ILL, is of the shape $_ \multimap \Phi$ with Φ a propositional variable. This means that, inside ILL, tensor formulas are 0-clean and we can apply Theorem 3.9. \square

This shows that TL can be seen as a *fragment* of LL (a subsystem defined by restricting the set of formulas) and, as a consequence, it can be studied inside LL.

4 Focusing

By using \neg instead of \neg_R , the image of $(_)^\bullet$ belongs to TL and one can see that Proposition 2.4 still holds: if $\vdash \Gamma$ is provable in LL, then $\Gamma^\bullet \vdash^t$ in TL. We are now going to optimise this translation by trying to minimise the use of the \neg connective.

4.1 Polarized Translation

First we partition linear formulas into two classes: *synchronous* and *asynchronous* formulas [2].

$$\begin{aligned}
S &::= X \mid 1 \mid F \otimes F \mid 0 \mid F \oplus F \mid !F \\
A &::= X^\perp \mid \perp \mid F \wp F \mid \top \mid F \& F \mid ?F.
\end{aligned}$$

We consider two mutually defined translations $(_)^+$ and $(_)^-$ from LL formulas to TL formulas:

$$\begin{array}{ll}
X^+ = X & (X^\perp)^- = X \\
1^+ = 1 & \perp^- = 1 \\
(F \otimes G)^+ = F^+ \otimes G^+ & (F \wp G)^- = F^- \wp G^- \\
0^+ = 0 & \top^- = 0 \\
(F \oplus G)^+ = F^+ \oplus G^+ & (F \& G)^- = F^- \& G^- \\
(!F)^+ = !\neg F^- & (?F)^- = !\neg F^+ \\
A^+ = \neg A^- & S^- = \neg S^+
\end{array}$$

This is similar to the focalized translation of [18], but $(!S)^+ = !\neg \neg S^+$ is crucial here (as also remarked in [5]). The case of $(?A)^-$ can be discussed (see below).

Lemma 4.1. For all F , $(F^\perp)^+ = F^-$ and $(F^\perp)^- = F^+$.

Lemma 4.2. For all F , $F^- \vdash^t F^\bullet$ in TL.

Proof. We first prove that for any F , $F^- \vdash^t F^\bullet$ implies $\neg F^+$, $\neg F^\bullet \vdash^t$. We consider the two cases, F synchronous and F asynchronous:

$$\frac{S^- \vdash^t S^\bullet}{\neg S^+, \neg S^\bullet \vdash^t} \neg L \qquad \frac{A^- \vdash^t A^\bullet}{A^-, \neg A^\bullet \vdash^t} \neg L \\
\frac{}{\neg A^\bullet \vdash^t A^+} \neg R \qquad \frac{}{\neg A^+, \neg A^\bullet \vdash^t} \neg L$$

We now prove the statement by induction on the formula F . Here are the main cases:

- $(F \otimes G)^- = \neg(F \otimes G)^+ = \neg(F^+ \otimes G^+)$ and $(F \otimes G)^\bullet = \neg(\neg F^\bullet \otimes \neg G^\bullet)$, and we have:

$$\frac{\text{IH} \quad \text{IH}}{\frac{\frac{\frac{F^- \vdash^t F^\bullet}{\neg F^+, \neg F^\bullet \vdash^t} \neg L \quad \frac{G^- \vdash^t G^\bullet}{\neg G^+, \neg G^\bullet \vdash^t} \neg L}{\neg(F^+ \otimes G^+), \neg F^\bullet \otimes \neg G^\bullet \vdash^t} \neg L}{\neg(F^+ \otimes G^+) \vdash^t \neg(\neg F^\bullet \otimes \neg G^\bullet)} \neg R} \text{Lemma 2.1 (i)}$$

- $(!F)^- = \neg(!F)^+ = \neg!\neg F^-$ and $(!F)^\bullet = \neg!\neg F^\bullet$, and we have:

$$\frac{\text{IH}}{\frac{\frac{F^- \vdash^t F^\bullet}{F^-, \neg F^\bullet \vdash^t} \neg L}{\neg F^\bullet \vdash^t \neg F^-} \neg R}{\frac{!\neg F^\bullet \vdash^t \neg F^-}{!\neg F^\bullet \vdash^t !\neg F^-} !L}{\frac{!\neg F^\bullet \vdash^t !\neg F^-}{!\neg F^\bullet \vdash^t !\neg F^-} !R}{\frac{!\neg F^\bullet \vdash^t !\neg F^-}{!\neg F^\bullet \vdash^t !\neg F^\bullet} \neg L} \neg R$$

- $(?F)^- = !\neg F^+$ and $(?F)^\bullet = !\neg \neg F^\bullet$, and we have:

$$\frac{\text{IH}}{\frac{\frac{F^- \vdash^t F^\bullet}{\neg F^+, \neg F^\bullet \vdash^t} \neg L}{\neg F^+ \vdash^t \neg \neg F^\bullet} \neg R}{\frac{!\neg F^+ \vdash^t \neg \neg F^\bullet}{!\neg F^+ \vdash^t \neg \neg F^\bullet} !L}{\frac{!\neg F^+ \vdash^t \neg \neg F^\bullet}{!\neg F^+ \vdash^t !\neg \neg F^\bullet} !R} \neg R$$

\square

Theorem 4.3 (Polarized Translation). If $\vdash \Gamma$ is provable in LL then $\Gamma^- \vdash^t$ is provable in TL.

Proof. By Proposition 2.4, we have $\Gamma^\bullet \vdash^t$ in TL. We then use cuts with the proofs from Lemma 4.2 to turn every F^\bullet into F^- . \square

While it is *not possible* to have $(!S)^+ = !S^+$ (it would break Theorem 4.3), for similar reasons as in Section 2.1, one can optimise $(?A)^- = !A^-$ while preserving Lemma 4.2 and Theorem 4.3, but we would loose symmetry (in particular it would lead to $(F^\perp)^- \neq F^+$ and thus the correspondence of Theorem 4.5 would not work so nicely for axioms and cuts). Moreover the additional use of \neg we impose here in $(?A)^-$ is necessary for the focusing analysis to come (it is responsible for the constraint of an empty focus in the asynchronous $(?d)$ rule).

The construction of the translations $(_)^-/(_)^+$ by first defining $(_)^\bullet$ and then using Lemma 4.2 is directly inspired by the focusing proof of S. Zimmerman for differential linear logic [26] which uses a similar decomposition in two steps.

$$\begin{array}{c}
 \frac{}{\vdash^f S^\perp | S} \text{ax} \quad \frac{\vdash^f \Gamma | S}{\vdash^f \Gamma, S |} \text{foc} \quad \frac{\vdash^f \Gamma | F \quad \vdash^f \Delta, F^\perp | \Pi}{\vdash^f \Gamma, \Delta | \Pi} \text{cut} \\
 \\
 \frac{\vdash^f \Gamma, F, G | \Pi}{\vdash^f \Gamma, F \wp G | \Pi} \wp \quad \frac{\vdash^f \Gamma | F \quad \vdash^f \Delta | G}{\vdash^f \Gamma, \Delta | F \otimes G} \otimes \\
 \\
 \frac{\vdash^f \Gamma | \Pi}{\vdash^f \Gamma, \perp | \Pi} \perp \quad \frac{}{\vdash^f | 1} 1 \\
 \\
 \frac{\vdash^f \Gamma, F | \Pi \quad \vdash^f \Gamma, G | \Pi}{\vdash^f \Gamma, F \& G | \Pi} \& \quad \frac{}{\vdash^f \Gamma, \top | \Pi} \top \\
 \\
 \frac{\vdash^f \Gamma | F}{\vdash^f \Gamma | F \oplus G} \oplus_1 \quad \frac{\vdash^f \Gamma | G}{\vdash^f \Gamma | F \oplus G} \oplus_2 \\
 \\
 \frac{\vdash^f ?\Gamma, F |}{\vdash^f ?\Gamma | !F} ! \quad \frac{\vdash^f \Gamma | F}{\vdash^f \Gamma, ?F |} ?d \\
 \\
 \frac{\vdash^f \Gamma | \Pi}{\vdash^f \Gamma, ?F | \Pi} ?w \quad \frac{\vdash^f \Gamma, ?F, ?F | \Pi}{\vdash^f \Gamma, ?F | \Pi} ?c
 \end{array}$$

 Figure 2. The LL_{foc} system

4.2 From TL to LL_{foc}

Weakly focused linear logic (LL_{foc}) [16] is a focused system for LL inspired from [10] and [2]. It relies on the distinction between *synchronous* and *asynchronous* formulas. Sequents are $\vdash^f \Gamma | \Pi$ where Π is either empty or a single synchronous formula S (called the *focus*). This means in particular that $\vdash^f \Gamma | A$ with A asynchronous is not a sequent of the system. However it is useful to define $\vdash^f \Gamma | A$ as a *notation* for $\vdash^f \Gamma, A |$, so that we can use the notation:

$$\vdash^f \Gamma | F = \begin{cases} \vdash^f \Gamma | F & \text{if } F \text{ is synchronous} \\ \vdash^f \Gamma, F | & \text{if } F \text{ is asynchronous} \end{cases}$$

for an arbitrary linear formula F . Rules are given on Figure 2.

Let us consider a proof of $\Gamma^- \vdash^t \Pi^+$ in TL. We can rely on the fact that all the $!$ connectives come with a \neg , to constrain slightly the structure of proofs. The system TL' is obtained from TL by replacing the rules ($!L$) and ($!R$) by:

$$\frac{\Gamma \vdash^t U}{\Gamma, !\neg U \vdash^t} !L' \quad \frac{!\Pi, U \vdash^t}{!\Gamma \vdash^t !\neg U} !R'$$

Lemma 4.4. *If $\Gamma^- \vdash^t \Pi^+$ is provable in TL, then it is provable in TL' as well.*

Proof. We start from a proof of $\Gamma^- \vdash^t \Pi^+$ in TL with atomic axioms. A ($!L$) rule must be of the shape:

$$\frac{\Gamma^-, \neg F^+ \vdash^t \Pi^+}{\Gamma^-, !\neg F^+ \vdash^t \Pi^+} !L$$

We can see that such an occurrence of rule can be commuted up with all rules until we reach the point where the \neg of $\neg F^+$ has been introduced, so that we can turn it into a ($!L'$) rule.

Let us now look at a ($!R$) rule. It must be of the shape:

$$\frac{\pi}{!\neg \Gamma^+ \vdash^t !\neg F^-} !R$$

Since π has atomic axioms, we can find in π the rules ($\neg R$) introducing the negation \neg of $\neg F^-$. They can be commuted down until we reach the ($!R$) rule (this corresponds to the reversibility of the ($\neg R$) rule). We then turn the sequence ($\neg R$)-($!R$) into ($!R'$). \square

The transformations of proofs of TL into proofs of TL' described in the proof of Lemma 4.4 can also be obtained by introducing cuts and by eliminating them appropriately in TL:

$$\frac{\frac{\frac{}{F^+ \vdash^t F^+} \text{ax}}{\neg F^+, F^+ \vdash^t} \neg L}{!\neg F^+, F^+ \vdash^t} !L}{!\neg F^+, F^+ \vdash^t} \neg R \quad \frac{\Gamma^-, \neg F^+ \vdash^t \Pi^+}{\Gamma^-, !\neg F^+ \vdash^t \Pi^+} !L}{!\neg F^+ \vdash^t !\neg F^+} !R \quad \frac{}{\Gamma^-, !\neg F^+ \vdash^t \Pi^+} \text{cut}$$

and

$$\frac{\frac{\frac{}{F^- \vdash^t F^-} \text{ax}}{\neg F^-, F^- \vdash^t} \neg L}{!\neg F^-, F^- \vdash^t} !L}{!\neg \Gamma^+ \vdash^t \neg F^-} \neg R \quad \frac{\Gamma^-, \neg F^- \vdash^t \Pi^+}{\Gamma^-, !\neg F^- \vdash^t \Pi^+} !L}{!\neg \Gamma^+ \vdash^t !\neg F^-} !R \quad \frac{}{!\neg \Gamma^+ \vdash^t !\neg F^-} \text{cut}$$

Theorem 4.5. *$\Gamma^- \vdash^t \Pi^+$ is provable in TL' if and only if $\vdash^f \Gamma | \Pi$ is provable in LL_{foc} .*

Proof. There is a one-to-one correspondence for almost all rules. We focus on the most tricky cases:

- The (ax) rule $S^+ \vdash^t S^+$ exactly corresponds to $\vdash^f S^\perp | S$ since $(S^\perp)^- = S^+$.
- The ($\neg R$) rule of TL' :

$$\frac{\Gamma^-, A^- \vdash^t}{\Gamma^- \vdash^t \neg A^-} \neg R$$

is the identity in LL_{foc} since it corresponds to the notation $\vdash^f \Gamma | A = \vdash^f \Gamma, A |$ by $\neg A^- = A^+$.

- The ($\neg L$) rule of TL' :

$$\frac{\Gamma^- \vdash^t S^+}{\Gamma^-, \neg S^+ \vdash^t} \neg L$$

corresponds to the (foc) rule of LL_{foc} since $\neg S^+ = S^-$.

- The ($!L'$) rule of TL' :

$$\frac{\Gamma^- \vdash^t F^+}{\Gamma^-, !\neg F^+ \vdash^t} !L'$$

corresponds to the ($?d$) rule of LL_{foc} .

- The ($!R'$) rule of TL' :

$$\frac{(? \Gamma)^-, F^- \vdash^t}{(? \Gamma)^- \vdash^t !\neg F^-} !R'$$

corresponds to the ($!$) rule of LL_{foc} since $(? \Gamma)^- = !\neg \Gamma^+$. \square

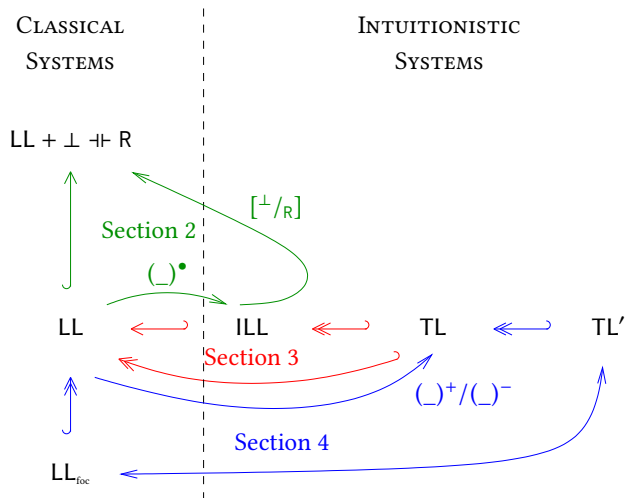
Once restricted to $!$ connectives only used in association with \neg : $!\neg$, TL (optimised into TL') can thus be seen as a focused system for LL and the “at most one active synchronous formula” constraint of focused sequents occurs as a particular case of the “at most one formula on the right” constraint of intuitionistic sequents.

Corollary 4.6 (Weak Focusing). *If $\vdash \Gamma$ is provable in LL then $\vdash^f \Gamma |$ is provable in LL_{foc} .*

For more discussions about various notions of focusing and how to deduce traditional focusing from this weak focusing property, see [16]. However weak focusing is the key step in focusing.

5 Conclusion

We can sum up the main results in the following picture:



We use \hookrightarrow for (almost) identity-on-formulas translations and \hookleftarrow for conservative such embeddings.

We have focused on propositional systems, but the results we have presented can be extended to first-order and second-order quantifiers (except Section 3 which breaks for second-order). The key ingredients to add, in second-order for example, are:

$$\begin{aligned} (\exists X.F)^\bullet &= \neg_R \exists X. \neg_R F^\bullet & (\forall X.F)^\bullet &= \exists X.F^\bullet \\ (\exists X.F)^+ &= \exists X.F^+ & (\forall X.F)^- &= \exists X.F^- \end{aligned}$$

with the following lemma which allows us to extend Proposition 2.4:

Lemma 5.1. *For all F, G and X , if $R[\neg_R G^\bullet / X] = R$ then $F^\bullet[\neg_R G^\bullet / X], \neg_R(F[G/X])^\bullet \vdash^i R$ in ILL.*

In polarization and focusing, the status of the exponential connectives has always been more difficult to understand than for the other connectives. The negations we are forced to introduce in translating the connective $!$ of LL into ILL or TL, can be justified by associating to $!$ the polarity $- \mapsto +$ (meaning that it turns a negative (asynchronous) formula into a positive (synchronous) formula) in LL, but the polarity $+ \mapsto +$ in TL. The $!$ of LL is then decomposed into the two operations $!$ and \neg of TL (which is coherent with a polarity $\neg : - \mapsto +$). Moreover when $!$ is applied to a synchronous formula, another negation has to be added. Such a decomposition of the $!$ of LL is suggested in [5]. This could also be described in systems using shift operators (\downarrow and \uparrow , see for example [18]).

As a final remark, let us mention that reintroducing the parameter R in negations, so that $!_R$ in LL is mapped to $!(_ \wp R)$ is what G. Munch proposed for working with delimited continuations [19].

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