

A Complete Axiomatisation of the ZX-Calculus for Clifford+T Quantum Mechanics

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Abstract

We introduce the first complete and approximately universal diagrammatic language for quantum mechanics. We make the ZX-Calculus, a diagrammatic language introduced by Coecke and Duncan, complete for the so-called Clifford+T quantum mechanics by adding two new axioms to the language. The completeness of the ZX-Calculus for Clifford+T quantum mechanics – also called the $\frac{\pi}{4}$ -fragment of the ZX-Calculus – was one of the main open questions in *categorical quantum mechanics*. We prove the completeness of this fragment using the recently studied ZW-Calculus, a calculus dealing with integer matrices. We also prove that the $\frac{\pi}{4}$ -fragment of the ZX-Calculus represents exactly all the matrices over some finite dimensional extension of the ring of dyadic rationals.

CCS Concepts • Theory of computation → Quantum computation theory; Logic; Semantics and reasoning;

Keywords Categorical Quantum Mechanics, Clifford+T, Completeness, ZX-Calculus

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1 Introduction

The ZX-Calculus is a powerful graphical language for quantum reasoning and quantum computing introduced by Bob Coecke and Ross Duncan [7]. The language comes with a way of interpreting any ZX-diagram as a matrix – called the *standard interpretation*. Two diagrams represent the same quantum evolution when they have the same standard interpretation. The language is also equipped with a set of axioms – transformation rules – which are sound, i.e. they preserve the standard interpretation. Their purpose is to explain how a diagram can be transformed into an equivalent one.

The ZX-calculus has several applications in quantum information processing [9] (e.g. measurement-based quantum computing [11, 15, 20], quantum codes [6, 10, 13, 14], foundations [4, 12]), and can be used through the interactive theorem prover Quantomatic [24, 25]. However, the main obstacle to wider use of the ZX-calculus

was the absence of a *completeness* result for a *universal* fragment of quantum mechanics, in order to guarantee that any true property is provable using the ZX-calculus. More precisely, the language would be complete if, given any two diagrams representing the same matrix, one could transform one diagram into the other using the axioms of the language. Completeness is crucial, it means in particular that all the fundamental properties of quantum mechanics are captured by the graphical rules.

ZX-Calculus has been proved to be incomplete in general [30], and despite the necessary axioms that have since been identified [23, 28], the language remained incomplete. However, several fragments of the language have been proved to be complete ($\frac{\pi}{2}$ -fragment [2]; π -fragment [16]; single-qubit $\frac{\pi}{4}$ -fragment [3]), but none of them are *universal* for quantum mechanics, even approximately. In particular all quantum algorithms expressible in these fragments are efficiently simulable on a classical computer.

As a consequence, most of the attention has been paid to find a complete axiomatisation of the $\frac{\pi}{4}$ -fragment of the ZX-Calculus for the Clifford+T quantum mechanics, the simplest approximately universal fragment of quantum mechanics, which is widely used in quantum computing.

Our Approach. In the following, we introduce the first complete axiomatisation of the ZX-calculus for Clifford+T quantum mechanics, thanks to the help of the ZW-Calculus, another graphical language – based on the interactions of the so-called GHZ and W states [8]. The ZW-Calculus has been proved to be complete [18] but its diagrams only represent matrices over \mathbb{Z} , and hence is not approximately universal. We introduce the $ZW_{1/2}$ -calculus, a simple extension of the ZW-Calculus which remains complete and in which any matrix over the dyadic rational numbers can be represented. We then introduce two interpretations from the ZX-Calculus to the $ZW_{1/2}$ -calculus and back. Thanks to these interpretations, we derive the completeness of the $\frac{\pi}{4}$ -fragment of the ZX-Calculus from the completeness of the $ZW_{1/2}$ -calculus. Notice that the interpretation of ZX-diagrams (which represent complex matrices) into $ZW_{1/2}$ -diagrams (which represent dyadic rationals) requires a non trivial encoding. Notice also that this approach provides a completion procedure. Roughly speaking each axiom of the $ZW_{1/2}$ -calculus generates an equation in the ZX-calculus: if this equation is not already provable using the existing axioms of the ZX-calculus one can treat this equality as an new axiom. A great part of the work has been to reduce all these equalities to only two additional axioms for the language.

Related works. The pre-print version [21] of the present paper has been uploaded on Arxiv in May 2017. In the following weeks, Hadzihanovic [19] independently introduced in his PhD thesis the $ZW_{\mathbb{C}}$ -calculus, an extension of the ZW-Calculus, which is universal and complete for complex matrices. Notice that the $ZW_{\mathbb{C}}$ -calculus does not capture the peculiar properties of Clifford+T quantum

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mechanics, and hence the use of the $ZW_{1/2}$ remains crucial in the proof of the completeness of the ZX-calculus for this fragment. Based upon our work and Hadzihasanovic's, Ng and Wang [26] have then introduced a complete axiomatisation of the ZX-calculus for the full quantum mechanics. Their approach consists in deriving the completeness of the ZX-calculus from the completeness of the $ZW_{\mathbb{C}}$ -calculus, using a completion procedure based on the back and forth interpretations. In [22], we improved this result, showing that a single additional axiom is sufficient to make the ZX-calculus complete in general, whereas 22 new axioms together with two additional generators are used in [26]. Shortly before the submission of the present paper, Ng and Wang uploaded a note [27] on Arxiv providing an alternative complete axiomatisation of the ZX-calculus for Clifford+T quantum mechanics. Their axiomatisation is using two additional generators and significantly more axioms than the axiomatisation given in the present paper.

The paper is structured as follows: A ZX-Calculus augmented with two new axioms is presented in Section 2. Section 3 gives a general overview of the completeness proof. In Section 4, we introduce an extension of the ZW-Calculus that deals with matrices over dyadic rational numbers $\mathbb{D} = \mathbb{Z}[1/2]$ and show its completeness. Sections 5 and 6 are presenting a back and forth translation between the ZX- and ZW-calculi, from which we deduce the completeness of the ZX-Calculus for Clifford+T quantum mechanics in Section 7. In Section 8, we characterise the exact expressive power of the $\frac{\pi}{4}$ -fragment of the ZX-Calculus: the diagrams of this fragment represent exactly the matrices over $\mathbb{D}[e^{i\frac{\pi}{4}}]$. In Section 9, we briefly discuss the interpretation of the two new axioms of the language.

2 ZX-Calculus

2.1 Diagrams and standard interpretation

A ZX-diagram $D : k \rightarrow l$ with k inputs and l outputs is generated by:

$R_Z^{(n,m)}(\alpha) : n \rightarrow m$		$R_X^{(n,m)}(\alpha) : n \rightarrow m$	
$H : 1 \rightarrow 1$		$e : 0 \rightarrow 0$	
$I : 1 \rightarrow 1$		$\sigma : 2 \rightarrow 2$	
$\epsilon : 2 \rightarrow 0$		$\eta : 0 \rightarrow 2$	

where $n, m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, and the generator e is the empty diagram. and the two compositions:

- Spatial Composition: for any $D_1 : a \rightarrow b$ and $D_2 : c \rightarrow d$, $D_1 \otimes D_2 : a + c \rightarrow b + d$ consists in placing D_1 and D_2 side by side, D_2 on the right of D_1 .
- Sequential Composition: for any $D_1 : a \rightarrow b$ and $D_2 : b \rightarrow c$, $D_2 \circ D_1 : a \rightarrow c$ consists in placing D_1 on the top of D_2 , connecting the outputs of D_1 to the inputs of D_2 . We say that time flows from top to bottom.

The standard interpretation of the ZX-diagrams associates to any diagram $D : n \rightarrow m$ a linear map $\llbracket D \rrbracket : \mathbb{C}^{2^n} \rightarrow \mathbb{C}^{2^m}$ inductively

defined as follows:

$$\begin{aligned} \llbracket D_1 \otimes D_2 \rrbracket &:= \llbracket D_1 \rrbracket \otimes \llbracket D_2 \rrbracket & \llbracket D_2 \circ D_1 \rrbracket &:= \llbracket D_2 \rrbracket \circ \llbracket D_1 \rrbracket \\ \llbracket \begin{array}{|c|} \hline \square \\ \hline \end{array} \rrbracket &:= (1) & \llbracket \begin{array}{|c|} \hline \square \\ \hline \end{array} \rrbracket &:= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ \llbracket \begin{array}{|c|} \hline \text{X} \\ \hline \end{array} \rrbracket &:= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \llbracket \begin{array}{|c|} \hline \cup \\ \hline \end{array} \rrbracket &:= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ \llbracket \begin{array}{|c|} \hline \odot \\ \hline \end{array} \rrbracket &:= (1 + e^{i\alpha}) & \llbracket \begin{array}{|c|} \hline \cup \\ \hline \end{array} \rrbracket &:= (1 \ 0 \ 0 \ 1) \end{aligned}$$

$$\llbracket \begin{array}{|c|} \hline \text{X} \\ \hline \end{array} \rrbracket := 2^m \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & e^{i\alpha} \end{pmatrix} \quad (n + m > 0)$$

For any $n, m \geq 0$ and $\alpha \in \mathbb{R}$:

$$\llbracket \begin{array}{|c|} \hline \text{X} \\ \hline \end{array} \rrbracket = \llbracket \begin{array}{|c|} \hline \square \\ \hline \end{array} \rrbracket^{\otimes m} \circ \llbracket \begin{array}{|c|} \hline \text{X} \\ \hline \end{array} \rrbracket \circ \llbracket \begin{array}{|c|} \hline \square \\ \hline \end{array} \rrbracket^{\otimes n}$$

(where $M^{\otimes 0} = (1)$ and $M^{\otimes k+1} = M \otimes M^{\otimes k}$ for any $k \in \mathbb{N}$).

To simplify, the red and green nodes will be represented empty when holding a 0 angle:

$$\begin{array}{|c|} \hline \dots \\ \hline \odot \\ \hline \dots \end{array} := \begin{array}{|c|} \hline \dots \\ \hline \circ \\ \hline \dots \end{array} \quad \text{and} \quad \begin{array}{|c|} \hline \dots \\ \hline \odot \\ \hline \dots \end{array} := \begin{array}{|c|} \hline \dots \\ \hline \circ \\ \hline \dots \end{array}$$

Also in order to make the diagrams a little less heavy, when n copies of the same sub-diagram occur, we will use the notation $(\cdot)^{\otimes n}$.

ZX-Diagrams are universal:

$$\forall A \in \mathbb{C}^{2^n} \times \mathbb{C}^{2^m}, \exists D : n \rightarrow m, \llbracket D \rrbracket = A$$

This implies dealing with an uncountable set of angles, so it is generally preferred to work with *approximate* universality – the ability to approximate any linear map with arbitrary accuracy – in which only a finite set of angles is involved. The $\frac{\pi}{4}$ -fragment – ZX-diagrams where all angles are multiples of $\frac{\pi}{4}$ – is one such approximately universal fragment, whereas the $\frac{\pi}{2}$ -fragment is not [1].

2.2 Calculus

The diagrammatic representation of a matrix is not unique in the ZX-Calculus. As a consequence the language comes with a set of axioms. Additionally to the axioms of the language described in Figure 1, one can:

- bend any wire of a ZX-diagram at will, without changing its semantics. This paradigm – the so-called **Only Topology Matters** – can be derived from the following axioms:

$$\begin{array}{l} \begin{array}{|c|} \hline \cup \\ \hline \end{array} = \begin{array}{|c|} \hline \cup \\ \hline \end{array} = \begin{array}{|c|} \hline \cup \\ \hline \end{array} \quad \begin{array}{|c|} \hline \text{X} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{X} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{X} \\ \hline \end{array} \quad \begin{array}{|c|} \hline \cup \\ \hline \end{array} = \begin{array}{|c|} \hline \cup \\ \hline \end{array} = \begin{array}{|c|} \hline \cup \\ \hline \end{array} \\ \begin{array}{|c|} \hline \cup \\ \hline \end{array} = \begin{array}{|c|} \hline \cup \\ \hline \end{array} = \begin{array}{|c|} \hline \cup \\ \hline \end{array} \quad \begin{array}{|c|} \hline \text{X} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{X} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{X} \\ \hline \end{array} \quad \begin{array}{|c|} \hline \cup \\ \hline \end{array} = \begin{array}{|c|} \hline \cup \\ \hline \end{array} = \begin{array}{|c|} \hline \cup \\ \hline \end{array} \end{array}$$

- apply the axioms to sub-diagrams. If $ZX \vdash D_1 = D_2$ then, for any diagram D with the appropriate number of inputs and outputs:
 - $ZX \vdash D_1 \circ D = D_2 \circ D$
 - $ZX \vdash D \circ D_1 = D \circ D_2$
 - $ZX \vdash D_1 \otimes D = D_2 \otimes D$
 - $ZX \vdash D \otimes D_1 = D \otimes D_2$
 where $ZX \vdash D_1 = D_2$ means that D_1 can be transformed into D_2 using the axioms of the ZX-Calculus.

Equalities between ZX-diagrams have the two following interesting properties:

- “Colour-swapping” (exchanging red and green dots) preserves the equality.
- Multiplying all the angles by -1 preserves the equality.

In the following, $ZX_{\pi/4}$ will denote the entire $\frac{\pi}{4}$ -fragment of the ZX-Calculus with the set of rules in figure 1.

2.3 What’s new?

We introduce in this paper a new axiomatisation of the ZX-Calculus. We briefly review here the differences with the previous version of the ZX-Calculus. Since we are only interested in the $\frac{\pi}{4}$ -fragment of the ZX-Calculus in this paper, all the axioms which are not expressible with angles multiple of $\pi/4$, like the cyclotomic supplementarity [23], are ignored. However, the rule (E), also introduced in [23], is specific to the $\frac{\pi}{4}$ -fragment, and hence appears in the set of rules. The two axioms ((C), (BW)) given in Figure 1 are new axioms, for which we do not know any derivation using the previous axiomatisations of the language.

The two rules are of a different kind. On the one hand, (BW) is specific to the $\frac{\pi}{4}$ -fragment, and is hardly understandable as is. On the other hand, (C) is parametrised by 3 different angles, and the rule holds whatever these angles are.

2.4 Soundness and Completeness

It’s routine to prove the soundness of the axioms of the ZX-Calculus given in Figure 1. The main result of the paper is the completeness of this axiomatisation for Clifford+T quantum mechanics:

Theorem 2.1. *The $\frac{\pi}{4}$ -fragment of the ZX-Calculus as presented in Figure 1 is complete: for any two diagrams D_1, D_2 in the $\frac{\pi}{4}$ -fragment of the ZX-Calculus, $\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket$ iff $ZX_{\pi/4} \vdash D_1 = D_2$.*

The five following sections of the paper are devoted to the proof of the theorem. A general overview of the proof is given in the next section.

3 A Bird’s Eye View of the Proof of Theorem 2.1

The proof uses the completeness result of the ZW-Calculus, a calculus dealing with matrices with integer coefficients. The syntax and semantics of the ZW-Calculus are presented in section 4.

We start by slightly changing the ZW-Calculus to obtain a new language, the $ZW_{1/2}$ -calculus, that is able to express matrices with dyadic rational coefficients, i.e. rational numbers of the form $p/2^q$. This is done merely by adding a symbolic inverse to the scalar 2. We then prove that this new language is complete:

Part 1 (Proposition 4.3). *The $ZW_{1/2}$ -calculus is complete: for two diagrams D_1, D_2 of the $ZW_{1/2}$ -calculus, we have $\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket$ iff $ZW_{1/2} \vdash D_1 = D_2$.*

This is done in subsection 4.3.

We now introduce two interpretations, from $ZX_{\pi/4}$ to $ZW_{1/2}$ and back.

First, we provide an interpretation $\llbracket \cdot \rrbracket_{XW}$ from $ZX_{\pi/4}$ to $ZW_{1/2}$ that transforms $ZX_{\pi/4}$ -diagrams of type $k \rightarrow l$ to $ZW_{1/2}$ -diagrams of type $k + 2 \rightarrow l + 2$. This interpretation is sound in the following sense:

Part 2 (Proposition 5.5). *Let D_1, D_2 be two diagrams of the $ZX_{\pi/4}$ -calculus.*

Then $\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket$ iff $\llbracket \llbracket D_1 \rrbracket_{XW} \rrbracket = \llbracket \llbracket D_2 \rrbracket_{XW} \rrbracket$.

The encoding is nontrivial as the $ZX_{\pi/4}$ -Calculus expresses matrices with complex coefficients, and the $ZW_{1/2}$ -calculus is only able to express matrices with dyadic rational coefficients. It turns out that coefficients involved in matrices of the $ZX_{\pi/4}$ -Calculus are actually in a vector space (more accurately a module) of dimension 4 over the set of dyadic rational numbers, so that every complex coefficient will be represented by a 4×4 matrix with dyadic rational coefficients. This encoding is done in section 5.

We then provide an interpretation $\llbracket \cdot \rrbracket_{WX}$ from $ZW_{1/2}$ to $ZX_{\pi/4}$. This interpretation preserves both semantics and provability:

Part 3 (Proposition 6.1 and 6.3). *Let D_1, D_2 be two diagrams of the $ZW_{1/2}$ -calculus.*

Then $\llbracket \llbracket D_i \rrbracket_{WX} \rrbracket = \llbracket D_i \rrbracket$.

Furthermore, if $ZW_{1/2} \vdash D_1 = D_2$ then $ZX_{\pi/4} \vdash \llbracket D_1 \rrbracket_{WX} = \llbracket D_2 \rrbracket_{WX}$

This is done in section 6.

The composition of the two interpretations does not give back the initial diagram (we obtain after all a diagram with two more inputs and outputs), but we can (provably) recover it. In fact

Part 4 (Corollary 7.2). *Let D_1, D_2 be a diagram of the ZX-Calculus. If $ZX_{\pi/4} \vdash \llbracket \llbracket D_1 \rrbracket_{XW} \rrbracket_{WX} = \llbracket \llbracket D_2 \rrbracket_{XW} \rrbracket_{WX}$ then $ZX_{\pi/4} \vdash D_1 = D_2$.*

Our main theorem is now obvious:

Proof of Theorem 2.1. Let D_1, D_2 be two diagrams of the $ZX_{\pi/4}$ -Calculus s.t. $\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket$. By Part 2, $\llbracket \llbracket D_1 \rrbracket_{XW} \rrbracket = \llbracket \llbracket D_2 \rrbracket_{XW} \rrbracket$.

By Part 1, the $ZW_{1/2}$ -calculus is complete and therefore $ZW_{1/2} \vdash \llbracket D_1 \rrbracket_{XW} = \llbracket D_2 \rrbracket_{XW}$.

By Part 3, $ZX_{\pi/4} \vdash \llbracket \llbracket D_1 \rrbracket_{XW} \rrbracket_{WX} = \llbracket \llbracket D_2 \rrbracket_{XW} \rrbracket_{WX}$.

By Part 4 this implies $ZX_{\pi/4} \vdash D_1 = D_2$. \square

This approach gives a completion procedure. It gives a set of equalities between $ZX_{\pi/4}$ -diagrams whose derivability proves the completeness of the language. Hence, the new rules of the $ZX_{\pi/4}$ -Calculus we introduced have obviously been chosen for Parts 4 and 3 to hold. However they have been greatly simplified from what one can obtain using the approach naively.

4 ZW-Calculus

4.1 Diagrams and Standard Interpretation

The ZW-Calculus has been introduced by Amar Hadzihasanovic in 2015 [18] and is based on the GHZ/W-Calculus [8]. We will present here the expanded version of this calculus. To stay coherent with the previous definition of the ZX-Calculus, we will assume that the time flows from top to bottom – which is the opposite of the original

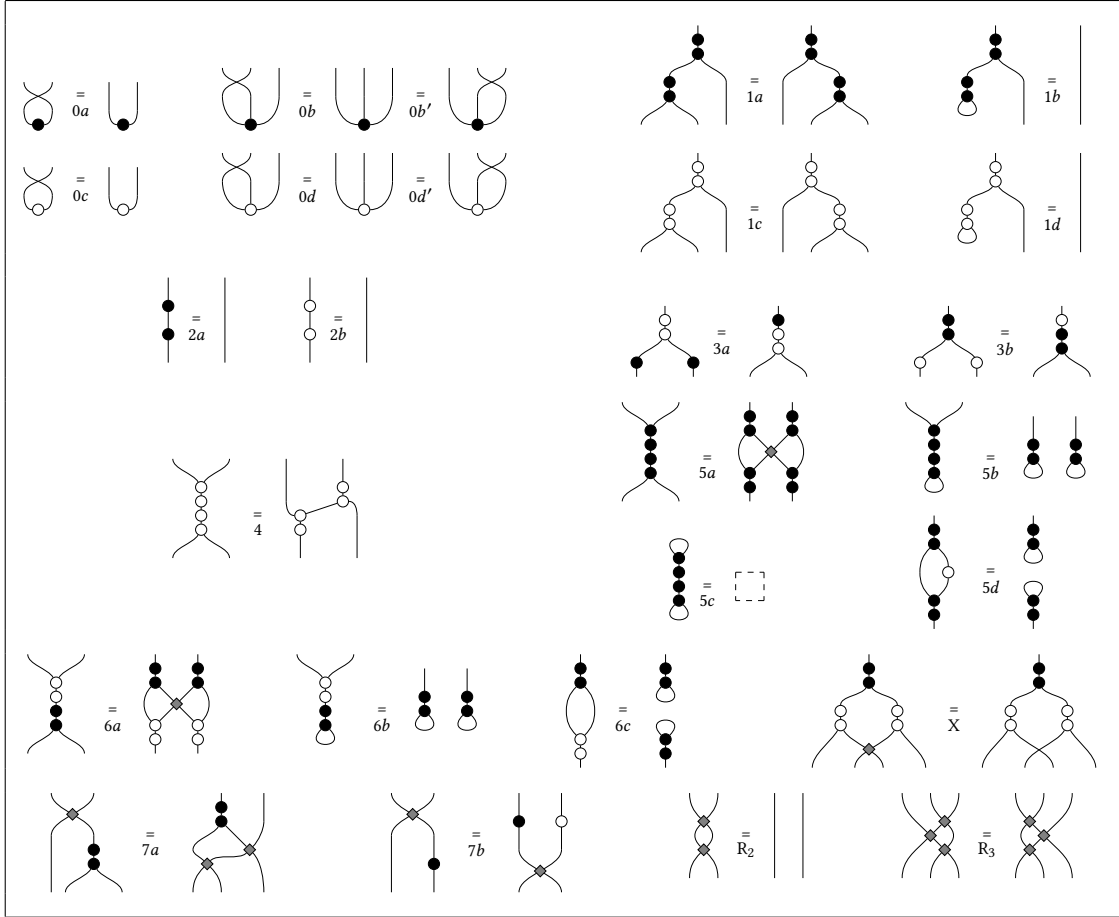


Figure 2. Set of rules for the ZW-Calculus.

In our case, $e^{i\frac{\pi}{4}}$ is a eighth primitive root of the unity, so that $\mathbb{Q}[e^{i\frac{\pi}{4}}]$ is a vector space of dimension 4, one basis being given by $1, e^{i\frac{\pi}{4}}, (e^{i\frac{\pi}{4}})^2, (e^{i\frac{\pi}{4}})^3$. In particular:

Proposition 5.1. *Every element of $\mathbb{Q}[e^{i\frac{\pi}{4}}]$ can be written in a unique way $a + be^{i\frac{\pi}{4}} + c(e^{i\frac{\pi}{4}})^2 + d(e^{i\frac{\pi}{4}})^3$ for some rational numbers a, b, c, d .*

For $x \in \mathbb{Q}[e^{i\frac{\pi}{4}}]$, let $\psi(x)$ be the function defined by $\psi(x) = y \mapsto xy$. For each x , $\psi(x)$ is a linear map and therefore can be given by a 4×4 matrix in the basis $(e^{i\frac{\pi}{4}})^3, (e^{i\frac{\pi}{4}})^2, e^{i\frac{\pi}{4}}, 1$. $\psi(1)$ is of course the identity matrix and

$$\psi(e^{i\frac{\pi}{4}}) = M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Notice that M^4 is the companion matrix of the polynomial $X^4 + 1$ which characterises $e^{i\frac{\pi}{4}}$ as an algebraic number.

Proposition 5.2. *The map:*

$$\psi : a + be^{i\frac{\pi}{4}} + c(e^{i\frac{\pi}{4}})^2 + d(e^{i\frac{\pi}{4}})^3 \mapsto aI_4 + bM + cM^2 + dM^3$$

is a homomorphism of \mathbb{Q} -algebras from $\mathbb{Q}[e^{i\frac{\pi}{4}}]$ to $M_4(\mathbb{Q})$

This homomorphism has a left-inverse: there exists f such that $f \circ \psi = Id$. Indeed, let

$$e_1 = (1 \ 0 \ 0 \ 0) \quad \text{and} \quad \theta = \begin{pmatrix} 1 \\ e^{i\frac{\pi}{4}} \\ (e^{i\frac{\pi}{4}})^2 \\ (e^{i\frac{\pi}{4}})^3 \end{pmatrix}$$

and if $f : M \mapsto e_1 M \theta$, then $f \circ \psi(x) = e_1 \psi(x) \theta = e_1 x \theta = x$.

With this morphism, we can see elements of $\mathbb{Q}[e^{i\frac{\pi}{4}}]$ as matrices over \mathbb{Q} .

Of course we can do the same with matrices over $\mathbb{Q}[e^{i\frac{\pi}{4}}]$.

Definition 5.3. Define:

$$\psi : A + Be^{i\frac{\pi}{4}} + C(e^{i\frac{\pi}{4}})^2 + D(e^{i\frac{\pi}{4}})^3 \mapsto A \otimes I_4 + B \otimes M + C \otimes M^2 + D \otimes M^3$$

ψ is injective and maps a matrix over $\mathbb{Q}[e^{i\frac{\pi}{4}}]$ of dimension $n \times m$ to a matrix over \mathbb{Q} of dimension $4n \times 4m$.

We use the same notation ψ as before, as the scalars (i.e. 1×1 matrices) are a particular case.

It is easy to see that Proposition 5.2 holds for the extended ψ in the sense that $\psi(qA) = q\psi(A)$ for q rational, $\psi(A+B) = \psi(A) + \psi(B)$, $\psi(AB) = \psi(A)\psi(B)$ whenever this makes sense.

Notice however that $\psi(A \otimes B)$ is not $\psi(A) \otimes \psi(B)$.

As before, ψ has a left inverse, as shown by:

Proposition 5.4. For all matrices X of dimension $n \times m$, $\psi(X)(I_m \otimes \theta) = X \otimes \theta$

While it is true that all coefficients of the standard interpretation of the $\pi/4$ fragment are in $\mathbb{Q}[e^{i\frac{\pi}{4}}]$, we can be more precise.

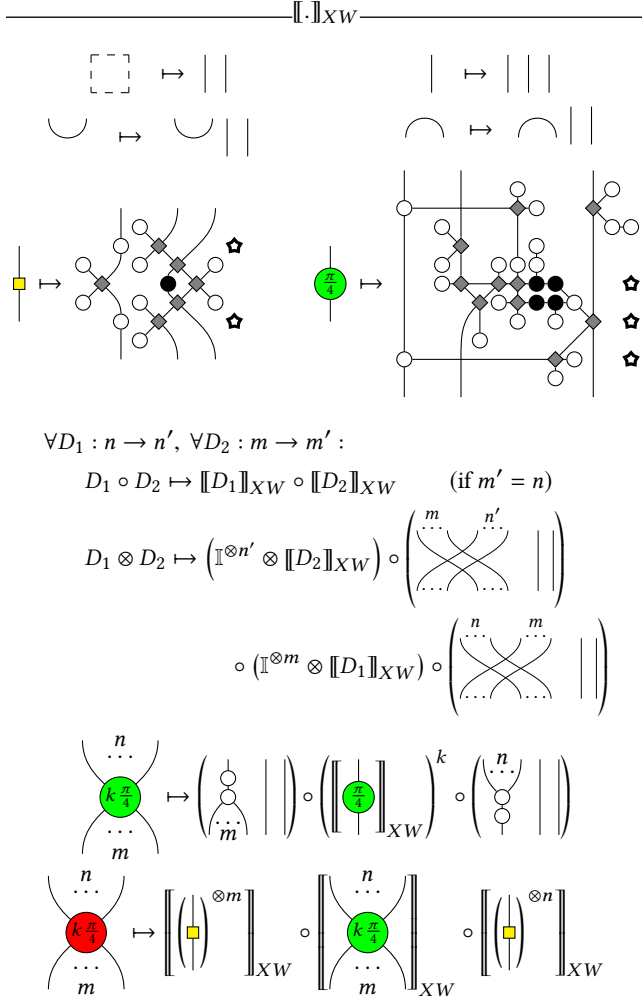
Let $\mathbb{D} = \mathbb{Z}[1/2]$ be the set of all dyadic rational numbers, i.e. rational numbers of the form $p/2^n$.

It is easy to see that any element of $\mathbb{D}[e^{i\frac{\pi}{4}}]$ can be written in a unique way $a + be^{i\frac{\pi}{4}} + c(e^{i\frac{\pi}{4}})^2 + d(e^{i\frac{\pi}{4}})^3$ for some dyadic rational numbers a, b, c, d . (It is NOT a consequence of the similar statement for \mathbb{Q} . We have to use here the additional property that $e^{i\frac{\pi}{4}}$ is not only an algebraic number, but also an algebraic integer).

Then it is clear that actually all coefficients of the $\pi/4$ fragment of the ZX-Calculus are in $\mathbb{D}[e^{i\frac{\pi}{4}}]$. As $\mathbb{D} \subset \mathbb{Q}$ all we said before still holds, and we actually obtain with ψ a map from matrices over $\mathbb{D}[e^{i\frac{\pi}{4}}]$ to matrices over \mathbb{D} .

5.2 Interpretation

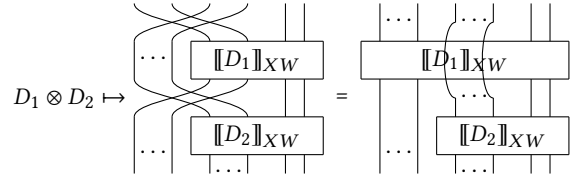
Based on the previous discussion, we define an interpretation $\llbracket \cdot \rrbracket_{XW}$ from $ZX_{\pi/4}$ -diagrams to $ZW_{1/2}$ -diagrams as follows:



This interpretation maps a $ZX_{\pi/4}$ -diagram to a $ZW_{1/2}$ -diagram with exactly 2 additional wires (on the right). They allow the encoding of any element of $\mathbb{D}[e^{i\frac{\pi}{4}}]$ as a 4×4 matrix (the extra space being

precisely provided by them). They can be seen as control wires (and will be called as such), for feeding them with the appropriate input will produce for instance the matrix with terms in $(e^{i\frac{\pi}{4}})^2$.

The interpretation of the spacial composition \otimes might seem a tad cryptic. It is in fact a way of putting “side-by-side” the interpretations of D_1 and D_2 , while at the same time making them share the two control wires. We can see it as:



In order for this to make sense, D_1 and D_2 should be able to commute on the control wires. This property is provided by the completeness of the ZW-Calculus, since it is semantically true.

One can check that $\left\| \left(\begin{array}{c} \dots \\ \dots \\ \dots \end{array} \right) \right\|_{XW} = \psi \left(\left(\begin{array}{c} \dots \\ \dots \\ \dots \end{array} \right) \right) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes (M - M^3)$

and $\left\| \left(\begin{array}{c} \dots \\ \dots \\ \dots \end{array} \right) \right\|_{XW} = \psi \left(\left(\begin{array}{c} \dots \\ \dots \\ \dots \end{array} \right) \right) = \begin{pmatrix} I_4 & 0 \\ 0 & M \end{pmatrix}$. More generally:

Proposition 5.5. Let D be a diagram of the $ZX_{\pi/4}$ -Calculus. Then

$$\llbracket \llbracket D \rrbracket_{XW} \rrbracket = \psi(\llbracket D \rrbracket)$$

In particular, if $\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket$ then $\llbracket \llbracket D_1 \rrbracket_{XW} \rrbracket = \llbracket \llbracket D_2 \rrbracket_{XW} \rrbracket$

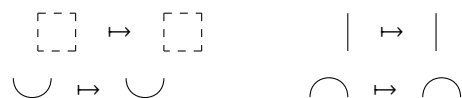
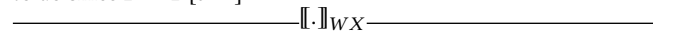
The proof is a straightforward induction using the fact that ψ is an homomorphism. Slight care has to be taken to treat the case of $D_1 \otimes D_2$.

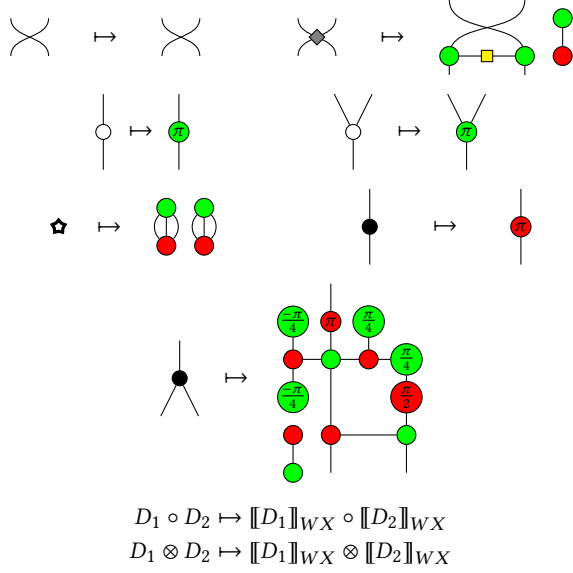
Suppose $\llbracket D_1 \rrbracket = \sum_{k=0}^3 A_k e^{i\frac{k\pi}{4}}$ and $\llbracket D_2 \rrbracket = \sum_{k=0}^3 B_k e^{i\frac{k\pi}{4}}$ are their unique decomposition, and that $\llbracket \llbracket D_i \rrbracket_{XW} \rrbracket = \psi(\llbracket D_i \rrbracket)$. Then:

$$\begin{aligned} \llbracket \llbracket D_1 \otimes D_2 \rrbracket_{XW} \rrbracket &= (I_{2n'} \otimes \psi(\llbracket D_2 \rrbracket)) \circ \left(\begin{array}{c} \dots \\ \dots \\ \dots \end{array} \right) \\ &\quad \circ \left(I_{2m} \otimes \sum_{k=0}^3 A_k \otimes M^k \right) \circ \left(\begin{array}{c} \dots \\ \dots \\ \dots \end{array} \right) \\ &= \left(\sum_{l=0}^3 I_{2n'} \otimes B_l \otimes M^l \right) \circ \left(\sum_{k=0}^3 A_k \otimes I_{2m} \otimes M^k \right) \\ &= \sum_{k,l} A_k \otimes B_l \otimes M^{k+l} \\ &= \psi \left(\sum_{k,l} (A_k \otimes B_l) e^{i\frac{(k+l)\pi}{4}} \right) = \psi(\llbracket D_1 \otimes D_2 \rrbracket) \end{aligned}$$

6 From $ZW_{1/2}$ to $ZX_{\pi/4}$ -Diagrams

We define here an interpretation $\llbracket \cdot \rrbracket_{WX}$ that transforms any diagram of the $ZW_{1/2}$ -Calculus into a $ZX_{\pi/4}$ -diagram, which is easy to do since $\mathbb{D} \subset \mathbb{D}[e^{i\frac{\pi}{4}}]$:



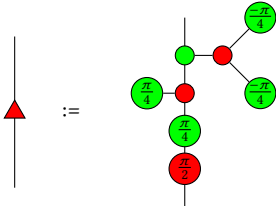


Proposition 6.1. *Let D be a diagram of the $ZW_{1/2}$ calculus. Then $[[[D]]_{WX}] = [D]$*

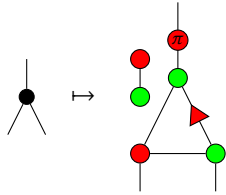
The proof is by induction on D .

This interpretation $[[\cdot]]_{WX}$ from the ZW -Calculus to the ZX -Calculus is pretty straightforward, except for the three-legged black node. This is where some syntactic sugar can come in handy.

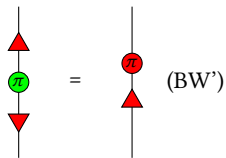
Definition 6.2. We define the “triangle node” as:



One can check that $[[[\text{triangle node}]]] = \begin{pmatrix} 1 & \\ 0 & 1 \end{pmatrix}$. Then the interpretation of the three-legged black dot is simplified:



as is the rule (BW):



This shortcut will be very useful in the technical proof of the completeness of the language for Clifford+T with the set of rules in Figure 1.

Proposition 6.3. *The interpretation $[[\cdot]]_{WX}$ preserves all the rules of the $ZW_{1/2}$ -Calculus:*

$$ZW_{1/2} \vdash D_1 = D_2 \implies ZX_{\pi/4} \vdash [[D_1]]_{WX} = [[D_2]]_{WX}$$

7 Completeness of the $\frac{\pi}{4}$ -fragment of the ZX-Calculus

To finish the proof it remains to compose the two interpretations:

Proposition 7.1. *We can retrieve any $ZX_{\pi/4}$ -diagram from its image under the composition of the two interpretations:*

$$\forall D \in ZX_{\pi/4}, \quad ZX_{\pi/4} \vdash D = \left(\dots \mid \begin{array}{c} \text{red dot} \\ \text{green dot} \end{array} \right) \circ [[D]]_{XW} \circ \left(\dots \mid \begin{array}{c} \text{green dot} \\ \text{red dot} \end{array} \right)$$

Corollary 7.2. *If $ZX_{\pi/4} \vdash [[D_1]]_{XW} = [[D_2]]_{XW}$ then $ZX_{\pi/4} \vdash D_1 = D_2$.*

8 Expressive power of the $ZX_{\pi/4}$ -diagrams

The ZW -Calculus is complete, and additionally any integer matrix can be represented in the ZW -Calculus [18]. A similar result follows immediately for the $ZW_{1/2}$ -calculus.

Proposition 8.1. *$ZW_{1/2}$ -Diagrams are universal for matrices of $\mathbb{D}^{2^n} \times \mathbb{D}^{2^m}$:*

$$\forall A \in \mathbb{D}^{2^n} \times \mathbb{D}^{2^m}, \exists D \in ZW_{1/2}, [D] = A$$

Regarding the expressive power of $ZX_{\pi/4}$ -diagrams, since the unitary matrices over $\mathbb{D}[e^{i\frac{\pi}{4}}]$ are representable with Clifford+T circuits [17], so are they with $ZX_{\pi/4}$ -diagrams. We actually show that any matrix over $\mathbb{D}[e^{i\frac{\pi}{4}}]$ can be represented by a $ZX_{\pi/4}$ -diagram:

Proposition 8.2. *The $\frac{\pi}{4}$ -fragment of the ZX-Calculus represents exactly matrices over $\mathbb{D}[e^{i\frac{\pi}{4}}]$:*

$$\forall A \in \mathbb{D}[e^{i\frac{\pi}{4}}]^{2^n \times 2^m}, \exists D \in ZX_{\pi/4}, [D] = A$$

Proof. Let $A \in \mathbb{D}[e^{i\frac{\pi}{4}}]^{2^n \times 2^m}$. We define $A' = \psi(A) \in \mathbb{D}^{2^{n+2} \times 2^{m+2}}$. Since $ZW_{1/2}$ -diagrams are universal for matrices over dyadic rationals: $\exists D \in ZW_{1/2}, [D] = A'$. Since $[[\cdot]]_{WX}$ preserves the semantics, we can define a ZX -diagram of the $\frac{\pi}{4}$ -fragment $D' = [[D]]_{WX}$ such that $[D'] = A'$.

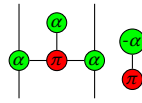
$$\text{Now, notice that } \theta = \begin{pmatrix} 1 \\ e^{i\frac{\pi}{4}} \\ (e^{i\frac{\pi}{4}})^2 \\ (e^{i\frac{\pi}{4}})^3 \end{pmatrix} = \left[\begin{array}{c} \text{green dot} \\ \text{red dot} \\ \text{green dot} \\ \text{red dot} \end{array} \right],$$

and $e_1 = (1 \ 0 \ 0 \ 0) = \left[\begin{array}{c} \text{red dot} \\ \text{red dot} \\ \text{green dot} \\ \text{red dot} \end{array} \right]$, so if we apply the second diagram at the two bottom right wires, and the first state on the two top right wires of D' , we end up with D'' such that $[D''] = A$. Indeed:

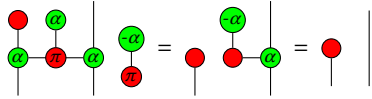
$$[[D'']] = (I \otimes e_1) \circ [[D']] \circ (I \otimes \theta) = (I \otimes e_1) \circ \psi(A) \circ (I \otimes \theta) = (I \otimes e_1) \circ (A \otimes \theta) = A \otimes (e_1 \circ \theta) = A \quad \square$$

9 Discussion on the New Rules

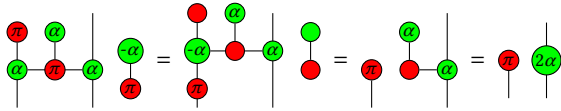
We can try and give an explanation of the rule (C), in terms of commutation of controlled operations. Consider the following diagram:



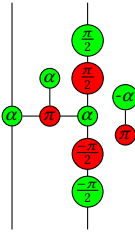
It is a *controlled operation*. Indeed, if \bullet is plugged on the left wire, we obtain the identity on the right one:



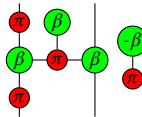
and if \bullet is plugged:



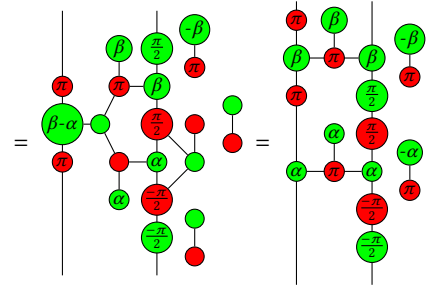
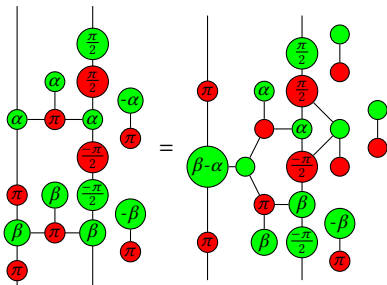
Hence, the diagram is actually the controlled Z -rotation $\Lambda R_Z(2\alpha)$. Similarly, one can show that the following diagram represents the controlled X -rotation $\Lambda R_X(2\alpha)$:



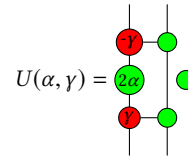
A controlled operation ΛU can be turned into an “*anti-controlled*” operation $\bar{\Lambda}U$ (where the role of \bullet and $\bar{\bullet}$ are exchanged) by adding two red π -nodes on the first wire, one on the input and one on the output. For instance, $\bar{\Lambda}R_Z(2\beta)$ is expressed as:



Controlled and anti-controlled operations obviously commute: for any operations U and V of the same size, $\Lambda U \circ \bar{\Lambda}V = \bar{\Lambda}V \circ \Lambda U$. To derive this result in the ZX-calculus for $\Lambda R_X(2\alpha)$ and $\bar{\Lambda}R_Z(2\beta)$ the use of rule (C) seems to be required:

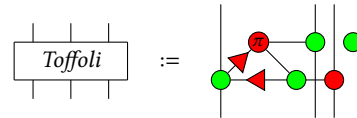


The rule (C) itself actually goes a step further and directly expresses the equality $\Lambda U(\alpha, \gamma) \circ \bar{\Lambda}(R_Z(2\beta) \otimes \mathbb{I}) = \bar{\Lambda}(R_Z(2\beta) \otimes \mathbb{I}) \circ \Lambda U(\alpha, \gamma)$ for

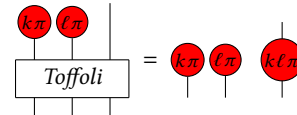


$\Lambda U(\alpha, \gamma)$ is a control of a 2-qubits operation, such as the Toffoli gate [31], which controls CNOT. Because the operation it controls (CNOT) is also a controlled operation, it can also be seen as an operation on one wire, controlled by two others.

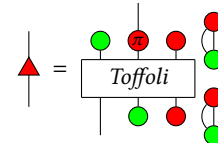
Since the Toffoli gate is represented by a matrix over \mathbb{Z} , it is possible to express it in the $\frac{\pi}{4}$ -fragment of the ZX-Calculus. Indeed, using the \blacktriangle -notation:



One can check that this diagram actually represents the Toffoli gate, for instance by deriving, for any $k, \ell \in \{0, 1\}$:



More surprisingly, by plugging particular states, operation and projectors on the Toffoli gate, one can recover the triangle node:



This shows a connection between the diagram denoted \blacktriangle and the Toffoli gate. As pointed out in section 6, the rule (BW) can be greatly simplified when using the \blacktriangle -notation. Hence, (BW) might help to find rules for a potential axiomatisation of the quantum circuits, or in fact any graphical language that contains the Toffoli gate.

We leave open the question of minimality of the axiomatisation in Figure 1. Many of the previous axioms were proven to be necessary (not derivable from the others) [5, 28], but some proofs may

not hold with the addition of the two new axioms, and there is currently no known proof of the necessity of (C) and (BW).

Acknowledgments

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