What's in a game?

A theory of game models

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Abstract

Game semantics is a rich and successful class of denotational models for programming languages. Most game models feature a rather intuitive setup, yet surprisingly difficult proofs of such basic results as associativity of composition of strategies. We seek to unify these models into a basic abstract framework for game semantics, *game settings*. Our main contribution is the generic construction, for any game setting, of a category of games and strategies. Furthermore, we extend the framework to deal with innocence, and prove that innocent strategies form a subcategory. We finally show that our constructions cover many concrete cases, mainly among the early models [5, 23] and the recent, sheaf-based ones [40].

CCS Concepts • **Theory of computation → Denotational semantics**; **Categorical semantics**;

Keywords Game semantics, category theory, presheaves, sheaves

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1 Introduction

Game semantics has provided adequate models for a variety of (idealised) programming languages. We will here mainly be concerned with the numerous variations on *arena games*. This comprises, e.g., models for PCF [23, 36], general references [4], finite nondeterminism [16, 18], control operators [26], and the recent model by Tsukada and Ong [40]. We will also briefly consider other models of PCF [5] and of linear logic [8].

In all these models, the types of the considered language are interpreted as games and programs as strategies. Games form the objects, and strategies the morphisms of a category, which is compared to the 'syntactic' category generated from the operational description of the language. However, as noted, e.g., in [17], a problem shared by all these models is the surprising difficulty of certain proofs like associativity of composition, or stability of innocent strategies under composition.

This raises the issue of unifying these models into a satisfactory theory, with an emphasis on factoring out such difficult proofs. This is an ambitious goal, because although game models clearly share a lot of ideas, they are also rather diverse. E.g., depending on the considered language, various constraints are imposed upon strategies, like *innocence* or *well-bracketing*. Further sources of diversity have appeared with recent extensions, e.g., sheaf-based innocence [40], nominal models [35], tensorial logic [33], or concurrent extensions [22, 37].

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This paper is an attempt at improving the situation. Our main contributions are as follows.

(1) We organise the basic data underlying typical game models into a simple categorical structure called a *game setting*, emphasising its simplicial features. We then show that each game setting gives rise to a category of games and strategies, whose construction relies on categorical techniques such as (presheaf) polynomial functors [13, 42] and exact squares [15].
(2) We extend the framework to deal with *innocence*, an emblematic constraint put on strategies to capture purely functional computation. For this, we enrich game settings with a notion of *view* and, under mild hypotheses, we derive a subcategory of innocent strategies. Our approach exploits the recent recasting of innocence as a sheaf condition [22, 40], and again relies on advanced category theory to give high-level proofs.
(3) We show our framework covers quite a few examples, namely variants of the original Hyland-Ong/Nickau (HON) model [16, 23, 28], AJM games [5] and Tsukada and Ong's model [40].
(4) We work out the limits of our techniques in two well-known dead ends of game semantics: non-associativity of composition in Blass games [3] and non-stability of innocent strategies under composition in the absence of determinism [16].

Let us elaborate a bit on *(1)* and *(2)*. Our framework in fact deals with various notions of composition and innocence, and establishes some links between them. In order to briefly explain these links, let us introduce some terminology that will only make rigorous sense later on. An important distinction among examples is whether plays are considered as a poset (with the prefix ordering) or as a category (for some refined notion of morphism [40]). Another is whether the considered strategies are general or boolean presheaves. We annotate composition and innocence with the following codes.

Plays		Strategies	
poset	category	boolean	general
p	c	b	s

Example 1.1. Standard strategies, being prefix-closed sets of plays, are boolean presheaves on the prefix ordering, so their composition is pb-composition. Moreover, standard innocent strategies are innocent pb-strategies. Similarly, Tsukada and Ong [40] use proper categories of plays and their strategies are presheaves, hence their composition and innocent strategies are cs-composition and innocent cs-strategies. Unspecified items denote either possibility. E.g., s-composition means composition of presheaves, in either the poset-based or category-based setting.

Much effort is put into linking the different variants together, as summarised in the following table.

Section	relates	to
2.8	s-composition	b-composition
4.2	cs-innocence	ps-innocence
4.3	cs-innocence	pb-innocence

The established links are of various nature. In Section 2.8, we express b-composition in terms of s-composition, and infer associativity of b-composition from associativity of s-composition. In Section 4.2, we define ps-innocence from cs-innocence, and derive that innocent ps-strategies form a subcategory from the fact that innocent cs-strategies do. Finally, in Section 4.3, we explain where our techniques cease to work when trying to pass to pb-innocence.

1.1 Related work

The main result of [20, 21] is close in spirit to our Theorem 2.19, though it only covers the prefix-based, non-innocent case. Another general approach to composition of strategies is Bowler's thesis [9], which covers simple games [17] and Conway games [24]. Two further abstract approaches to game semantics are our [11, 22]. Both papers focus on the link between naive and innocent strategies as well as the interpretation of programs, but elude composition of strategies. Beyond such attempts at abstraction, significant work has been devoted to giving efficient proofs in particular models [17, 30, 31, 33]. Notably, the use of polynomial functors and exact squares in game semantics dates back to Hatat's unpublished thesis [19].

1.2 Plan

In Section 2, we gradually introduce game settings, following the successive steps for constructing a typical game model. We then state our main results for the basic setup. We remain very informal about game semantics, and only start to consider the particulars of various game models in Section 3, where we establish that the announced game models fit into our framework. We then refine game settings to deal with innocence in Section 4, covering in passing Tsukada and Ong's model [40]. Finally, we conclude in Section 5. In order to provide a feel for our techniques, we include a proof sketch for stability of innocent strategies under compostion (Theorem 4.11), but otherwise most proofs are omitted.

1.3 Notation and prerequisites

For all $n \in \mathbb{N}$, $[n]$ denotes the finite ordinal with n elements, i.e., the set $\{0, \ldots, n-1\}$, and we sometimes use just n to denote the set $\{1, \ldots, n\}$. We assume some basic knowledge of category theory, namely categories, functors and natural transformations, as well as adjunctions. The category of *presheaves* over any category $\mathcal{C}$ is the functor category $[\mathcal{C}^{op}, \mathrm{Set}]$ of contravariant functors to sets and natural transformations between them, which we denote by $\widehat{\mathcal{C}}$. For any presheaf $X \colon \mathcal{C}^{op} \to \mathrm{Set}$, objects $c, c' \in \mathcal{C}$, morphism $f \colon c' \to c$, and element $x \in X(c)$, we use a right action notation $x \cdot f$ for $X(f)(x)$. By functoriality, we have $x \cdot f \cdot g = x \cdot (f \circ g)$, for any $g \colon c'' \to c'$. Replacing Set with 2, the ordinal [2] viewed as a category, we get the category $\widetilde{\mathcal{C}}$ of *boolean presheaves*.

2 Game settings

2.1 Categories of plays in game semantics

In this section, we sketch several notions of play typically involved in the construction of a game model. We do this without referring to any particular model. In the next sections, we will organise this data into a coherent categorical structure, which we will then exploit to give an abstract construction of a category of games and strategies.

The construction of a typical game model relies on the definition of notions of play involving increasingly many players. There is first a notion of *game*. Each game A involves two players O (Opponent) and P (Proponent), and features in particular a set of *plays* $\mathbb{P}_A$, which may be endowed with the prefix ordering or with a more sophisticated notion of morphism, thus forming a category of plays. Such two-player games form the basis of the model.

Example 2.1. We will provide more precise definitions later on, but for now, to fix intuition, in HON-style games (without bracketing), games are *arenas*, and $\mathbb{P}_A$ consists of all *justified sequences* (of any finite length, potentially non-alternating). This might be more liberal than expected, but we will see below why it is necessary.

The crucial step to view strategies as morphisms is to consider the *arrow* game $A \to B$, which intuitively describes the interaction of a middle player M acting as Opponent against a left player L and as Proponent against a right player R, as in

$$\begin{array}{ccc} L & M & R \\ \mathbb{B} & \longrightarrow & \mathbb{B} \\ & & q^R \\ q^M & & \\ \mathsf{t}^L & & \\ & & \mathsf{f}^M. \end{array} \qquad (1)$$

In this example, M plays like the negation function on booleans: R asks its return value by playing the move q^R; M in turn asks L for its argument by playing q^M, to which L answers 'true' by playing t^L; M eventually answers the original question by playing f^M. The two players L and R are sometimes thought of as a single player respresenting the environment.

However, there is a subtlety: one often needs to restrict attention to a certain subcategory $\mathbb{P}_{A,B} \hookrightarrow \mathbb{P}_{A \to B}$. One then obtains projections to $\mathbb{P}_A$ and $\mathbb{P}_B$.

Example 2.2. In HON-style games, $\mathbb{P}_{A,B}$ would consist of alternating sequences of even length of $\mathbb{P}_{A \to B}$. The projections of a play in $\mathbb{P}_{A,B}$ to A and B may not be alternating or have odd length, so it is crucial to be liberal in the choice of $\mathbb{P}_A$.

For composition of strategies, the situation (1) is then scaled up to combinations of two such situations in which a first middle player M_1 plays on the right with a second one, say M_2, as in

$$\begin{array}{ccccc} L & M_1 & & M_2 & R \\ \mathbb{B} & \longrightarrow & \mathbb{B} & \longrightarrow & \mathbb{B}. \end{array} \qquad (2)$$

Plays in such combinations are standardly called *interaction sequences*, and typically form a subcategory $\mathbb{P}_{A,B,C} \hookrightarrow \mathbb{P}_{(A \to B) \to C}$. An important point is that interaction sequences admit projections to $\mathbb{P}_{A,B}$, $\mathbb{P}_{B,C}$ and $\mathbb{P}_{A,C}$, which satisfy the obvious equations w.r.t. further projections, e.g., the following square commutes:

$$\begin{array}{ccc} \mathbb{P}_{A,B,C} & \longrightarrow & \mathbb{P}_{A,C} \\ \downarrow & & \downarrow \\ \mathbb{P}_{B,C} & \longrightarrow & \mathbb{P}_{C}. \end{array}$$

Example 2.3. In HON games, $\mathbb{P}_{A,B,C}$ typically consists of alternating justified sequences on $(A \to B) \to C$ which end in A or C and whose projections to $A \to B$ and $B \to C$ are plays.

Finally, in order to prove associativity of composition, one defines *generalised interaction sequences* as a subcategory $\mathbb{P}_{A,B,C,D} \hookrightarrow \mathbb{P}_{((A \to B) \to C) \to D}$, with projections satisfying the obvious equations.

2.2 Plays as a category-valued presheaf

Let us now organise all this data ($\mathbb{P}_A, \mathbb{P}_{A,B}, \mathbb{P}_{A,B,C}, \mathbb{P}_{A,B,C,D}$) into a simple categorical structure. First, as suggested by our notation, for all lists $L = (A_0, \ldots, A_{n-1})$ of games, we construct a category $\mathbb{P}_L$.

Example 2.4. In the HON case, we may take $\mathbb{P}_L$ to consist of alternating justified sequences on $(\ldots(A_0 \to A_1) \to \ldots) \to A_{n-1}$ whose projection to each $A_i \to A_{i+1}$ is a play, and which end in A_0 or A_{n-1}, the rightmost arena.

For the same reasons as before, we get projections $\delta_k \colon \mathbb{P}_L \to \mathbb{P}_{L \setminus A_k}$ (for 'delete k'), for all $k \in [n]$. A similar construction, relevant for defining identity strategies (so-called *copycat* strategies), is *insertions* $\iota_k \colon \mathbb{P}_L \to \mathbb{P}_{L^{+k}}$ (for 'insert k'), where $k \in [n]$ and L^{+k} denotes L with the kth game duplicated. E.g., $\iota_1 \colon \mathbb{P}_{A,B} \to \mathbb{P}_{A,B,B}$. Intuitively, this functor maps any play u in $\mathbb{P}_{A,B}$ to the interaction sequence in $\mathbb{P}_{A,B,B}$ which duplicates all moves on B. So in a situation like (2), M_2 would act as a 'proxy' between M_1 and R, repeating M_1's moves to R and conversely. For a precise definition and an example in the case of HON games, see Section 3.1.

Projections and insertions on lists of games may all be packed up into the comma category $\Delta/\mathbb{G}$, or more precisely $\mathrm{i}/\ulcorner\mathbb{G}\urcorner$, where

- $\mathbb{G}$ denotes the set of games;
- $\ulcorner\mathbb{G}\urcorner \colon 1 \to \mathrm{Set}$ is the functor picking $\mathbb{G}$;
- $\mathrm{i} \colon \Delta \hookrightarrow \mathrm{Set}$ is the embedding of the simplicial category Δ into sets.

Let us recall that Δ has non-empty finite ordinals $[n]$ as objects, with monotone maps as morphisms. So concretely, $\Delta/\mathbb{G}$ has non-empty, finite lists of games as objects, i.e., maps $L \colon [n] \to \mathbb{G}$ for some $n = \{0, \ldots, n-1\}$, and as morphisms $(n, L) \to (n', L')$ all monotone maps f making the following triangle commute:

$$\begin{array}{ccc} [n] & \xrightarrow{\quad f \quad} & [n'] \\ & {\scriptstyle L}\searrow \quad \swarrow{\scriptstyle L'} & \\ & \mathbb{G}. & \end{array}$$

Example 2.5. Let $\mathrm{d}^n_k \colon [n] \to [n+1]$ miss $k \in [n]$, i.e., $\mathrm{d}^n_k(i) = i$ for $i < k$ and $\mathrm{d}^n_k(j) = j + 1$ for $j \geq k$. E.g., d^2_1 yields a map $(A, C) \to (A, B, C)$ for all games A, B, C. Similarly, consider $\mathrm{i}^n_k \colon [n+1] \to [n]$ which collapses $k \in [n] \subseteq [n+1]$ and $k + 1 \in [n+1]$. E.g., for $n = 2$ and $k = 0$, it yields a map $(A, A, B) \to (A, B)$ for all A and B.

As promised, this yields a way to organise the categories of plays involved in a typical game model into a coherent categorical structure: we will show below that, for quite a few game models, the assignment $L \mapsto \mathbb{P}_L$ induces a category-valued presheaf on $\Delta/\mathbb{G}$, i.e., a functor $(\Delta/\mathbb{G})^{op} \to \mathrm{Cat}$. Furthermore, the maps δ_k and ι_k introduced earlier will respectively be given by $\mathbb{P}(\mathrm{d}_k)$ and $\mathbb{P}(\mathrm{i}_k)$.

In the following, we will only need to use this structure up to lists of length 4:

Definition 2.6. For any $p \leq q$ and set $\mathbb{G}$, let $\mathbb{G}_{[p,q]}$ denote the full subcategory of $\Delta/\mathbb{G}$ spanning lists of length between p and q.

In the next sections, we will define strategies, composition and copycat strategies abstractly, based on the category-valued presheaf $\mathbb{P}$ on $\mathbb{G}_{[1,4]}$. This is quite demanding, but we are rewarded with a high-level view of composition, which yields abstract proofs of associativity and unitality, under mild hypotheses on $\mathbb{P}$. We will define a game setting to consist of a set $\mathbb{G}$ and a category-valued presheaf satisfying these hypotheses.

2.3 Notions of strategy

Let us start our reconstruction of a game model from an arbitrary category-valued presheaf $\mathbb{P}$ on $\mathbb{G}_{[1,4]}$. Our first step is to define strategies. Standardly, a strategy $\sigma \colon A \to B$ is a prefix-closed set of plays in $\mathbb{P}_{A,B}$ (generally required to be non-empty). Equivalently, it is a functor $\mathbb{P}^{op}_{A,B} \to 2$, the ordinal 2 viewed as a category. In Tsukada and Ong's model [40], $\mathbb{P}_{A,B}$ is a proper category, and strategies are generalised to *presheaves* on $\mathbb{P}_{A,B}$, i.e., functors $\mathbb{P}^{op}_{A,B} \to \mathrm{Set}$. This is indeed a generalisation because 2 embeds into Set (see Section 2.8). The basis of our approach will be the general notion:

Definition 2.7. Let the category of *strategies* from A to B be $\widehat{\mathbb{P}_{A,B}}$. The category of *boolean strategies* is $\widetilde{\mathbb{P}_{A,B}}$.

2.4 Polynomial functors

The next step in our reconstruction of a game model from $\mathbb{G}$ and $\mathbb{P}$ is to define identities and composition. This will rely on polynomial functors, which we now briefly recall.

Notation 2.8. *Any functor $F \colon \mathcal{C} \to \mathcal{D}$ induces a restriction functor $\Delta_F \colon \widehat{\mathcal{D}} \to \widehat{\mathcal{C}}$ mapping any $X \colon \mathcal{D}^{op} \to \mathrm{Set}$ to $X \circ F^{op}$, where $F^{op} \colon \mathcal{C}^{op} \to \mathcal{D}^{op}$ acts just as F but on opposite categories. When $\mathcal{C}$ and $\mathcal{D}$ are small, this restriction functor has both a left and a right adjoint, which we respectively denote by $\sum_F$ and $\prod_F$, as in*

$$\widehat{\mathcal{C}} \underset{\prod_F}{\overset{\sum_F}{\rightleftarrows}} \widehat{\mathcal{D}}, \qquad \textstyle\sum_F \dashv \Delta_F \dashv \prod_F, \quad \Delta_F \colon \widehat{\mathcal{D}} \to \widehat{\mathcal{C}}.$$

The left and right adjoints are respectively given by left and right extension, and enjoy explicit descriptions, both in terms of *coends* and *ends* and in terms of colimits and limits [27, 38].

Definition 2.9. A functor is *polynomial* iff it is isomorphic to some finite composite of functors of the form Δ_F, $\prod_F$ and $\sum_F$.

This is essentially the closure of Fiore's polynomial functors [13] under composition, though in fact all considered polynomial functors will be so in the restricted sense of Weber [42, Remark 2.12].

2.5 Copycat as a polynomial functor

As a warm-up before considering composition, let us start with our abstract definition of identities, which are standardly given by *copycat* strategies. A natural way to define the copycat strategy $id_A \colon A \to A$ is to decree that it accepts all plays in $\mathbb{P}_{A,A}$ that are in the image of the insertion functor $\iota_0 \colon \mathbb{P}_A \to \mathbb{P}_{A,A}$. Indeed, recalling (1) and according to the discussion of insertions, right after Example 2.4, such plays are precisely those in which M acts as a proxy between L and R, which agrees with the standard definition of copycat strategies.

This definition has the advantage of concreteness, but as announced let us give an equivalent, polynomial definition. Because an object of a category $\mathbb{C}$ is the same as a functor $1 \to \mathbb{C}$, we may define id_A as a functor $1 \to \widehat{\mathbb{P}_{A,A}}$. Furthermore, 1 is a presheaf category: indeed it is $\widehat{\emptyset}$, presheaves over the empty category. So we may view copycat over A as a polynomial functor:

Definition 2.10. Let the *copycat* strategy e_A over any game A be the unique presheaf in the image of $\widehat{\emptyset} \xrightarrow{\prod_!} \widehat{\mathbb{P}_A} \xrightarrow{\Sigma_{\iota_0}} \widehat{\mathbb{P}_{A,A}}$.

In order for this definition to agree with id_A, we will need to assume that the insertion functor $\iota_0 \colon \mathbb{P}_A \to \mathbb{P}_{A,A}$ is a discrete fibration. Let us recall the definition:

Definition 2.11. A functor $p \colon \mathcal{E} \to \mathcal{B}$ is a *discrete fibration* when for all objects $e \in \mathcal{E}$ and morphisms $f \colon b \to p(e)$ there exists a unique morphism $u \colon e' \to e$ such that $p(u) = f$. Such a morphism is called a *cartesian lifting* of e along f. Let $\mathsf{DFib}_{\mathcal{B}}$ denote the full subcategory of $\mathsf{Cat}/\mathcal{B}$ spanning discrete fibrations.

For us, the relevant property of discrete fibrations is the following characterisation of left extension along them:

Lemma 2.12. *For any discrete fibration $p \colon \mathcal{E} \to \mathcal{B}$, presheaf $X \in \widehat{\mathcal{E}}$, and object $b \in \mathcal{B}$, we have $\sum_p(X)(b) \cong \sum_{e \mid p(e)=b} X(e)$, where $\sum$ means left extension on the left and disjoint union on the right.*

We obtain as promised:

Proposition 2.13. *If the insertion functor $\iota_0 \colon \mathbb{P}_A \to \mathbb{P}_{A,A}$ is a discrete fibration, then we have $\mathsf{e}_A \cong id_A$.*

2.6 Composition as a polynomial functor

The next step is to express composition of strategies using the same language of polynomial functors. Let us first recall the standard definition in the boolean case: the composite $\sigma;\tau$ of two boolean strategies σ and τ over (A, B) and (B, C) respectively, is defined to accept all plays $p \in \mathbb{P}_{A,C}$ for which there exists $u \in \mathbb{P}_{A,B,C}$ such that $\delta_1(u) = p$ and

$$\delta_2(u) \in \sigma \qquad\text{and}\qquad \delta_0(u) \in \tau.$$

In [40], this is extended to a functor $\widehat{\mathbb{P}_{A,B}} \times \widehat{\mathbb{P}_{B,C}} \to \widehat{\mathbb{P}_{A,C}}$ that is essentially a proof-relevant version of the above one:

Definition 2.14. The composite $\sigma;\tau$ of two strategies σ and τ, over (A, B) and (B, C) respectively, maps any play $p \in \mathbb{P}_{A,C}$ to the set of triples (u, x, y) where $u \in \mathbb{P}_{A,B,C}$ is such that $\delta_1(u) = p$ and

$$x \in \sigma(\delta_2(u)) \qquad\text{and}\qquad y \in \tau(\delta_0(u)).$$

Let us present this polynomially. First, by universal property of coproduct we have $\widehat{\mathbb{P}_{A,B}} \times \widehat{\mathbb{P}_{B,C}} \cong \widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}}$, so we reduce to defining a functor $\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \to \widehat{\mathbb{P}_{A,C}}$. Here is our candidate:

Definition 2.15. Let $\mathsf{m}_{A,B,C}$ denote the polynomial functor

$$\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \xrightarrow{\Delta_{\delta_2+\delta_0}} \widehat{\mathbb{P}_{A,B,C} + \mathbb{P}_{A,B,C}} \xrightarrow{\prod_{[id,id]}} \widehat{\mathbb{P}_{A,B,C}} \xrightarrow{\Sigma_{\delta_1}} \widehat{\mathbb{P}_{A,C}}.$$

We often omit subscripts in $\mathsf{m}_{A,B,C}$ when clear from context. This definition is legitimated by:

Proposition 2.16. *If δ_1 is a discrete fibration, then* m *agrees with Definition 2.14, i.e., for all σ and τ, we have $(\sigma;\tau) \cong \mathsf{m}[\sigma,\tau]$.*

Proof sketch. For all $p \in \mathbb{P}_{A,C}$, we have:

$$\begin{array}{ll}
\mathsf{m}_{A,B,C}[\sigma,\tau](p) & \\
= \sum_{\delta_1} \left(\prod_{\nabla} \left(\Delta_{\delta_2+\delta_0}[\sigma,\tau]\right)\right)(p) & \\
\cong \sum_{\delta_1(u)=p} \prod_{\nabla} \left(\Delta_{\delta_2+\delta_0}[\sigma,\tau]\right)(u) & \text{(by Lemma 2.12)} \\
\cong \sum_{\delta_1(u)=p} \left(\Delta_{\delta_2+\delta_0}[\sigma,\tau](\mathrm{inl}\, u) \times \Delta_{\delta_2+\delta_0}[\sigma,\tau](\mathrm{inr}\, u)\right) & \\
& \text{(by a result dual to Lemma 2.12)} \\
\cong \sum_{\delta_1(u)=p} \sigma(\delta_2(u)) \times \tau(\delta_1(u)), &
\end{array}$$

where the first $\sum$ denotes left extension along δ_1 and the others disjoint union. □

Remark 2.17. *Discrete fibredness of δ_1 and ι_0 holds in most game models, with the notable exception of the saturated interpretation of AJM games (see Section 3.3).*

2.7 Game settings, associativity, and unitality

We have now expressed copycat strategies and composition abstractly, relying only on the postulated category-valued presheaf. Let us now consider associativity. It has become standard in game semantics to prove associativity of composition using a *zipping* result [6] stating that, for all A, B, C, D, both squares

$$\begin{array}{ccc}
\mathbb{P}_{A,B,C,D} & \xrightarrow{\delta_2} & \mathbb{P}_{A,B,D} \\
{\scriptstyle\delta_0}\downarrow \lrcorner & & \downarrow{\scriptstyle\delta_0} \\
\mathbb{P}_{B,C,D} & \xrightarrow[\delta_1]{} & \mathbb{P}_{B,D}
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{P}_{A,B,C,D} & \xrightarrow{\delta_1} & \mathbb{P}_{A,C,D} \\
{\scriptstyle\delta_3}\downarrow \lrcorner & & \downarrow{\scriptstyle\delta_2} \\
\mathbb{P}_{A,B,C} & \xrightarrow[\delta_1]{} & \mathbb{P}_{A,C}
\end{array}
\qquad (3)$$

are pullbacks. We will require this to hold in game settings, which will suffice for associativity. However, we will need a bit more in order to prove that copycat strategies are identities for composition. Suppose that u in $\mathbb{P}_{A,A,B}$ is such that $\delta_2(u) = \iota_0(s)$ for some sequence s in $\mathbb{P}_A$. Then we intuitively want to have $u = \iota_0(\delta_0(u))$, which does not hold in general. We thus require that both squares

$$\begin{array}{ccc}
\mathbb{P}_{A,B} & \xrightarrow{\delta_1} & \mathbb{P}_{A} \\
{\scriptstyle\iota_0}\downarrow \lrcorner & & \downarrow{\scriptstyle\iota_0} \\
\mathbb{P}_{A,A,B} & \xrightarrow[\delta_2]{} & \mathbb{P}_{A,A}
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{P}_{A,B} & \xrightarrow{\delta_0} & \mathbb{P}_{B} \\
{\scriptstyle\iota_1}\downarrow \lrcorner & & \downarrow{\scriptstyle\iota_0} \\
\mathbb{P}_{A,B,B} & \xrightarrow[\delta_0]{} & \mathbb{P}_{B,B}
\end{array}
\qquad (4)$$

be pullbacks, which is a slight generalisation of this intuition.

Definition 2.18. A *game setting* consists of a set $\mathbb{G}$ (whose elements we call *games*) and a category-valued presheaf $\mathbb{P}$ on $\mathbb{G}_{[1,4]}$ such that all projections $\mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$ and insertions $\mathbb{P}_A \to \mathbb{P}_{A,A}$ are discrete fibrations, and all squares (3) and (4) are pullbacks.

We call the four squares in (3) and (4) the *zipping* squares of $\mathbb{P}$. Our first main result is:

Theorem 2.19. *In any game setting, games and isomorphism classes of strategies form a category with copycats as units.*

2.8 The boolean case

Let us conclude this section by treating the boolean case: until now, our strategies were given by general presheaves (Definition 2.7). We would like to derive from Theorem 2.19 that boolean strategies also form a category.

The bridge to the boolean case is given by the embedding $\mathsf{r} \colon 2 \hookrightarrow \mathsf{Set}$ mapping $0 \leq 1$ to $\emptyset \to 1$. This functor has a left adjoint l mapping $\emptyset$ to 0 and collapsing all non-empty sets to 1. Furthermore, r being fully faithful, we have in fact a full reflection, which induces a further one between presheaves and boolean presheaves:

Proposition 2.20. *For any small category $\mathbb{C}$, post-composition by* l *and* r *yield a full reflection, i.e., an adjunction*

$$[\mathbb{C}^{op}, \mathrm{Set}] \underset{\mathsf{r}_!}{\overset{\mathsf{l}_!}{\rightleftarrows}} [\mathbb{C}^{op}, 2] \quad (\bot)$$

with $\mathsf{r}_!$ *fully faithful. The left adjoint* $\mathsf{l}_!$ *is called* booleanisation.

Because 2 is complete and cocomplete, replacing Set with 2 in Notation 2.8 yields a notion of *boolean* polynomial functor.

Notation 2.21. *Any functor $F \colon \mathcal{C} \to \mathcal{D}$ induces restriction, left extension and right extension functors between boolean presheaf categories $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{D}}$, respectively denoted by $\overline{\Delta_F}$, $\overline{\sum_F}$ and $\overline{\prod_F}$. Accordingly, the boolean version of any polynomial functor P will be denoted by $\overline{P}$.*

We may thus transfer our polynomial definitions of copycat and composition to boolean strategies. Concrete examples of game settings will be considered in Section 3, for which we have:

Proposition 2.22. *In all the game settings of Section 3, $\overline{\mathsf{m}}$ coincides with standard composition.*

As desired, we obtain:

Proposition 2.23. *In any game setting, composition of boolean strategies is associative and unital (on the nose).*

Remark 2.24. *Please note that we have not claimed that boolean composition agrees with general, set-based composition, i.e., commutation of the diagram below.*

$$\begin{array}{ccc} \widetilde{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} & \xrightarrow{\mathsf{m}} & \widetilde{\mathbb{P}_{A,C}} \\ \uparrow{\scriptstyle \mathsf{r}_!} & & \uparrow{\scriptstyle \mathsf{r}_!} \\ \overline{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} & \xrightarrow[\overline{\mathsf{m}}]{} & \overline{\mathbb{P}_{A,C}} \end{array}$$

In fact it does not in general, and this is the main cause for the failure of stability of boolean, innocent strategies under composition (see Section 4.3 and [16, Section 3.7.2]). What does hold, however, is

- *commutation of booleanisation with composition as on the left below (this is the main idea for the proof of Proposition 2.23),*
- *the characterisation of boolean composition given below right, as set-based composition followed by booleanisation.*

$$\begin{array}{ccc} \widetilde{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} & \xrightarrow{\mathsf{m}} & \widetilde{\mathbb{P}_{A,C}} \\ \downarrow{\scriptstyle \mathsf{l}_!} & & \downarrow{\scriptstyle \mathsf{l}_!} \\ \overline{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} & \xrightarrow[\overline{\mathsf{m}}]{} & \overline{\mathbb{P}_{A,C}} \end{array} \qquad \begin{array}{ccc} \widetilde{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} & \xrightarrow{\mathsf{m}} & \widetilde{\mathbb{P}_{A,C}} \\ \uparrow{\scriptstyle \mathsf{r}_!} & & \downarrow{\scriptstyle \mathsf{l}_!} \\ \overline{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} & \xrightarrow[\overline{\mathsf{m}}]{} & \overline{\mathbb{P}_{A,C}} \end{array} \quad (5)$$

Let us move on to exhibit a few concrete game settings. We will return to the boolean case in Section 4.3, to deal with innocence.

3 Applications

In this section, we show that a number of standard game models fit into our framework. In Section 3.1, we consider HON games, in their p-form first, which by the results of Section 2 yields categories of ps and pb-strategies. We then refine our results by considering variants in which some constraints are imposed on strategies (or equivalently plays): a first, local form of constraint is treated in Section 3.2, followed by a slightly more involved form, obtained by enriching games with validity predicates on plays. These variants are shown to form game settings (hence yield categories of ps and pb-strategies). AJM games are considered in Section 3.3, and also shown to form a game setting. Finally, we explain in Section 3.4 why Blass games do not.

3.1 Hyland-Ong games and strategies

Let us now consider HON games in more detail, and show that they form a game setting. We mostly follow Harmer's [16] presentation. For simplicity, we adopt the following innocuous modification of the standard notion of arena:

Definition 3.1. An *arena* is a simple, countable, directed acyclic graph A such that for all vertices m, the lengths of all paths from m to some initial vertex have the same parity. We denote by $\sqrt{A}$ the set of *initial* vertices, or *roots* of A.

In particular, simple, upside-down forests form arenas. The intuition is that vertices of an arena are moves in a two-player game, and that an edge $m \to m'$ in the forest means that m is *enabled*, or *justified* by m'. If the path from m to some root has even length, then O (for *Opponent*) is playing; otherwise P (for *Proponent*) is. E.g., all roots are played by O.

Example 3.2. The boolean type $\mathbb{B}$ may be interpreted as the arena

$$\mathsf{t} \swarrow q \searrow \mathsf{f}.$$

Now that we have defined arenas, let us move on to define plays. The idea, explained at length, e.g., in [29], is that plays are sequences of moves in which O and P take turns. But a subtlety is that moves may be played several times. So for any edge $m \to m'$ in the considered arena, there may be several occurrences of m and m'. We thus decorate sequences of moves with justification pointers matching those of the considered arena.

Definition 3.3. A *justified sequence* on any arena A consists of a natural number $n \in \mathbb{N}$, equipped with maps $f \colon n \to \mathrm{ob}(A)$ and $\varphi \colon n \to \{0\} \uplus n$ (recalling Section 1.3), where $\mathrm{ob}(A)$ denotes the set of vertices of A, such that, for all $i \in n$,

- $\varphi(i) < i$,
- if $\varphi(i) = 0$ then $f(i) \in \sqrt{A}$, and
- if $\varphi(i) \neq 0$, then there is an edge $f(\varphi(i)) \to f(i)$ in A.

Let $\mathbb{P}_A$ denote the poset of justified sequences on A, with prefix ordering (as in Example 2.1).

We will draw justified sequences (n, f, φ) as the sequence of their $f(i)$'s, with arrows to denote φ, as is standard in game semantics.

Example 3.4. Here is a justified sequence in the boolean arena: $q \; q \; \mathsf{t} \; \mathsf{f}$, where times flows to the right.

Recalling (1), game semantics proceeds by letting a middle player M play on two arenas A and B, with specific restrictions. For this, we form the *arrow* arena $A \to B$:

Definition 3.5. For any two arenas A and B, let $A \to B$ denote the arena obtained by taking the disjoint union of A and B as directed graphs, adding an edge $m \to m'$ for all $m \in \sqrt{A}$ and $m' \in \sqrt{B}$ (if B is not empty; otherwise we take $A \to B$ to be empty).

The constraints mentioned above are implemented by considering a subposet of $\mathbb{P}_{A \to B}$:

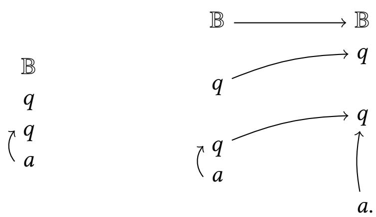
Definition 3.6. For any two arenas A and B, let $\mathbb{P}_{A,B}$ denote the poset of *plays* on (A, B), i.e., *alternating* justified sequences of even length on $A \to B$.

Alternation here means that, for any $s = (n, f, \varphi)$, $f(i)$ is played by O iff i is odd.

We are now in a position to define the insertion functors $\iota_0 : \mathbb{P}_A \to \mathbb{P}_{A,A}$: they send any justified sequence $m_1 \dots m_n$ to some play in which P mimics the behaviour of O. The technical definition is not particularly illuminating, but the example following it should convince the reader that nothing really difficult is going on here.

Definition 3.7. For any justified sequence $p = (m_1 \dots m_n)$ on A, let $\iota_0(p)$ be the sequence $(m_1, k_1)(m_1, l_1) \dots (m_n, k_n)(m_n, l_n)$, where k_i and l_i denote either 0 or 1, according to the component of $A \to A$ in which the move is played. If m_i is an O-move, then $k_i = 1$ and $l_i = 0$; and otherwise $k_i = 0$ and $l_i = 1$. Pointers are as in p except for initial moves on the left, i.e., moves of the form $(m, 0)$ with $m \in \sqrt{A}$, which are justified by the corresponding $(m, 1)$.

Example 3.8. The justified sequence on $\mathbb{B}$ below left, which is not alternating, yields the copycat play on the right:



The next step is interaction sequences, for which the basic idea is: any play in $(A \to B) \to C$ may be projected to $\mathbb{P}_{A \to B}$, $\mathbb{P}_{B \to C}$, and even $\mathbb{P}_{A \to C}$, by prolongating pointers (i.e., $a \to b \to c$ becomes $a \to c$). Following Example 2.3, we put:

Definition 3.9. An *interaction sequence* is a justified sequence on $(A \to B) \to C$ ending in A or C, whose projections to $A \to B$ and $B \to C$ are plays. Let $\mathbb{P}_{A,B,C}$ denote the poset of interaction sequences with prefix ordering.

As desired, the projection to $A \to C$ is also a play, and we have monotone maps $\delta_k : \mathbb{P}_{A_0,A_1,A_2} \to \mathbb{P}_{A_i,A_j}$ with $i < j$ in $\{0, 1, 2\} \setminus \{k\}$.

The category $\mathbb{P}_{A,B,C,D}$ of *generalised interaction sequences* is defined similarly, and we obtain:

Proposition 3.10. *The category-valued presheaf* $\mathbb{P}$ *defined by respectively taking* $\mathbb{P}_A$, $\mathbb{P}_{A,B}$, $\mathbb{P}_{A,B,C}$ *and* $\mathbb{P}_{A,B,C,D}$ *to be the posets of all justified sequences, plays, interaction sequences and generalised interaction sequences, for all arenas* A, B, C, D*, with projections and insertions as above, forms a game setting.*

Proof. Copycat plays form a full subcategory and are closed under prefix, hence insertions $\iota_0 : \mathbb{P}_A \to \mathbb{P}_{A,A}$ are discrete fibrations. Projections $\delta_1 : \mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$ are discrete fibrations: the restriction of any $u \in \mathbb{P}_{A,B,C}$ along any $p \leq \delta_1(u)$ may be taken to be the shortest prefix of u whose projection is p (longer such prefixes do not end in A or C). The fact that squares (3) (resp. (4)) are pullbacks is a variation on the standard zipping lemma (resp. obvious). □

3.2 Constraining strategies

In the previous section, we consider a rather rough notion of play. Standardly, further constraints are considered on strategies, such as P-visibility, O-visibility, well-threadedness, and well-bracketing (when arenas are equipped with an appropriate question-answer discipline). E.g., a P-visible strategy is one which only accepts P-visible plays. One then needs to prove that such constraints are *robust*, i.e., are preserved by composition and satisfied by identities. This is done in a very clean and modular way in Harmer's thesis [16, Chapter 3]. In order for our framework to apply to such constrained strategies, we may start from the game setting for unconstrained plays and convert the proof of robustness of constraint c into the construction of a sub-game setting $\mathbb{P}^c$. Very briefly:

Proposition 3.11. *For any set* $c \subseteq \{P\text{-}vis, O\text{-}vis, wb, wt\}$ *(for* P*-visibility,* O*-visibility, well-bracketing and well-threadedness), if* $(O\text{-}vis \in c) \Rightarrow (P\text{-}vis \in c)$*, then* c *gives rise to a game setting* $\mathbb{P}^c$ *and an embedding* $\mathrm{c} : \mathbb{P}^c \hookrightarrow \mathbb{P}$ *of category-valued presheaves. Therefore, for any such set of constraints* c*, arenas and strategies satisfying these constraints form a category.*

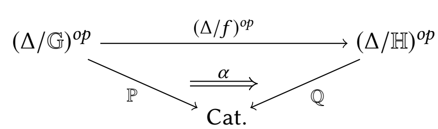
Beyond the constraints mentioned above, a similar result may be proved for the refined notion of game in McCusker's thesis [28]. McCusker's games A are just like arenas, except that they come equipped with an abstract *validity* predicate P_A, which is a subset of the set L_A of *legal plays*, i.e., alternating, well-bracketed, P- and O-visible justified sequences. This predicate should satisfy a few technical conditions. McCusker then defines $P_{A \to B}$ to consist of legal plays in $L_{A \to B}$ whose projections to A and B are in P_A and P_B (instead of simply L_A and L_B), respectively. He finally proceeds in a similar way to define interaction sequences and generalised interaction sequences.

In order to organise McCusker's games into a game setting, we should use as a base not mere arenas, but the set $\mathbb{G}^p$ of pairs (A, P_A) of an arena and a predicate on legal plays satisfying the above conditions. From the first projection $\mathrm{p} : \mathbb{G}^p \to \mathbb{G}$, we derive a functor $\Delta/\mathrm{p} : \Delta/\mathbb{G}^p \to \Delta/\mathbb{G}$, and for any $L \in (\Delta/\mathbb{G}^p)$, we define $\mathbb{P}^p_L$ to be the full subcategory of $\mathbb{P}_{(\Delta/\mathrm{p})(L)}$ spanning plays whose projections satisfy the required predicates. We thus obtain:

Proposition 3.12. *The pair* $(\mathbb{G}^p, \mathbb{P}^p)$ *forms a game setting.*

Beyond Propositions 3.11 and 3.12, we would like to prove that composition and identity in the constrained game settings agree with the original. Let us do this, by considering the general case.

Definition 3.13. Given game settings $(\mathbb{G}, \mathbb{P})$ and $(\mathbb{H}, \mathbb{Q})$, a *morphism* between them consists of a pair of a map $f : \mathbb{G} \to \mathbb{H}$ and a natural embedding



(6)

In such a situation, the functor $\sum_{\alpha_{A,B}} : \widehat{\mathbb{P}_{A,B}} \to \widehat{\mathbb{Q}_{f(A),f(B)}}$ maps strategies in the sense of $(\mathbb{G}, \mathbb{P})$ to strategies in the sense of $(\mathbb{H}, \mathbb{Q})$. Let us prove that under mild hypotheses this functor commutes with composition.

Definition 3.14. The tuple $(\mathbb{G}, \mathbb{P}, f, \alpha)$ above forms a *local* morphism on $(\mathbb{H}, \mathbb{Q})$ iff (1) $\alpha_{A,B} : \mathbb{P}_{A,B} \to \mathbb{Q}_{f(A),f(B)}$ is a discrete fibration for all $A, B \in \mathbb{G}$ and (2) any $u \in \mathbb{Q}_{f(A),f(B),f(C)}$ is essentially in the image of α if $\delta_2(u)$ and $\delta_0(u)$ are.

Remark 3.15. *Locality may be expressed as a sheaf condition.*

Proposition 3.16. *For any local morphism* $\alpha : \mathbb{P} \hookrightarrow \mathbb{Q} \circ (\Delta/f)^{op}$*, the following square commutes up to isomorphism.*

$$\begin{array}{ccc}
\widehat{\mathbb{P}_{A,B}} \times \widehat{\mathbb{P}_{B,C}} & \xrightarrow{\mathrm{m}^{\mathbb{G}}} & \widehat{\mathbb{P}_{A,C}} \\
{\scriptstyle \Sigma_{\alpha_{A,B}} \times \Sigma_{\alpha_{B,C}}}\downarrow & & \downarrow{\scriptstyle \Sigma_{\alpha_{A,C}}} \\
\widehat{\mathbb{Q}_{f(A),f(B)}} \times \widehat{\mathbb{Q}_{f(B),f(C)}} & \xrightarrow[\mathrm{m}^{\mathbb{H}}]{} & \widehat{\mathbb{Q}_{f(A),f(C)}}
\end{array}$$

Corollary 3.17. *Composition in $\mathbb{P}^c$ (resp. $\mathbb{P}^p$) commutes with embedding into $\mathbb{P}$, for all sets of constraints c as in Proposition 3.11.*

There are other kinds of constraints like innocence or single-threadedness, which may not be treated this way. We will deal with innocence in Section 4.

3.3 AJM games: a partial answer

Let us now briefly consider AJM games [5], an alternative approach to game semantics. On the one hand, this approach is more elementary than HON's in that games do not feature justification pointers. So, e.g., composition of strategies is significantly simpler. On the other hand, games feature a partial equivalence relation between plays, which needs to be dealt with at the level of strategies.

In order to organise such games into a game setting, we have two sensible choices for the notion of morphism between plays: beyond the prefix ordering, we may also incorporate equivalence between plays. Presheaves then amount to so-called *saturated* strategies. We adopt Harmer's presentation [16].

Definition 3.18. A *game A* consists of two sets O_A and P_A, respectively of *Opponent* and *Proponent moves*, equipped with a partial, prefix-closed equivalence relation $\approx$ on alternating sequences of moves started by Opponent, for which any two equivalent plays have the same length, and such that

if $s \approx t$ and $sa \approx sa$, then there exists a' such that $sa \approx ta'$.

Let $\mathbb{P}_A$ consist of all alternating sequences of moves s started by Opponent, such that $s \approx s$. Then, for any games A and B, we form the game $A \to B$, which has $O_{A\to B} = P_A + O_B$ and $P_{A\to B} = O_A + P_B$ and $s \approx_{A\to B} t$ iff s and t play in the same component at each stage and their projections are equivalent in A, resp. B. The poset $\mathbb{P}_{A,B}$ may then be defined as the set of plays of even length in $A \to B$ equipped with the prefix ordering. We define $\mathbb{P}_{A,B,C}$ and $\mathbb{P}_{A,B,C,D}$ similarly, but restricting to plays of even length ending in the rightmost or leftmost game. This slightly departs from standard definitions, but yields the same composition functor. We get:

Proposition 3.19. *AJM games form a game setting.*

Proof. Squares (3) (resp. (4)) being pullbacks is again a variation on the standard zipping lemma (resp. obvious). Showing that projections $\delta_1 : \mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$ are discrete fibrations follows just as for HON games. □

For saturated strategies, the idea is to incorporate for all A, B the partial equivalence relations $\approx_A$ and $\approx_B$ into the category of plays.

Proposition 3.20. *AJM games form a category-valued presheaf by mapping each list of games to the corresponding set of plays with as morphisms between any two plays u and v:*

- *a singleton when there exists some play w such that $u \approx w \leq v$, or equivalently there exists w such that $u \leq w \approx v$;*
- *none otherwise.*

However, the obtained category-valued presheaf is not a game setting, because projections $\mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$ and insertions $\mathbb{P}_A \to \mathbb{P}_{A,A}$ are not discrete fibrations in general. We hint in Section 5 at a potential generalisation of our results to cover this example.

3.4 A non-example: Blass games

In the previous sections, we have shown that several approaches to game semantics form game settings, with the exception of the saturated AJM setting. It may be instructive to consider Blass games [7, 8], as they are well-known for their non-associative composition. Our account essentially follows Abramsky [3, Section 3], through the lens of game settings.

Definition 3.21. A Blass game consists of a family of rooted trees, together with a *polarity* in $\{P, O\}$.

Vertices are thought of as positions in the game, with alternating polarities. The given polarity indicates which player is to start the game, by choosing the initial position. The fact that Proponent may start is a notable difference with arena games. Another difference is that the given family of trees genuinely represents the 'game tree' – no move may be played twice. This determines the definition of $\mathbb{P}_A$, for any game $A = (T, \pi)$: it is the poset consisting of positions (i.e., vertices of T, plus a formal initial position), with $x \leq y$ when x is above y in T.

For $\mathbb{P}_{A,B}$ things become a bit more complicated. Strategies should be based on the *linear implication* game $A \to B$, which is constructed much as in, e.g., AJM games. First, let $A^\perp$ denote the game with the same family of trees as A but with opposite polarity. Then, define $A \to B$ by interleaving moves from $A^\perp$ and B with natural switching conditions: Opponent is to play as soon as possible. In other words, if the respective polarities in $A^\perp$ and B are *OP*, *PO*, or *OO*, then O is to play; otherwise P is. There is a catch, however: if the polarity is *OO* and Opponent plays, say in B, we reach a position with polarity *OP*, and Opponent is to play again, which breaks alternation. This is rectified by defining $A \to B$ to comprise *compound* moves from *OO* to *PP*, for each pair of moves in $A^\perp$ and B. This settles the definition of $\mathbb{P}_{A,B}$, up to some technicalities.

The next level is to define $\mathbb{P}_{A,B,C}$. Glossing over the details, this should consist of sequences of moves in A, B and C, whose projections to $\mathbb{P}_{A,B}$, $\mathbb{P}_{B,C}$ and $\mathbb{P}_{A,C}$ are well-defined. However, we may show that with these definitions, the squares (3) cannot both be pullbacks in general. Indeed, consider the case where the respective polarities of A, B, C and D are O, P, O and P, and A is non-empty. Then, let $\mathbb{P}^l_{A,B,C,D}$ denote the left-hand pullback and $\mathbb{P}^r_{A,B,C,D}$ denote the right-hand one. We will show that both pullbacks cannot be the same category by exhibiting a play in $\mathbb{P}^l_{A,B,C,D}$ which is not in $\mathbb{P}^r_{A,B,C,D}$. First, let us observe that the initial polarities from the respective points of view of $A \to B$, $B \to C$ and $C \to D$ are like so:

$$\begin{array}{cccccc}
A & \longrightarrow B & B & \longrightarrow C & C & \longrightarrow D \\
A^\perp & B & B^\perp & C & C^\perp & D \\
P & P & O & O & P & P.
\end{array}$$

Letting a denote any root of A, the sequence a is then legal in $\mathbb{P}_{A,B,D}$ (the polarities are *PP* both in $A \to B$ and $A \to D$) and the empty sequence is legal in $\mathbb{P}_{B,C,D}$. Thus, a is legal in $\mathbb{P}^l_{A,B,C,D}$ by the left-hand pullback. However, if the two pullbacks were isomorphic, then by the properties of projections $a \in \mathbb{P}^l_{A,B,C,D}$ would be mapped to $a \in \mathbb{P}^r_{A,B,C,D}$ under the isomorphism. But $\mathbb{P}^r_{A,B,C,D}$ cannot contain a because this play is illegal in $\mathbb{P}_{A,B,C}$ (because the polarity is *PO* in $A \to C$).

4 Innocence

4.1 Concurrent innocence

In the previous sections, we have constructed a category of games and strategies parameterised over an arbitrary game setting which unifies a number of such categories as instances of the same construction. However, in game models of purely functional languages, the relevant category is the identity-on-objects subcategory of innocent strategies. We now extend game settings with a notion of view, which allows us to construct a subcategory of innocent strategies.

In order to achieve this, we will use the recent recasting of innocence as a sheaf condition [22, 40]. Starting from HON games, the first step is to refine the posets $\mathbb{P}_A, \mathbb{P}_{A,B}, \ldots$ into proper categories (with exactly the same objects), say $\mathbb{P}^+_A, \mathbb{P}^+_{A,B}, \ldots$, with the crucial feature that for any play $p \in \mathbb{P}^+_{A,B}$ and move $m \in p$, there is a morphism $\lceil p \rceil_m \to p$ from the so-called *P-view* of m to p.[1] This of course does not hold with the prefix ordering, as the view is rarely a prefix. This idea was introduced by Melliès [31] in a slightly different setting.

Passing from $\mathbb{P}$ to $\mathbb{P}^+$ raises the issue of how to extend the abstract framework. Should it now contain two category-valued presheaves? Or should we simply forget about prefix-based strategies and accept $\mathbb{P}^+$ as the new basic set up? We do not make any definitive choice here, but for simplicity and modularity reasons, we choose to first work with $\mathbb{P}^+$ only, and introduce $\mathbb{P}$ in a second stage.

Indeed, perhaps surprisingly, we have:

Proposition 4.1. *Tsukada and Ong's $\mathbb{P}^+$ forms a game setting.*

Proof. Follows from Lemmas 39, 46 and 47 of [39]. □

Returning to the abstract setting, the new data thus merely consists of a full subcategory $\mathrm{i}_{A,B} : \mathbb{V}_{A,B} \hookrightarrow \mathbb{P}_{A,B}$, for all A, B, whose objects are called *views*.

Definition 4.2. The category of *innocent* strategies is the essential image of $\prod_{\mathrm{i}_{A,B}} : \widehat{\mathbb{V}_{A,B}} \to \widehat{\mathbb{P}_{A,B}}$. The domain $\widehat{\mathbb{V}_{A,B}}$ is the category of *behaviours*.

The reason for using the term 'sheaf' in this context (and the only thing the reader will need to know about them) is:

Proposition 4.3. *Each embedding $\mathrm{i}_{A,B}$ induces a (Grothendieck) topology on $\mathbb{P}_{A,B}$ such that $\prod_{\mathrm{i}_{A,B}}$ is equivalent to the embedding of sheaves into presheaves.*

We now would like to establish that in any game setting equipped with such full embeddings, innocent strategies form a subcategory. However, our proof relies on two additional properties. The first, already observed in [40, Lemma 32], states that one can reconstruct uniquely any interaction sequence from its projection to $\mathbb{P}_{A,C}$, say u, together with a compatible family, for each view v of u of an interaction sequence projecting to v. The second property essentially says that any morphism $v \to \delta_2(u)$ from a view $v \in \mathbb{V}_{A,B}$ to the projection of some $u \in \mathbb{P}_{A,B,C}$ factors canonically through the projection of some view (and similarly for δ_0).

Let us introduce both properties in more detail.

The first property essentially says that interaction is local. To state it, we need to recall the following well-known fact (recalling Definition 2.11).

[1] We omit the definition of views, as it is unnecessary for understanding the rest.

Proposition 4.4. *There is an adjoint equivalence*

$$\widehat{\mathbb{C}} \underset{\text{sing}}{\overset{\pi}{\rightleftarrows}} \mathsf{DFib}_{\mathbb{C}}, \quad (\bot) \qquad \textit{where}$$

- *$\pi(X)\colon \mathrm{el}(X) \to \mathbb{C}$ denotes the projection (indeed a discrete fibration) from the* category of elements $\mathrm{el}(X)$ *of any $X \in \widehat{\mathbb{C}}$, which has as objects all pairs (c, x) with $x \in X(c)$ and as morphisms $(c, x) \to (c', x')$ all morphisms $f : c \to c'$ such that $x' \cdot f = x$;*
- *$\mathrm{sing}(p)$ denotes the presheaf $c \mapsto p^{-1}(c)$, for any discrete fibration $p\colon \mathbb{E} \to \mathbb{C}$ and $c \in \mathbb{C}$, with action on morphisms given by cartesian lifting.*

Let us now state the first property we need to impose on game settings with embeddings $\mathrm{i}_{A,B} : \mathbb{V}_{A,B} \hookrightarrow \mathbb{P}_{A,B}$. The projection $\delta_1\colon \mathbb{P}_{A,B,C} \to \mathbb{P}_{A,C}$, as a discrete fibration, induces a presheaf $\mathrm{sing}(\delta_1)$ on $\mathbb{P}_{A,C}$ which we will require to be a sheaf for the topology induced by the embedding $\mathbb{V}_{A,C} \to \mathbb{P}_{A,C}$. By Proposition 4.3, this is equivalent to being in the essential image of $\prod_{\mathrm{i}_{A,C}}$. Similarly, we require the presheaf induced by $\iota_0\colon \mathbb{P}_A \to \mathbb{P}_{A,A}$ to be a sheaf for the topology induced by the embedding $\mathbb{V}_{A,A} \to \mathbb{P}_{A,A}$.

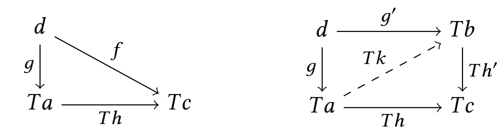
Definition 4.5. A game setting $(\mathbb{G}, \mathbb{P})$ with full embeddings $\mathrm{i}_{A,B}\colon \mathbb{V}_{A,B} \hookrightarrow \mathbb{P}_{A,B}$ is *local* iff $\mathrm{sing}(\delta_1)$ and $\mathrm{sing}(\iota_0)$ are sheaves.

Proposition 4.6. *Tsukada and Ong's $\mathbb{P}^+$ is local.*

Proof. For δ_1, the result is precisely [40, Lemma 32]. For ι_0, just observe that a play is copycat iff all its views are. □

So locality is the first property we need to require of our game settings with views. The second property has to do with projections $\delta_0\colon \mathbb{P}_{A,B,C} \to \mathbb{P}_{B,C}$ and $\delta_2\colon \mathbb{P}_{A,B,C} \to \mathbb{P}_{A,B}$. For δ_2, it essentially says that any morphism $v \to \delta_2(u)$ with $v \in \mathbb{V}_{A,B}$ and $u \in \mathbb{P}_{A,B,C}$ factors 'canonically' through some $\delta_2(w)$ with $w \in \mathbb{V}_{A,B,C}$, where $\mathbb{V}_{A,B,C}$ denotes the full subcategory of $\mathbb{P}_{A,B,C}$ projecting to $\mathbb{V}_{A,C}$ (or otherwise said, $\mathbb{V}_{A,B,C} = \mathbb{P}_{A,B,C} \times_{\mathbb{P}_{A,C}} \mathbb{V}_{A,C}$). In order to define such canonicity, we appeal to the theory of analytic functors [25, 41, 42].

Definition 4.7 (Weber [41, 42])**.** A functor $T\colon \mathbb{C} \to \mathbb{D}$ admits *generic factorisations* relative to an object $d \in \mathbb{D}$ iff any $f : d \to Tc$ admits a factorisation as below left



such that for all commuting squares as the exterior above right, there exists a lifting k as shown making the diagram commute, or more precisely such that $g' = Tk \circ g$ and $h = h'k$. The middle object a is called the *arity* of f – all generic factorisations share the same a up to isomorphism.

For all subcategories $\mathbb{B} \hookrightarrow \mathbb{C}$ and $\mathbb{E} \hookrightarrow \mathbb{D}$, a functor $\mathbb{C} \to \mathbb{D}$ admitting generic factorisations relative to all objects of $\mathbb{E}$ with arities in $\mathbb{B}$ is called $(\mathbb{B}, \mathbb{E})$-*analytic* [14].

Definition 4.8. A game setting $(\mathbb{G}, \mathbb{P})$ equipped with full embeddings $\mathrm{i}_{A,B}\colon \mathbb{V}_{A,B} \hookrightarrow \mathbb{P}_{A,B}$ is *view-analytic* iff δ_2 is $(\mathbb{V}_{A,B,C}, \mathbb{V}_{A,B})$-analytic and δ_0 is $(\mathbb{V}_{A,B,C}, \mathbb{V}_{B,C})$-analytic.

Proposition 4.9. *Tsukada and Ong's $\mathbb{P}^+$ is view-analytic.*

We may now state our main result about innocence:

Definition 4.10. An *innocent game setting* is a game setting $(\mathbb{G}, \mathbb{P})$ equipped with full embeddings $\mathsf{i}_{A,B} \colon \mathbb{V}_{A,B} \hookrightarrow \mathbb{P}_{A,B}$, which is both local and view-analytic.

Theorem 4.11. *In any innocent game setting, innocent strategies form a subcategory.*

Proof sketch. First of all, in most proofs, we use a slightly different, yet isomorphic, definition of composition: instead of $\mathsf{m}_{A,B,C}$ as in Definition 2.15, we use the polynomial functor

$$\widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} \xrightarrow{\Pi} \widehat{\mathbb{P}_{(A,B),(B,C)}} \xrightarrow{\Delta} \widehat{\mathbb{P}_{A,B,C}} \xrightarrow{\Sigma_{\delta_1}} \widehat{\mathbb{P}_{A,C}},$$

where $\mathbb{P}_{(A,B),(B,C)}$ intuitively embeds and relates $\mathbb{P}_{A,B}$, $\mathbb{P}_{B,C}$ and $\mathbb{P}_{A,B,C}$. E.g., it contains a morphism $\delta_2(u) \to u$, for all $u \in \mathbb{P}_{A,B,C}$.

Returning to the theorem, because a strategy X, say on (A, B), is innocent iff it is essentially in the image of $\prod_{\mathsf{i}_{A,B}}$, it suffices to show that the perimeter of

$$\begin{array}{ccccccc} \widehat{\mathbb{V}_{A,B} + \mathbb{V}_{B,C}} & \xrightarrow{\Pi} & \widehat{\mathbb{V}_{(A,B),(B,C)}} & \xrightarrow{\Delta} & \widehat{\mathbb{V}_{A,B,C}} & \xrightarrow{\Sigma} & \widehat{\mathbb{V}_{A,C}} \\ \Pi \downarrow & & \Pi\downarrow & & \Pi\downarrow & & \downarrow\Pi \\ \widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} & \xrightarrow[\Pi]{} & \widehat{\mathbb{P}_{(A,B),(B,C)}} & \xrightarrow[\Delta]{} & \widehat{\mathbb{P}_{A,B,C}} & \xrightarrow[\Sigma]{} & \widehat{\mathbb{P}_{A,C}} \end{array} \qquad (7)$$

commutes up to isomorphism. We proceed by tiling the diagram as above, which requires the introduction of $\mathbb{V}_{(A,B),(B,C)}$, similar to $\mathbb{P}_{(A,B),(B,C)}$ but with views instead of general plays.

Now, the left square commutes because the underlying square does, and so does the middle square by [15, Theorem 1.2], using view-analyticity. Our main proofs all follow this pattern: after tiling the considered diagram, we get a few commutations for free, a few more from [15], and are then left with a distributivity result to prove, in the obvious sense of $\prod$ and $\sum$ commuting. We have designed a sufficient condition for this to hold, which covers all needed instances [12]. In the present case, the condition reduces to $\mathrm{sing}(\delta_1)$ being a sheaf, which holds by locality. □

4.2 Prefix-based innocence

In the previous section, we have shown that innocent strategies behave well in any innocent game setting. However, our only concrete example of an innocent game setting for now is Tsukada and Ong's $\mathbb{P}^+$. There is in fact a further example, given by enriching arenas with bracketing information and restricting $\mathbb{P}^+_{A,B}$ to well-bracketed plays [40, Section VII]. This shows that innocence is stable under cs-composition. How about pb-composition? As mentioned before, innocence is not stable under pb-composition in general [16, Section 3.7.2]. In an attempt to better understand this phenomenon, we first move in this section from cs-composition to ps-composition, and prove that innocence remains stable. In the next section, we will explain why this does not carry over to pb-composition, although, as is well-known, it does on deterministic strategies.

We here proceed by first defining innocent, prefix-based strategies in an extended framework and then showing that our definition agrees with the standard one (which is only defined on boolean behaviours). We then show that ps-innocent strategies include copycats and are closed under composition.

Definition 4.12. Consider game settings $(\mathbb{G}, \mathbb{P}^+)$ and $(\mathbb{G}, \mathbb{P})$ with the same set of games and $\mathbb{V}$ making $\mathbb{P}^+$ innocent, further equipped with a componentwise identity-on-objects natural embedding $\mathsf{k} \colon \mathbb{P} \hookrightarrow \mathbb{P}^+$ such that $\mathsf{i} \colon \mathbb{V} \hookrightarrow \mathbb{P}^+$ factors through k. Let a presheaf on $\mathbb{P}_{A,B}$ be *innocent via* $\mathbb{P}^+$, or $\mathbb{P}^+$*-innocent*, iff it is in the essential image of $\widehat{\mathbb{V}_{A,B}} \xrightarrow{\prod_{\mathsf{i}_{A,B}}} \widehat{\mathbb{P}^+_{A,B}} \xrightarrow{\Delta_{\mathsf{k}_{A,B}}} \widehat{\mathbb{P}_{A,B}}$. Similarly, let a presheaf on $\mathbb{P}_{A,A}$ be $\mathbb{P}^+$*-copycat* iff it is in the essential image of $1 \cong \widehat{\emptyset} \xrightarrow{\prod_!} \widehat{\mathbb{P}^+_A} \xrightarrow{\Sigma_{\iota_0}} \widehat{\mathbb{P}^+_{A,A}} \xrightarrow{\Delta_{\mathsf{k}_{A,A}}} \widehat{\mathbb{P}_{A,A}}$.

In such a setting, one could consider studying $(\mathbb{V}, \mathbb{P})$ directly. However, it will rarely form an innocent game setting (essentially only when k is an isomorphism). Let us mention the following sanity check, denoting composition in $\mathbb{P}^+$ by m^+.

Proposition 4.13. *Composition in the game setting $\mathbb{P}$ agrees with composition in $\mathbb{P}^+$, in the sense that for all games A, B, C, the following square commutes up to isomorphism.*

$$\begin{array}{ccc} \widehat{\mathbb{P}^+_{A,B} + \mathbb{P}^+_{B,C}} & \xrightarrow{\mathsf{m}^+_{A,B,C}} & \widehat{\mathbb{P}^+_{A,C}} \\ \Delta_{\mathsf{k}_{A,B}+\mathsf{k}_{B,C}} \downarrow & & \downarrow \Delta_{\mathsf{k}_{A,C}} \\ \widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} & \xrightarrow[\mathsf{m}_{A,B,C}]{} & \widehat{\mathbb{P}_{A,C}} \end{array}$$

Proposition 4.14. *A strategy in the standard HON sense is innocent iff it is $\mathbb{P}^+$-innocent. It is copycat iff it is $\mathbb{P}^+$-copycat.*

By the proposition, we may understand ps-innocence through cs-innocence. Let us now state the transfer result.

Proposition 4.15. *In the setting of Definition 4.12, if for all A, B, C*

$$\begin{array}{ccc} \mathbb{P}_{A,B,C} & \longrightarrow & \mathbb{P}_{A,C} \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{P}^+_{A,B,C} & \longrightarrow & \mathbb{P}^+_{A,C} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbb{P}_{A} & \longrightarrow & \mathbb{P}_{A,A} \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{P}^+_{A} & \longrightarrow & \mathbb{P}^+_{A,A} \end{array}$$

are pullbacks, then $\mathbb{P}^+$-innocent strategies are closed under composition and comprise $\mathbb{P}^+$-copycat strategies.

Of course, both hypotheses are satisfied by HON games.

4.3 Boolean innocence

We finally consider boolean innocence. As mentioned before, innocent pb-strategies are not closed under composition. One usually either imposes a further determinism constraint, or relaxes the innocence constraint. It might be instructive to see how trying to derive the boolean case from the set-based one using our methods directly points to the problem.

Indeed, suppose given any innocent game setting $(\mathbb{G}, \mathbb{P}, \mathbb{V}, \mathsf{i})$ (though the argument also applies in the setting of Section 4.2). We would like to show that two boolean polynomial functors, say

$$\overline{P_1}, \overline{P_2} \colon \widehat{\mathbb{V}_{A,B} + \mathbb{V}_{B,C}} \to \widehat{\mathbb{P}_{A,C}}$$

coincide. Here $\overline{P_1}$ is right extension to plays followed by composition and $\overline{P_2}$ is the same, followed by innocentisation (restriction to views, then right extension to plays). Following the proof method of Proposition 2.23, we consider the commutation of

$$\begin{array}{ccccc} \widehat{\mathbb{V}_{A,B} + \mathbb{V}_{B,C}} & \xrightarrow{\prod_{\mathsf{i}_{A,B}+\mathsf{i}_{B,C}}} & \widehat{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} & \xrightarrow{Q_i} & \widehat{\mathbb{P}_{A,C}} \\ \mathsf{l}_! \downarrow & & \downarrow \mathsf{l}_! & & \downarrow \mathsf{l}_! \\ \overline{\mathbb{V}_{A,B} + \mathbb{V}_{B,C}} & \xrightarrow[\prod_{\mathsf{i}_{A,B}+\mathsf{i}_{B,C}}]{} & \overline{\mathbb{P}_{A,B} + \mathbb{P}_{B,C}} & \xrightarrow[Q_i]{} & \overline{\mathbb{P}_{A,C}}, \end{array}$$

where Q_1 is composition and Q_2 is the same followed by innocentisation. In particular, for this proof method to work, we need innocentisation to commute with booleanisation, which does not hold in general, as already observed in [40, Section VII.A].

Another possible proof strategy would be to show that each $\overline{P_i}$ factors as $\mathbb{V}_{\widecheck{A,B} + \mathbb{V}_{B,C}} \xrightarrow{\mathsf{r}_!} \mathbb{V}_{\widehat{A,B} + \mathbb{V}_{B,C}} \xrightarrow{P_i} \widehat{\mathbb{P}_{A,C}} \xrightarrow{\mathsf{l}_!} \widecheck{\mathbb{P}_{A,C}}$, where $\mathsf{r}_!$ and $\mathsf{l}_!$ are as in Proposition 2.20. Indeed, because we have already shown that $P_1 \cong P_2$, we would then automatically get $\overline{P_1} \cong \overline{P_2}$ as desired. However, this fails because the square

$$\begin{array}{ccc} \widehat{\mathbb{P}_{A,B,C}} & \xrightarrow{\sum \delta_1} & \widehat{\mathbb{P}_{A,C}} \\ {\scriptstyle \mathsf{r}_!}\uparrow & & \uparrow{\scriptstyle \mathsf{r}_!} \\ \widecheck{\mathbb{P}_{A,B,C}} & \xrightarrow[\overline{\sum \delta_1}]{} & \widecheck{\mathbb{P}_{A,C}} \end{array}$$

does not commute in general.

Remark 4.16. *Boolean composition disagrees with set-based composition in general for much the same reason (as noted in Remark 2.24).*

Standardly, the problem is overcome by restricting to deterministic strategies, for which innocentisation does commute with booleanisation. Similarly, copycats are deterministic, so we have:

Proposition 4.17. *In any innocent game setting* $(\mathbb{G}, \mathbb{P}, \mathbb{V}, \mathrm{i})$*, boolean copycat strategies are innocent. This extends to the setting of Proposition 4.15, so that, as is standard, copycats are innocent* pb*-strategies.*

5 Conclusion and perspectives

We have introduced game settings and their innocent variant, a categorical framework aiming at unifying existing game models and facilitating the construction of new ones. A lot remains to be done, starting with covering more game models. Although we have not covered them here, we know that *simple games*, which are the basis of [17], do form a game setting. Furthermore, the saturated view of AJM games (Section 3.3) seems at hand, though it will involve significantly more advanced category theory, as Street fibrations and stacks will replace discrete fibrations and sheaves. Less obvious is the treatment of more exotic game models [10, 31–34, 37]. We also plan to go beyond mere categories of games and strategies and construct structured categories of various kinds, depending on the considered language. These could be, e.g., cartesian closed, symmetric monoidal closed, linear, or Freyd categories. Another direction is categorification: instead of reasoning up to isomorphism, we could refine our point of view and prove that games and strategies in fact form a bicategory, as, e.g., in [37]. Finally, beyond game models, we should investigate game semantics, i.e., the correspondence with operational semantics, as initiated in [11] in a different setting.

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