Optimization over the Boolean Hypercube via Sums of Nonnegative Circuit Polynomials*

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1 Abstract

An *optimization problem over a boolean hypercube* is an n-variate (constrained) polynomial optimization problem where the feasibility set is restricted to some vertices of an n-dimensional hypercube. This class of optimization problems belongs to the core of theoretical computer science. However, it is known that solving them is NP-hard in general, since one can easily cast, e.g., the Independent Set problem in this framework.

One of the most promising approaches in constructing theoretically efficient algorithms is the *sum of squares (SOS) hierarchy*, also known as *Lasserre relaxation*. The method is based on a Positivstellensatz result saying that the polynomial f, which is nonnegative over the feasibility set, can be expressed as a sum of squares times the constraints defining the set. Bounding a maximum degree of a polynomial used in a representation of f provides a family of algorithms parametrized by an integer d. Finding a degree d SOS certificate for nonnegativity of f can be performed by solving a *semidefinite programming* (SDP) formulation of size $n^{O(d)}$. Finally, for every (feasible) n-variate hypercube optimization problem, with constraints of degree at most d, there exists a degree $2(n + \lceil d/2 \rceil)$ SOS certificate.

The SOS algorithm is a frontier method in algorithm design. It provides the best available algorithms used for a wide variety of combinatorial optimization problems as well as problems in dictionary learning, tensor completion and decomposition and robust estimation.

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On the other hand it is known that the SOS algorithm admits certain weaknesses. Lower bounds on the effectiveness for this method were proved for e.g. Knapsack problem, planted clique problem and some CSP problems. Finally, SOS has hard time proving global nonnegativity, as first proved by Hilbert. Later an explicit example was given by Motzkin. Moreover, as shown by Blekherman, there are significantly more nonnegative polynomials than SOS polynomials.

In this article, we initiate an analysis of hypercube optimization problems via *sums of nonnegative circuit polynomials (SONC)*. SONCs are a nonnegativity certificate, which are independent of sums of squares. This means particularly that certain polynomials like the Motzkin polynomial, which have no SOS certificate for global nonnegativity, can be certified as nonnegative via SONCs. Moreover, SONCs generalize polynomials which are certified to be nonnegative via the arithmetic-geometric mean inequality. Similarly as Lasserre relaxation for SOS, a Schmüdgen-like Positivstellensatz yields a converging hierarchy of lower bounds for polynomial optimization problems with compact constraint set. These bounds can be computed via a convex optimization program called *relative entropy programming*. Our main question in this article is:

> *Can SONC certificates be an alternative for SOS methods for optimization problems over the hypercube?*

We answer this question affirmatively in the sense that we prove SONC complexity bounds for boolean hypercube optimization analogous to the SOS bounds mentioned above. More specifically, we show:

1. For every polynomial which is nonnegative over an n-variate hypercube with constraints of degree at most d there exists a SONC certificate of nonnegativity of degree at most $n + d$.
2. If a polynomial f admits a degree d SONC certificate of nonnegativity over an n-variate hypercube, then the polynomial f admits also a *short* degree d SONC certificate that includes at most $n^{O(d)}$ nonnegative circuit polynomials.

Furthermore, we show some structural properties of SONCs:

1. We give a simple, constructive example showing that the SONC cone is not closed under multiplication. Subsequently we use this construction to show that the SONC cone is neither closed under taking affine transformations of variables. We discuss this surprising property both from algebraic and optimization perspective.
2. We address an open problem asking whether the Schmüdgen-like Positivstellensatz for SONCs can be improved to an equivalent of Putinar's Positivstellensatz. We answer this question negatively by showing an explicit hypercube optimization example, which provably does not admit a Putinar representation for SONCs.