Two complete axiomatisations of pure-state qubit quantum computing

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Abstract

Categorical quantum mechanics places finite-dimensional quantum theory in the context of compact closed categories, with an emphasis on diagrammatic reasoning. In this framework, two equational diagrammatic calculi have been proposed for pure-state qubit quantum computing: the ZW calculus, developed by Coecke, Kissinger and the first author for the purpose of qubit entanglement classification, and the ZX calculus, introduced by Coecke and Duncan to give an abstract description of complementary observables. Neither calculus, however, provided a complete axiomatisation of their model.

In this paper, we present extended versions of ZW and ZX, and show their completeness for pure-state qubit theory, thus solving two major open problems in categorical quantum mechanics. First, we extend the original ZW calculus to represent states and linear maps with coefficients in an arbitrary commutative ring, and prove completeness by a strategy that rewrites all diagrams into a normal form. We then extend the language and axioms of the original ZX calculus, and show their completeness for pure-state qubit theory through a translation between ZX and ZW specialised to the field of complex numbers. This translation expands the one used by Jeandel, Perdrix, and Vilmart to derive an axiomatisation of the approximately universal Clifford+T fragment; restricting the field of complex numbers to a suitable subring, we obtain an alternative axiomatisation of the same theory.

CCS Concepts • **Theory of computation → Quantum computation theory**;

Keywords Quantum Computation, String Diagrams, Monoidal Categories, Categorical Quantum Mechanics

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1 Introduction

Categorical quantum mechanics, a research area initiated by [1], studies finite-dimensional quantum theory, in particular the structures relevant to quantum computing, as an abstract process theory whose model is a dagger compact closed category [10, 23]. Since its inception, it has used the corresponding string-diagrammatic language [24] both as a calculational tool, and as a heuristic for determining algebraic structures that fit naturally in the framework [12].

In [6], Coecke and Duncan proposed an axiomatisation of complementary quantum observables in terms of a pair of special Frobenius algebras whose monoid forms a bialgebra with the comonoid of the other [4]. These structures, with the addition of phases [8], seemed to capture enough interesting aspects of pure-state quantum theory, such as *non-locality* [7], that the question arose whether they could be the basis of a complete equational axiomatisation of the relevant monoidal categories. The resulting partial diagrammatic axiomatisations have been called *ZX calculi*.

Compared to matrix calculus, which has been compared to "programming with bit strings" [11], string diagrams are a higher-level language, allowing one to focus on the connections between gates and on the flow of information. Complete axiomatisations of fragments of quantum theory would provide quantum programmers with the possibility of understanding the behaviour of a circuit entirely within this language, without resorting to linear algebra.

Most attention has been devoted to qubit computing, and a ZX calculus complete for the stabiliser fragment [14], and one complete for single-qubit processes in the approximately universal Clifford+T fragment have been presented by Backens [2, 3]. Completeness for the whole of pure-state qubit theory has remained an open problem.

Observing that the components of the ZX calculus seemed ill-suited to analysing finer properties of entangled qubits, such as their separation in SLOCC classes [13], Coecke and Kissinger proposed a variant where one Frobenius algebra is replaced with one satisfying an "anti-specialness" equation [9]. In [15], the first author extended this theory into a calculus modelled on ZX calculi, the *ZW calculus*, and proved its completeness for maps of qubits that have only integer coefficients. In [18], Jeandel, Perdrix, and Vilmart used a non-trivial translation of the ZW calculus into the ZX calculus to obtain a complete axiomatisation of the entire Clifford+T fragment.

In this paper, we bring both calculi to their intended limit, by presenting inter-translatable versions of the ZW calculus and of the ZX calculus that are complete for pure-state qubit theory. In practice, this means that the equality of the interpretation of any two circuits as linear maps of qubits can be decided by rewriting string diagrams, possibly with the help of graph rewriting software [20].

While it may seem redundant to have two separate axiomatisations for the same theory, we believe that the two have different advantages, which make them a natural choice in different settings: while the ZX calculus has a strong connection with the Clifford+T fragment, and the related circuit model, the ZW calculus has a deep connection with fermionic quantum computing [5], which has recently be developed into an independent calculus in [17]. Thus, the availability of the translation between ZW and ZX gives a platform for comparing different models.

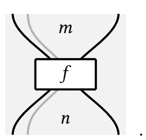
After discussing the common technical framework in Section 2, in Section 3 we introduce versions ZW_R of the ZW calculus parametrised by a commutative ring R, and prove their completeness for analogues of qubits with coefficents in R. Similarly to the original ZW calculus, recovered in the case $R = \mathbb{Z}$, this is achieved by the introduction of a normal form for diagrams, and a normalisation strategy which rewrites any diagram in its normal form. The quantum case is then obtained by the choice $R = \mathbb{C}$.

Then, in Section 4, we describe two mutually inverse translations of the generators of ZW into ZX, and vice versa, and show that the axioms of one calculus are recoverable from the axioms of the other, which proves that they define isomorphic categories. The translation from ZW to ZX, restricted to the parameter-free fragment, is essentially the same as the one used by Jeandel, Perdrix, and Vilmart in [18]; the inverse translation is original, and demonstrates a direct equivalence between the two calculi. Finally, by restricting the field $\mathbb{C}$ to a sub-ring corresponding to the Clifford+T fragment, we obtain an alternative axiomatisation of this fragment.

Future developments. Having settled the completeness problem for pure-state qubit computing, future directions include: improved normalisation strategies, potentially tailored for specific classes of circuits; their complexity analysis, and implementation in Quantomatic [20]; versions of the calculi for qudits with d other than 2 (a generalisation of the ZW generators to qudits is discussed in [16, Section 5.3], while rules for a qutrit ZX calculus have been proposed in [25]); and axiomatisations of mixed-state qubit computing, possibly in the style of the mixed quantum-classical calculus of [10, Chapter 8].

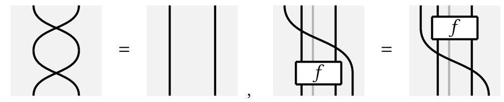
2 Preliminary

The ambient category in which the graphical calculi live is the free *self-dual, compact closed PROP* [21]. A *products and permutations category (PROP)* is a symmetric strict monoidal category whose objects are finitely iterated monoidal products of a single object X. A morphism in a PROP $f : X^{\otimes m} \to X^{\otimes n}$ is depicted diagrammatically as a box labelled f with m inputs and n outputs:



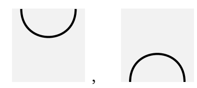
We will draw a diagram with lines of various shades; a darker shade means that that portion of the diagram is the focus of the discussion while a lighter shade means otherwise. A lighter shade lines can also indicate a repeated pattern. Occasionally, we may zoom in on the diagram while not showing the rest of it. It should be clear from the context what the lighter shade of lines mean or that we have zoomed in on the diagram while the rest remains the same.

In this diagrammatic language, composing morphisms is "plugging" the wires of the diagrams together and tensoring of morphisms is placing the diagrams side by side. The monoidal unit $1 \xrightarrow{e} 1$ is an empty diagram, the identity morphism $X \xrightarrow{\mathbb{I}} X$ is a straight wire, and the *swap*[1] morphism $X \otimes X \xrightarrow{\sigma} X \otimes X$ is two intersecting wires satisfying

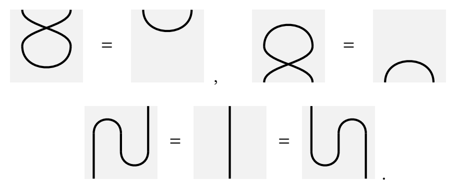


for all morphisms f.

A self-dual, compact closed PROP is a PROP with two morphisms, $X \otimes X \xrightarrow{\varepsilon} 1$ and $1 \xrightarrow{\eta} X \otimes X$, depicted as



respectively, satisfying the rules



The ZW and ZX calculi are self-dual compact closed PROPs, generated by a set of morphisms T, quotiented by an equivalence relation E on morphisms respecting the number of inputs and outputs. *Completeness* of the calculi for a PROP **C** corresponds to showing a monoidal equivalence between **C** and the PROP presented by generators and relations.

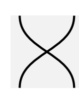
Definition 1. The PROP R**bit** is a PROP whose generating object is $X := R \oplus R$ for a commutative ring R, the monoidal product is the tensor product of R-modules, and the morphisms are R-module homomorphisms. The PROP **Qubit** is R**bit** for $R := \mathbb{C}$, the ring of complex numbers.

We will show that the ZW_R calculus is complete for R**bit**, and the ZX calculus is complete for **Qubit** and the Clifford+T fragment of **Qubit**.

We will use *bra-ket* notation, and denote the two generators of $R \oplus R$ as $|\,0\,\rangle$ and $|\,1\,\rangle$. An element in $R \oplus R$ (a *state*) is written as $r|\,0\,\rangle + s|\,1\,\rangle$ for some $r, s \in R$; for tensor products, we write $|\,b_1 \ldots b_n\,\rangle := |\,b_1\,\rangle \otimes \cdots \otimes |\,b_n\,\rangle$ for $b_i \in \{0, 1\}$. As for morphisms, for instance, $|\,1\,\rangle \mapsto |\,01\,\rangle + |\,10\,\rangle$ is written as $|\,01\,\rangle\langle\,1\,| + |\,10\,\rangle\langle\,1\,|$.

We interpret the basic generators of a self-dual, compact closed PROP in R**bit** as

 $\mapsto |\,0\,\rangle\langle\,0\,| + |\,1\,\rangle\langle\,1\,|$,

 $\mapsto |\,00\,\rangle\langle\,00\,| + |\,01\,\rangle\langle\,10\,| + |\,10\,\rangle\langle\,01\,| + |\,11\,\rangle\langle\,11\,|$,

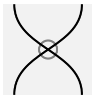
 $\mapsto \langle\,00\,| + \langle\,11\,|$,

 $\mapsto |\,00\,\rangle + |\,11\,\rangle$.

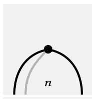
[1]The swap morphism in [15] is depicted as braided wires.

3 The ZW Calculus

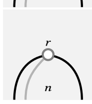
The ZW calculus has the following set T_{ZW} of generators and interpretations in $R\mathbf{bit}$:



$$\mapsto \quad |00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| - |11\rangle\langle 11|,$$



$$\mapsto \quad \sum_{k=1}^{n} |\underbrace{0\dots0}_{k-1}1\underbrace{0\dots0}_{n-k}\rangle,$$



$$\mapsto \quad |\underbrace{0\dots0}_{n}\rangle + r|\underbrace{1\dots1}_{n}\rangle,$$

for some $r \in R$. In quantum theory, the n-ary black vertex corresponds to the W-state, and the n-ary white vertex[2] corresponds to the GHZ-state (modulo normalisation).

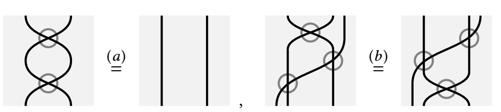
The first generator is called the *crossing*. It can be thought of as a fermionic swap [17], introducing a phase to the wavefunction of a pair of fermions under exchange of particles, where the states $|0\rangle$ and $|1\rangle$ are seen as the vacuum and occupied states of a fermionic oscillator. There is a fragment of the ZW calculus which captures precisely the physical maps of fermionic quantum computing; see [17] for the details.

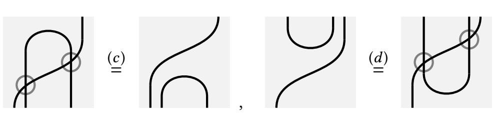
Remark 2. The crossing is almost like a braiding: it fails to be natural for some maps of $R\mathbf{bit}$, for example, the map $|0\rangle\langle 1| + |1\rangle\langle 0|$. However, the crossing can be considered as a proper symmetric braiding if we restrict to *even* morphisms, which are expressible as linear combinations of terms with an even number of 1s. Similarly, we speak of *odd* morphisms, and *mixed parity* morphisms which are neither even nor odd. Calculi with two different braidings have been studied in virtual knot theory [19].

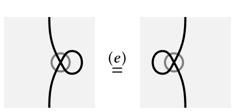
We will now present a set E_{ZW} of axioms relating the morphisms in the ZW calculus. We claim that the axioms are sound, meaning that, through the interpretation, they express true equalities in $R\mathbf{bit}$, so the interpretation defines a monoidal functor $ZW_R \to R\mathbf{bit}$, where ZW_R is the monoidal category presented by the generators T_{ZW} and the relations E_{ZW}. This can be easily verified by calculation.

ZW Axioms

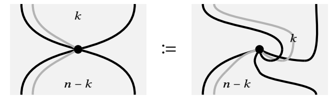
1. The following are the axioms for the crossings:



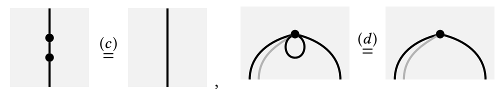


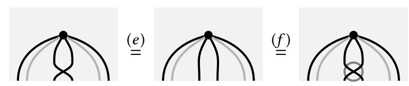


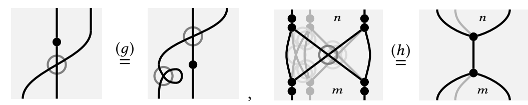
2. Defining



the following are the axioms for the black vertices:

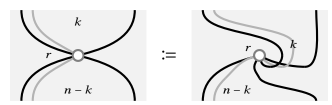




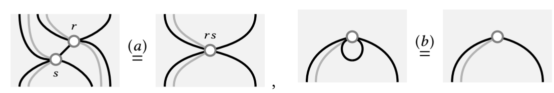


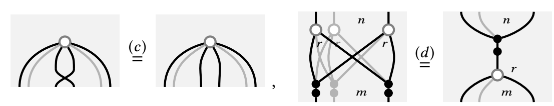
for all $m, n \in \mathbb{N}$.

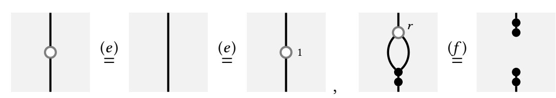
3. Given a commutative ring R and defining

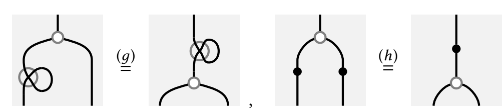


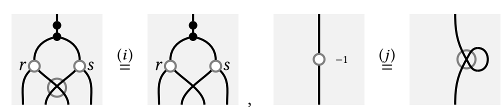
the following are the axioms for the white vertices:











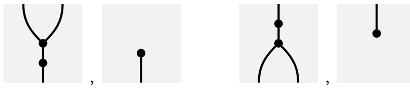
for $r, s \in R$, $m, n \in \mathbb{N}$ such that either $m = n = 0$ or $m > 0$.

Discussion of the ZW axioms

The axioms in group 1 say that the crossing behaves like a symmetric braiding, except that it fails to be natural for morphisms of mixed parity.

The axioms in group 2 say that the black vertices are symmetric with respect to the swap and crossing $(2(e, f))$. They fuse together as long as there is a binary black vertex between them $(2(d))$, which allows us to construct a monoid and a comonoid

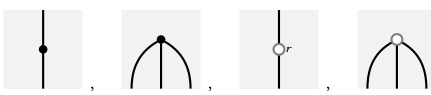
[2] This white vertex differs from the one in [15] by a phase.



satisfying the *bialgebra axiom* ($2(h)$) (in fact they form a *Hopf algebra* with the "self crossing" as the antipode). Morphisms with an even number of black vertices are natural with respect to the crossing ($2(a)$) while the binary black vertex (odd morphism) induces a self-crossing when slid past a crossing ($2(g)$). The binary black vertex is also an involution ($2(c)$).

The axioms in group 3 say that the R-labelled white vertices form a special Frobenius algebra with the labels forming an abelian group ($3(a, b)$). They are symmetric with respect to the swap ($3(c)$), and they satisfy the bialgebra equation with the black vertices ($3(d, f)$) (although they do not form a Hopf algebra). The self crossing is a comodule homomorphism ($3(g)$) while the binary black vertex is a comonoid homomorphism ($3(h)$) of the white vertices, and composition and convolution by the black Hopf algebra corresponds to the ring operations on the R-labels ($3(e, j, k)$). The rule $3(i)$ is used to eliminate crossings from the normal form, as we shall later see. Due to the symmetry of the black and white vertices, except for the crossing, we can see this as a calculus of undirected multigraphs, and we are free to "pull" the wires as we please.

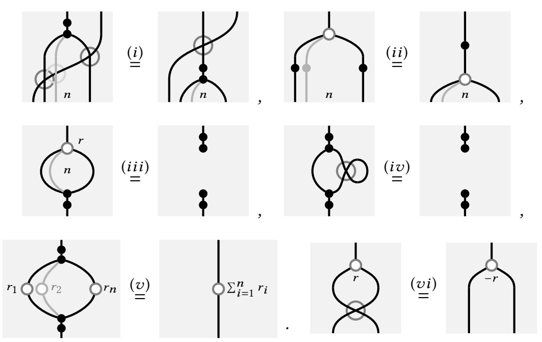
We have listed a set of axioms for the ZW calculus which we claim is complete. We used a presentation with vertices of arbitrary arity, but we note that there is an equivalent presentation whose generators are binary and ternary black and white vertices only:



Completeness of the ZW calculus

We state some useful equations that can be derived from the axioms.

Proposition 3. *The following are derived rules:*



for all $n \in \mathbb{N}$ in rules (i, ii, v), $n \geq 2$ in rule (iii), and $r, r_i \in R$ in rule (v, vi).

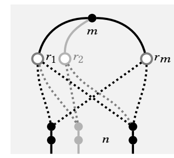
The rule (iv) is the Hopf equation mentioned in the discussion of the ZW axioms. The rule (vi) is a handy rule for expressing the crossing in normal form, as we will later see.

We will start proving the completeness of the ZW calculus by first proving its *universality*.

Theorem 3.1 (Universality of the ZW calculus)**.** *The interpretation functor $ZW_R \to R\mathbf{bit}$ is full.*

Proof. Due to the presence of self-duality (known as the Choi-Jamiołkowski isomorphism in quantum information theory), every morphism in $R\mathbf{bit}$ can be written as a partial transpose of a state. Hence it suffices to prove that for every state in $R\mathbf{bit}$ there exists a corresponding ZW diagram.

Write an arbitrary n-partite state as $\sum_{i=1}^{m} r_i | b_{i_1} \ldots b_{i_n} \rangle$, where $r_i \neq 0$ and no two kets in the sum are the same. We claim that it is the image of the diagram



(1)

where the i-th white vertex has one connection to the j-th output if and only if $b_{i_j} = 1$, for $i = 1, \ldots m$, $j = 1, \ldots n$. The claim can be proved by a direct calculation. □

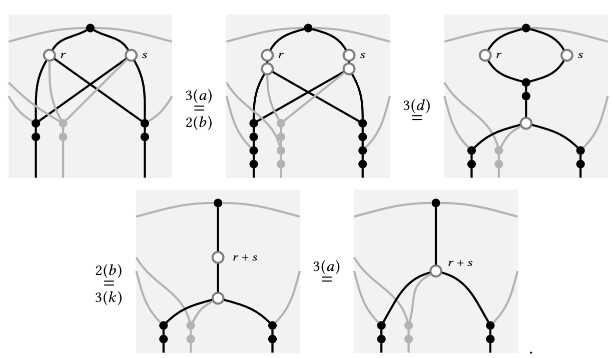
The proof of universality showed that any state is the image of the diagram in (1). We will call this diagram the *normal form.* From how the normal form is constructed, it is clear that it is unique up to a permutation of the white vertices. It is possible to give an ordering to the white vertices, but due to the symmetry of the black vertex with respect to swap it does not matter what ordering we take.

Allowing diagram (1) to have two white vertices with the same connections, and r_i to be 0 for some i, we obtain what we call a diagram in *pre-normal form.* The pre-normal form can be reduced to a normal form, as we will now show. Then, the structure of our completeness proof is as follows:

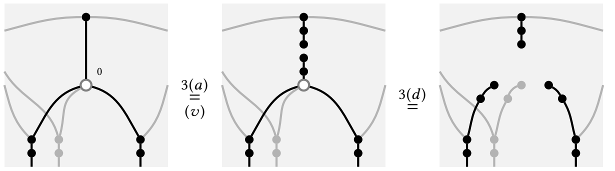
- any composite of two diagrams in normal form can be rewritten to pre-normal form:
  - the juxtaposition of diagrams in normal form can be rewritten to pre-normal form;
  - the plugging of an output of a diagram in normal form into another (self-plugging) can be rewritten to pre-normal form;
  - an arbitrary composition of two diagrams can be factored as a juxtaposition, followed by a self-plugging;
- all generators can be rewritten to normal form.

Lemma 3.2. *A ZW diagram in pre-normal form can be rewritten in normal form.*

Proof. Suppose a diagram is in pre-normal form. If two white vertices have the same connections, then we can "sum" them by



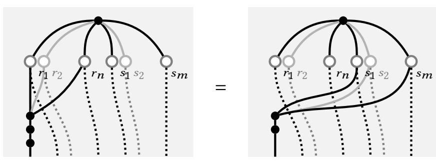
If $r_i = 0$ for some i, then we can eliminate that white vertex by



,

and use axiom $2(b)$ to simplify the diagram. ◻

Lemma 3.3 (Negation)**.** *Given a diagram in pre-normal form, the diagram obtained by composing an output with a binary black vertex can be rewritten in pre-normal form, which complements the connections of that output to the white vertices; that is, locally,*



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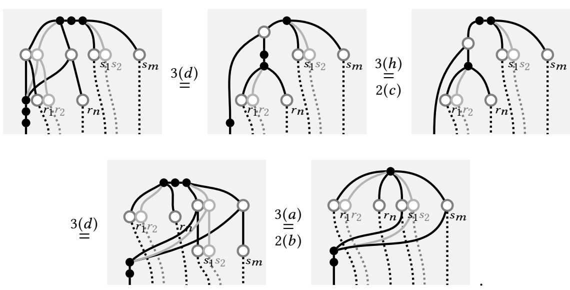
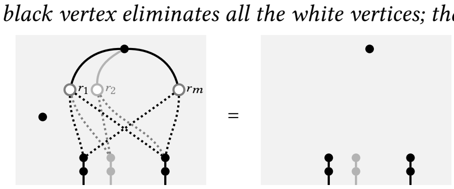
Proof. First isolate the first n white vertices on the left hand diagram using axioms $3(a)$ and $2(b)$, then proceed as follows:



. ◻

A nullary black vertex is interpreted as the 0 element of the ring.

Lemma 3.4 (Absorption)**.** *For all diagrams in pre-normal form, a nullary black vertex eliminates all the white vertices; that is,*



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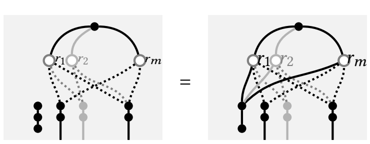
Proof. Use axiom $2(b)$ to expand the nullary black vertex and apply the negation lemma:



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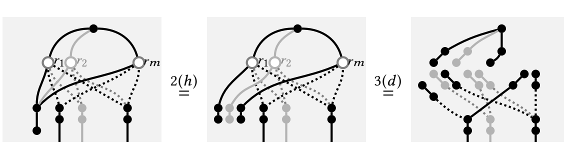
Then apply axioms $2(h)$ and $3(d)$, which eliminate all the white vertices:



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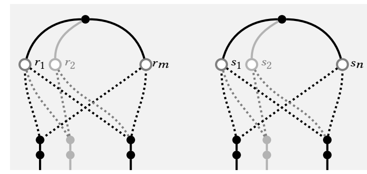
The final diagram can be simplified using axiom $2(b)$ which merges the black vertices to get the desired result. ◻

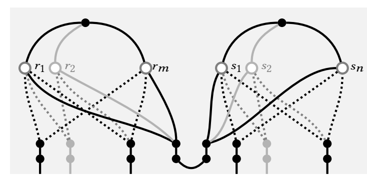
Lemma 3.5 (Juxtaposition)**.** *The juxtaposition of two diagrams in pre-normal form can be rewritten to pre-normal form.*

Proof. Consider the following juxtaposition of diagrams,



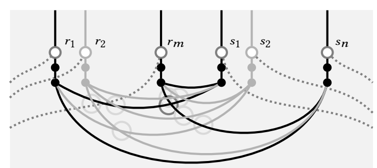
.

Using the axiom $2(h)$, we can produce a pair of connected black vertices, and using the negation lemma we can connect the pair of black vertices to the diagrams:



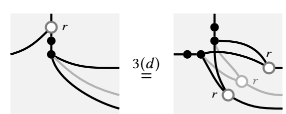
.

Applying axiom $2(h)$ again on the pair of black vertices (after eliminating the middle two black vertices using the $2(c)$ axiom) gives the following diagram (zooming in to the relevant part of the diagram):



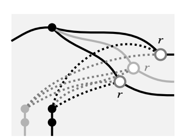
.

We can then push all the white vertices through the bottom black vertices using axiom $3(d)$, for instance,



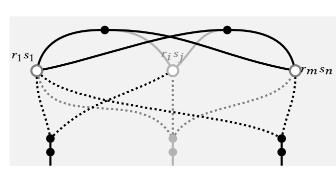
.

We would want to merge the white vertices, but they cannot pass through the crossing. However, using $2(b)$ in the final diagram, the higher black vertex merges with the top black vertex and the lower black vertex merges with some outputs, for instance



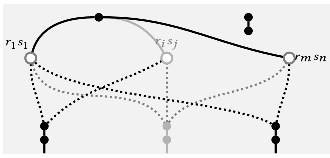
.

The white vertices are all connected to either of the two top black vertices and now we can eliminate all the crossings using $3(i)$. After merging white vertices pairwise, there is a white vertex with label $r_i s_j$ for each $i = 1, \ldots, m$, $j = 1, \ldots, n$



.

Finally, one of the top black vertices can be eliminated with the negation lemma

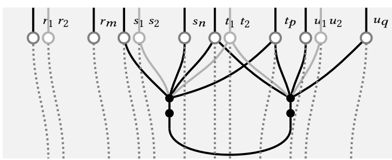


,

and the floating pair of black vertices is eliminated using $2(h)$. □

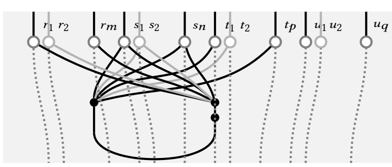
Lemma 3.6 (Trace)**.** *A self-plugging on a diagram in pre-normal form can be rewritten in pre-normal form.*

Proof. The proof involves some tactful use of the negation lemma. Suppose that we have the following self-plugging diagram, only zooming in to the white vertices and the pair of self-plugged outputs,



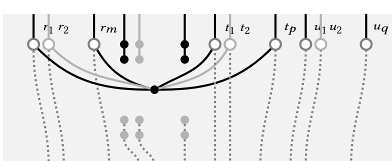
,

where the r white vertices are not connected to either of the outputs, the s white vertices are connected to the left output only, the u white vertices are connected to the right output only, and the t white vertices are connected to both outputs. From the interpretation in R**bit**, we expect the r and t labelled white vertices to survive while the s and u labelled white vertices are eliminated. This can be shown diagrammatically by first performing a negation to obtain the following diagram:



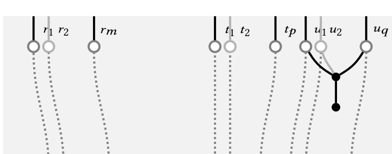
.

We can merge the black vertices using the axiom $2(b)$ and the s labelled white vertices have two connections with the black vertex. This means that we can apply $3(f)$ to eliminate all the s labelled white vertices:



.

We can apply negation lemma again to obtain



.

Finally, we can eliminate all the u labelled white vertices as in the absorption lemma, which completes the proof. □

With the juxtaposition and trace lemma, we have proved the following theorem:

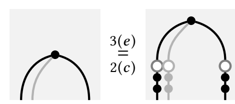
Theorem 3.7. *Any composition of two ZW diagrams in normal form can be rewritten in pre-normal form.*

Theorem 3.7 shows that the assignment R**bit** $\to ZW_R$ defined by the normal form is functorial: every morphism in R**bit** is mapped to a ZW normal form with some transposing of outputs, and composition of morphisms in R**bit** (tensoring and vertical composition) is composition of normal forms in ZW (juxtaposition and plugging), which can be rewritten in normal form. This functor is a right inverse to the interpretation $ZW_R \to R$**bit**. It remains to show that it is a two-sided inverse.

Theorem 3.8 (Completeness of the ZW calculus)**.** *The interpretation $ZW_R \to R$**bit** is an isomorphism of PROPs.*

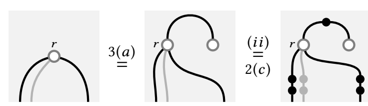
Proof. It suffices to show that every generator of ZW can be rewritten to normal form.

For the black vertices, the nullary vertex is already in normal form while the n-ary vertices for $n > 0$ can be rewritten in normal form as follows:



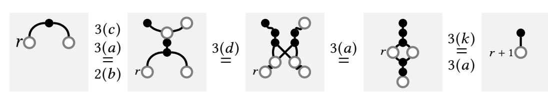
.

For the white vertices, the n-ary vertices for $n > 0$ can be rewritten in normal form as follows:



.

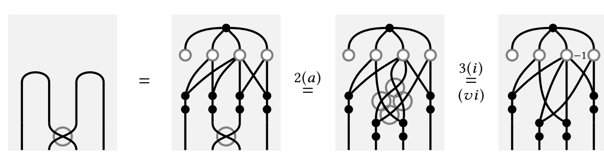
We can apply the same procedures to the nullary vertex, but it is not in normal form yet. It requires a few more steps to rewrite it in normal form:



.

as required. The self-duality morphisms are instances of a binary white vertex (after applying axiom $3(e)$).

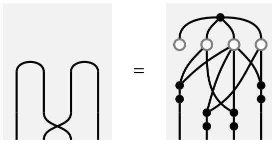
We rewrite the crossing as follows:



,

where in the first step we have rewritten the juxtaposition of two self-duality morphisms in normal form, as made possible by the juxtaposition lemma.

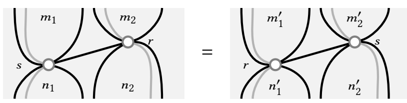
Finally, the case for swap is similar to the case for crossing, except that there are no crossings in the third diagram. Hence the normal form is simply:



.

□

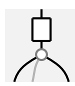
We have now proved the completeness of the ZW calculus for R**bit**. The calculus features an n-ary R-labelled white vertex for each $r \in R$. Given a presentation of R, we could in fact have just a label for each generator of R. The calculus remains largely the same; we just have to add an axiom for each relation the generators satisfy, and also a slight modification to some of the axioms involving the ring operations on the white vertices. For instance the axiom $3(k)$ can be slightly modified to suit the generators, and $3(a)$ can be replaced with



,

where r, s are some generators, $m_1, m_2, n_1, n_2, m_1', m_2', n_1', n_2'$ are natural numbers, and $m_1 + m_2 = m_1' + m_2'$, $n_1 + n_2 = n_1' + n_2'$.

The completeness proof also remains largely the same since any composition of diagrams in pre-normal form can be rewritten in pre-normal form. We can also modify the normal form by replacing the R-labelled white vertices with

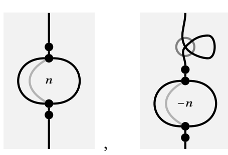


,

where the box is some fixed expression for the ring elements as sums and products of generators, expressed by convolution with black vertices and vertical composition.

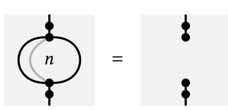
We will look at some different commutative rings as examples.

If R is the free commutative ring on one generator $\mathbb{Z}$, we recover the result of [15]. We can express $n \in \mathbb{Z}$ as



depending on whether n is positive or negative.

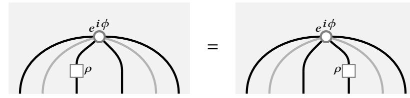
If $R := \mathbb{Z}_n$, then we just need to add one axiom for the relation $n = 0$,



.

In particular, for $\mathbb{Z}_2$**bit**, also called modal quantum theory in [22], the crossing is equal to the swap and many of the axioms regarding the crossing become redundant.

If we take $R := \mathbb{C}$, the ring of complex numbers, we obtain completeness for **Qubit**. It might be convenient to separate the phase part and the length part and express each element in $\mathbb{C}$ as $\rho e^{i\phi}$ for ρ the positive reals and $\phi \in [0, 2\pi)$. Then all we need to modify is to have an n-ary white vertex for $e^{i\phi}$, a binary white vertex for ρ which we require is a (co)module homomorphism with respect to the n-ary white vertices



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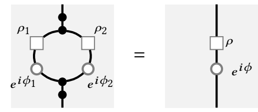
and change the axiom $3(k)$ to

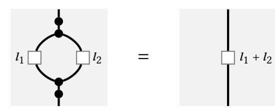


where $\rho e^{i\phi} = \rho_1 e^{i\phi_1} + \rho_2 e^{i\phi_2}$. We will call this example the $ZW_{\mathbb{C}}$ calculus.

As a final example, we let $R := \mathbb{Z}\left[\frac{1}{2}, e^{i\pi/4}\right]$. It is possible to have an n-ary white vertex with label $e^{i\frac{\pi}{4}}$ and a binary white vertex with label $\frac{1}{2}$, and some axioms for the generators. However, for convenience, we can use the same convention as in the complex case: an n-ary white vertex with labels $e^{i\phi}$ for $\phi = k\frac{\pi}{4}, k = 0, 1, \ldots, 7$, and a binary one with labels $0 < l \in \mathbb{Z}\left[\frac{1}{2}\right]$. Then an expression for an arbitrary ring element is

$$\sum_{k=0}^{7} l_k e^{ki\frac{\pi}{4}}$$

for $0 < l_k \in \mathbb{Z}\left[\frac{1}{2}\right]$, $k = 0, 1, \ldots, 7$. The axiom $3(k)$ is now



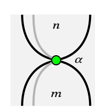
,

for $0 < l_1, l_2 \in \mathbb{Z}\left[\frac{1}{2}\right]$. The $\mathbb{Z}\left[\frac{1}{2}, e^{i\frac{\pi}{4}}\right]$ **bit** corresponds to the Clifford + T fragment of **Qubit** as proved in [18]. We will call this example the $\mathrm{ZW}_{\frac{\pi}{4}}$ calculus.

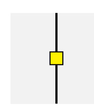
The last two examples, where $R = \mathbb{C}$ and $R = \mathbb{Z}\left[\frac{1}{2}, e^{i\frac{\pi}{4}}\right]$, are of great importance to the completeness of the ZX calculus for **Qubit** and the Clifford+T fragment, respectively. The proof for the completeness results is via a direct translation from ZX to ZW calculus, which we will detail in the next section.

4 The ZX Calculus

The ZX calculus has the following set T_{ZX} of generators and interpretations in **Qubit**:

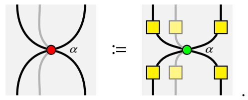


$$\mapsto \quad |\underbrace{0\ldots0}_{m}\rangle\langle\underbrace{0\ldots0}_{n}| + e^{i\alpha}|\underbrace{1\ldots1}_{m}\rangle\langle\underbrace{1\ldots1}_{n}|,$$



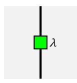
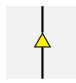
$$\mapsto \quad |+\rangle\langle 0| + |-\rangle\langle 1|,$$

where $\alpha \in [0, 2\pi)$, $|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. In quantum theory, the green vertices correspond to GHZ states (modulo normalisation and map-state duality) and phase gates for the computational basis. The pair $\{|+\rangle, |-\rangle\}$ is called the X basis, and forms a mutually unbiased pair with the computational basis $\{|0\rangle, |1\rangle\}$, also known as the Z basis. The yellow box is called the Hadamard gate, and corresponds to a change of basis between Z and X. Hence, we can define a GHZ-state in the X basis as:



The main focus of the calculus is the interaction of GHZ states in the two bases.

Although the ZX calculus is universal for **Qubit** as proved in [6], it is convenient to extend the language and include two more generators:

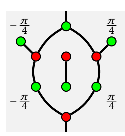
 $\mapsto |0\rangle\langle 0| + \lambda|1\rangle\langle 1|$,  $\mapsto |0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 1|$,

where $0 < \lambda \in \mathbb{R}$. The green box, labelled with the norm of a complex number, is the counterpart to the circular green vertex, labelled with the angle. The triangle is closely related to the *Toffoli gate* in quantum circuits, as shown in Chapter 12 of [10]. We will state a representation for the green box and the triangle in terms of the red and green vertices.

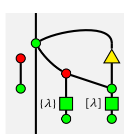
Remark 4. Although technically the green box is also a vertex, we will refer to it as the green box, while calling the circular green vertex simply the green vertex.

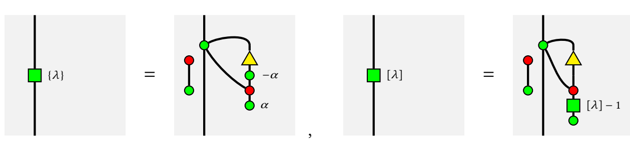
In practice, it may be convenient to define a red box just like the red vertex, but we will not use it in this paper.

A representation of the triangle has been given in [10, 18]. We will state the one given by the former simply because it appeared first:



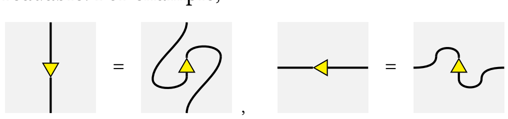
To represent the green box, we first write $\lambda = \{\lambda\} + [\lambda]$, where $[\lambda]$ is the integer part and $\{\lambda\}$ the remainder of λ. Then, the green box can be expressed as





where $\{\lambda\} = e^{i\alpha} + e^{-i\alpha}$, and $[\lambda]$ is expanded recursively until we reach 1. If there is no integer part, then the diagram is simply the remainder diagram.

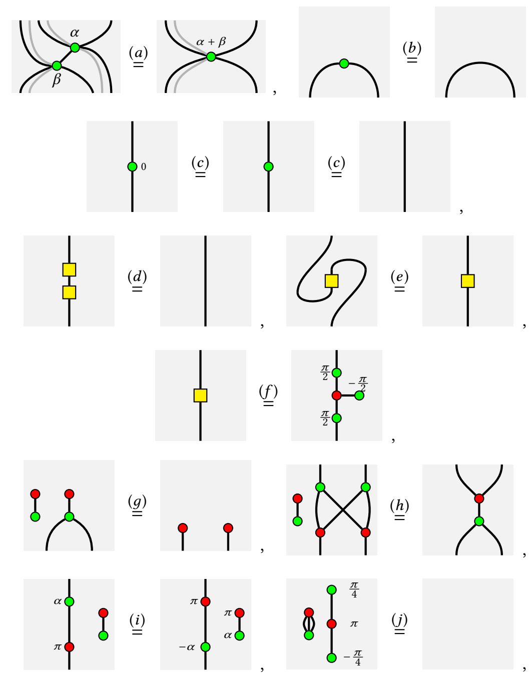
We will draw the triangle in various angles to make the diagrams more readable. For example,



We will now present a set E_{ZX} of axioms relating the morphisms in the ZX calculus. As with the ZW calculus, we claim that the axioms are sound, and define a monoidal functor $ZX \to$ **Qubit**, which can be verified by a direct calculation.

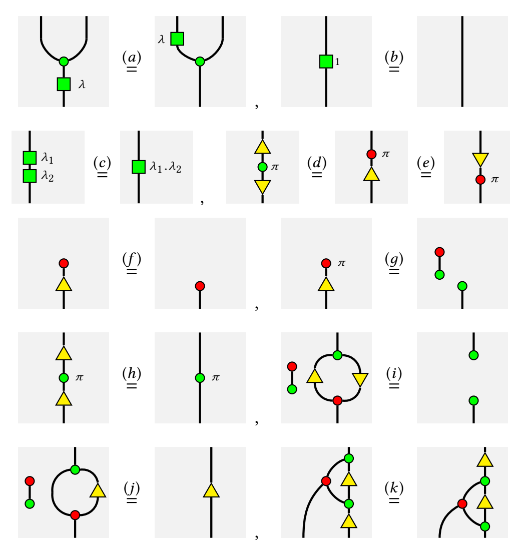
ZX Axioms

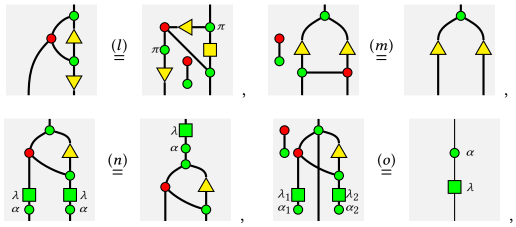
1. The following are the axioms for the basic generators of the ZX calculus:



for $\alpha, \beta \in [0, 2\pi)$. The axioms also hold if flipped upside-down. It is derivable that the axioms are also true for the interchange of the red and green colours, and for simplicity we will give them the same axiom labels.

2. The following are the axioms for the extended generators of the ZX calculus:





for $0 < \lambda, \lambda_1, \lambda_2 \in \mathbb{R}$, $\alpha, \alpha_1, \alpha_2 \in [0, 2\pi)$, and in (o), $\lambda e^{i\alpha} = \lambda_1 e^{i\alpha_1} + \lambda_2 e^{i\alpha_2}$.

Discussion of the ZX axioms

The axioms in group 1 say that the $\mathbb{C}$-labelled green vertices form a Frobenius algebra with the labels forming an abelian group $(1(a, b))$. In fact, the derived rules shown in the next section show that it is symmetric (ii) and special. The Hadamard gate is an involution and self-transpose $(1(d, e))$, and can be decomposed in terms of the red and green vertices $(1(f))$. The red and green vertices form a bialgebra $(1(g, h))$ (in fact they form a *Hopf algebra* with the identity wire as the antipode (iii)). Axiom $1(i)$ shows how a green vertex moves through a red π vertex, and $1(j)$ is called the scalar axiom. There is another scalar relation in (iv). So far, there is no directionality of the wires because of the special commutative Frobenius structure of the green vertices, and the Hadamard gate is self-transpose (hence this is true for red vertices too). Therefore we are free to deform the diagrams as we like, as long as the connectivity of the vertices is unchanged.

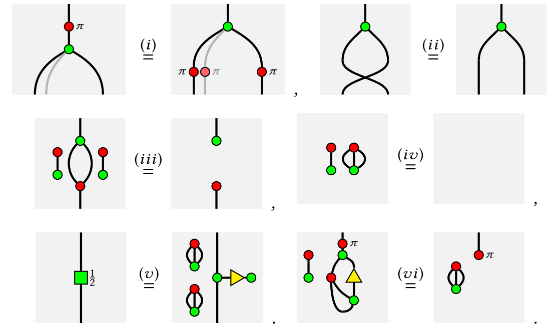
The axioms in group 2 say that the green box acts in the same way as the green vertex. The rest of group 2 characterises the triangle by axioms with a small number of nodes. Unlike the green and red vertices and Hadamard gate, the triangle has an implicit directionality, that is, it has a distinct input and output which cannot be swapped around. We can still treat the diagrams as pseudo-undirected; we just have to make sure that the input and output of the triangle are connected to the right vertices. It is possible to mitigate this as suggested by axiom $2(e)$, but we will leave that out for now.

The introduction of the triangle vertex and its axioms is inspired by [18], in which the completeness of a version of ZX for the Clifford+T fragment was proved via a translation from and into the parameter-free ZW calculus, enriched with the scalar $\frac{1}{2}$ [15]. We have employed a similar translation from ZW to ZX; however, our translation from the ZX calculus to the ZW calculus is different, and the two are inverse to each other, which was not the case in [18]. Having recognised that the proof relies heavily on the interaction of the triangle with other vertices, we chose to axiomatise the triangle as a generator, instead of using it as a shorthand for a diagram of red and green vertices. This is similar to the use of the crossing in the ZW calculus. The specific decomposition of the triangle is not crucial and different choices lead to different axiomatisations.

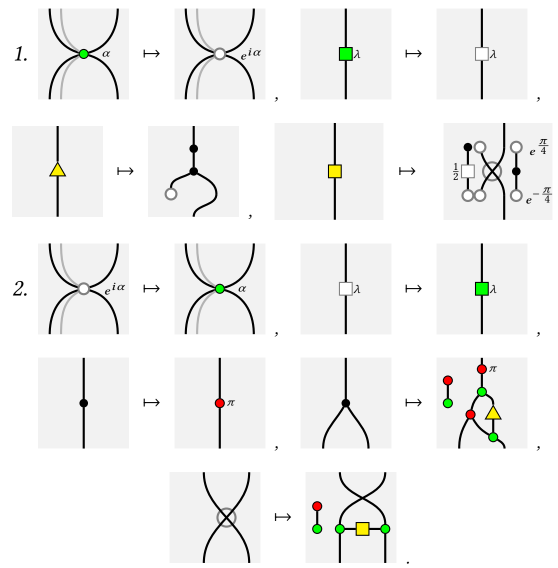
Completeness of the ZX calculus

The completeness of the ZX calculus for **Qubit** relies on the completeness of the $ZW_{\mathbb{C}}$ calculus. We will construct a direct translation of the diagrams between the two calculi which respects the interpretation in **Qubit**, and under this translation show that all diagrammatic equations in one of the calculi can be derived in the other. This will imply the completeness of the ZX calculus.

Proposition 5. *The following are derived rules:*



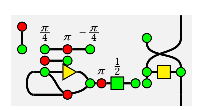
Lemma 4.1. *Let t_1, t_2 be the following assignments of diagrams in $ZW_{\mathbb{C}}$ to the generators of ZX, and vice versa:*



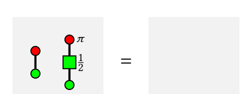
Then t_1 and t_2 respect the interpretations of diagrams in **Qubit**, *and for all generators g of ZX, and g' of $ZW_{\mathbb{C}}$, we have $t_2(t_1(g)) = g$, and $t_1(t_2(g')) = g'$.*

Proof. It is easy to check that the assignments respect the interpretations. Then, $t_1(t_2(g')) = g'$ follows from completeness of ZW and soundness of ZX.

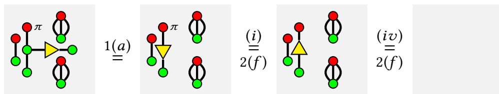
The claim that $t_2(t_1(g)) = g$ for all generators of ZX is trivial to check for the green vertex and green box. Checking the triangle is a simple application of $1(a)$ and $1(c)$ to eliminate the red π vertex, $1(g)$ to open the loop, and finally $1(a)$ to simplify the diagram. For the Hadamard gate, we will get



and after some simplifications using the derived rule (vi), axiom $1(a)$ and $1(j)$, we are left to show



This can be done by applying rule (v) to replace the green box, then



. ◻

Theorem 4.2 (Completeness of the ZX calculus)**.** *The functor* $ZX \to$ **Qubit** *is an isomorphism of PROPs.*

Proof. We only need to show that extending t_2, as defined in Lemma 4.1, to composite diagrams defines a monoidal functor; it will then automatically be an isomorphism. For this, it suffices to check that the translations of all axioms of $ZW_{\mathbb{C}}$ can be derived from the ZX axioms. The details are tedious and are left out of this paper. ◻

The Clifford+T ZX calculus

The Clifford+T fragment of quantum mechanics is traditionally defined by restricting Z basis phases to integer multiples of $\frac{\pi}{4}$. A version of the ZX calculus complete for this fragment was first produced in [18], but we can derive a different axiomatisation. As a result of our design choices, our axiomatisation has a larger number of axioms, which, on the other hand, involve a smaller number of vertices. We do not know, at the moment, whether all our axioms are mutually independent.

The generators of the $\mathrm{ZX}_{\frac{\pi}{4}}$ calculus are those of the ZX calculus, where labels of green vertices are restricted to multiples of $\frac{\pi}{4}$. We can extend this calculus to include the triangle as it is expressible in terms of the $\frac{\pi}{4}$ phases. From the derived rule (v), it follows that the green box can also be defined for $0 < \lambda \in \mathbb{Z}\left[\frac{1}{2}\right]$.

Lemma 4.3. *The functor* $ZX_{\frac{\pi}{4}} \to \mathbb{Z}\left[\frac{1}{2}, e^{i\frac{\pi}{4}}\right]$ **bit** *is full.*

Proof. It is clear from the interpretation of the generators that the $\mathrm{ZX}_{\frac{\pi}{4}}$ calculus is interpreted in the subcategory $\mathbb{Z}\left[\frac{1}{\sqrt{2}}, e^{i\frac{\pi}{4}}\right]$ **bit** $=$ $\mathbb{Z}\left[\frac{1}{2}, e^{i\frac{\pi}{4}}\right]$ **bit** of **Qubit**. Furthermore, restricting the functor defined in Lemma 4.1 we obtain a full functor $ZX_{\frac{\pi}{4}} \to ZW_{\frac{\pi}{4}}$, which then implies that $ZX_{\frac{\pi}{4}} \to \mathbb{Z}\left[\frac{1}{2}, e^{i\frac{\pi}{4}}\right]$ **bit** is full. ◻

The $\mathrm{ZX}_{\frac{\pi}{4}}$ calculus features the same axioms as the ZX calculus, with restricted phases $\alpha = k\frac{\pi}{4}$, $k = 0, 1, \ldots 7$, lengths $0 < \lambda \in \mathbb{Z}\left[\frac{1}{2}\right]$; moreover, the conditions of axiom $2(o)$ are changed to $0 < \lambda, \lambda_1, \lambda_2 \in \mathbb{Z}\left[\frac{1}{2}\right]$, $\alpha \equiv \alpha_1 \equiv \alpha_2 (\mathrm{mod}\ \pi)$. The proof of completeness goes through the $ZW_{\frac{\pi}{4}}$ calculus in exactly the same way as the unrestricted ZX calculus.

Theorem 4.4 (Completeness of the $\mathrm{ZX}_{\frac{\pi}{4}}$ calculus)**.** *The interpretation* $ZX_{\frac{\pi}{4}} \to \mathbb{Z}\left[\frac{1}{2}, e^{i\frac{\pi}{4}}\right]$ **bit** *is an isomorphism of PROPs.*

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