Tree-depth, quantifier elimination, and quantifier rank

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Abstract

For a class K of finite graphs we consider the following three statements. (i) K has bounded tree-depth. (ii) First-order logic FO has an effective generalized quantifier elimination on K. (iii) The parameterized model checking for FO on K is in para-AC^0. We prove that (i) $\Rightarrow$ (ii) and (ii) $\Leftrightarrow$ (iii). All three statements are equivalent if K is closed under taking subgraphs, but not in general.

By a result due to Elberfeld et al. [12] monadic second-order logic MSO and FO have the same expressive power on every class of graphs of bounded tree-depth. Hence the implication (i) $\Rightarrow$ (iii) holds for MSO, too; it is the analogue of Courcelle's Theorem for tree-depth (instead of tree-width) and para-AC^0 (instead of FPT). In [13] it was already shown that the model-checking for a fixed MSO-property on a class of graphs of bounded tree-depth is in AC^0.

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1. Introduction

If a problem Q can be defined by a sentence φ of first-order logic FO, then Q can be computed by a family of circuits $(\mathsf{C}_n)_{n\in\mathbb{N}}$ of bounded depth and size polynomial in n. Even more, the depth of the circuits C_n can be linearly bounded by the quantifier rank $\mathrm{qr}(\varphi)$ of the formula φ. Therefore, $O(\mathrm{qr}(\varphi))$ bounds the running time of a *parallel* computer simulating the circuits C_n. Hence, finding an FO-sentence with small quantifier rank defining Q has pleasant computational consequences.

Let us look at the *k-vertex-cover problem*, where k is a natural number. That is, given a graph $\mathcal{G}$, we want to decide whether there is a k-vertex-cover, i.e., a set of k vertices of $\mathcal{G}$ which contains at least one end of every edge in $\mathcal{G}$. It is straightforward to write an FO-sentence of quantifier rank $k+2$ for the k-vertex-cover problem. Perhaps surprisingly, in [10] it was shown that (for every k) this can be improved to quantifier rank 17 (see below for the precise statement). The main difficulty was to show how to express the existence of a k-element subset with a small number of quantifiers. Other graph-theoretic problems have been shown to exhibit similar phenomena [2, 4, 10].

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Central to this paper is the question for which classes K of graphs

every FO*-definable problem can be defined by an* FO*-formula φ whose quantifier rank is bounded by a constant depending only on K?* (1)

In classical model theory "in a few fortunate cases" [7] a first-order theory (or equivalently, its class K of models) allows the elimination of quantifiers (i.e., we can choose 0 as constant in (1)). The theory of the ordered field of the reals or the theory known as Presburger arithmetic are prominent examples.

For the class of all graphs without edges already the existence of at least k vertices cannot be expressed by an FO-sentence of quantifier rank $k-1$. It is reasonable (see below) to circumvent this difficulty by allowing in (1) the use of built-in arithmetic. If then (1) holds for the class K, we say that FO *has generalized quantifier elimination on K* (see Definition 3.3 for the precise definition).

Already for the result on the k-vertex-cover problem mentioned above we use built-in arithmetic: In [10] a sentence φ_k of quantifier rank 17 is presented such that for every graph $\mathcal{G}$ with vertex set $G=[n]$ $(:=\{0,1,\dots,n-1\})$,

$$\mathcal{G} \text{ has a } k\text{-vertex-cover} \iff (\mathcal{G},<,+,\times,1,\dots,k') \models \varphi_k, \qquad (2)$$

where $<$ is the natural ordering on $[n]$ and $+$ and $\times$ are ternary relations representing addition and multiplication on $[n]$, and where $1,\dots,k'$ are constants with k' only depending on k.

Built-in arithmetic is a basic tool in the study of the descriptive complexity of FO. Barrington et al. [5] showed that with built-in arithmetic, the expressive power of FO coincides exactly with dlogtime-uniform AC^0. This logical characterization can be extended to para-AC^0, the parameterized version of dlogtime-uniform AC^0.

In [10] it was shown that a parameterized problem is in para-AC^0 if and only if it is slicewise definable with bounded quantifier rank in FO with built-in arithmetic. The equivalence (2) shows that the kth slice of the parameterized vertex cover problem is definable with quantifier rank 17 and hence, the parameterized vertex cover problem is in para-AC^0 [2].

For a class K of graphs the parameterized model-checking problem p-MC(K,FO) asks, for a graph $\mathcal{G}\in K$, vertices $\bar{u}$ of $\mathcal{G}$, and an FO-formula $\varphi(\bar{x})$, whether $\bar{u}$ has the property φ in $\mathcal{G}$. We prove that the effective version of (1) is equivalent to p-MC$(K,\mathrm{FO})\in$ para-AC^0 (see Theorem 3.1).

Furthermore we prove that p-MC$(K,\mathrm{FO})\in$ para-AC^0 if the class K has bounded tree-depth. The tree-depth $\mathrm{td}(\mathcal{G})$ of a graph $\mathcal{G}$ measures how close $\mathcal{G}$ is to a star [17]. Various model-theoretic results on classes of bounded tree-depth relevant to our context have been obtained. For example, in [12], it was shown that monadic second-order logic MSO and FO have the same expressive power on any such class. Combined with the result p-MC$(K,\mathrm{FO})\in$ para-AC^0 just mentioned, we get that the model-checking problem for MSO-sentences on any class of graphs of bounded tree-depth is in para-AC^0 (see Theorem 5.1). This means that for every $d\in\mathbb{N}$ there exists a uniform family $(\mathrm{C}_{n,k})_{n,k\in\mathbb{N}}$ of circuits such that for all graphs $\mathcal{G}$ with $\mathrm{td}(\mathcal{G}) \le d$ and all MSO-sentences φ the circuit $\mathrm{C}_{\|\mathcal{G}\|,|\varphi|}$ decides whether $\mathcal{G}$ is a model of φ (here $\|\mathcal{G}\|$ denotes the size of $\mathcal{G}$ and $|\varphi|$ the length of φ). Moreover, the depth of the circuits is bounded by a fixed constant and the size of $\mathrm{C}_{n,k}$ is bounded by $f(k)\cdot n^{O(1)}$ for some computable function $f:\mathbb{N}\to\mathbb{N}$. This is the analogue of Courcelle's Theorem for tree-depth (instead of tree-width) and para-AC^0 (instead of FPT).

In [13] the authors prove that for a fixed sentence of MSO the model-checking problem on any class K of bounded tree-depth is in AC^0. Contrary to our result, the depth of the circuits depends on the MSO-sentence. The same applies to the result obtained in [3] where it is shown that the MSO model-checking on any class of graphs can be done by a uniform family of circuits of depth bounded by $f(\mathrm{td}(\mathcal{G})+|\varphi|)$ and size bounded by $f(\mathrm{td}(\mathcal{G})+|\varphi|)\cdot|G|^{O(1)}$, where φ is any MSO-sentence and f is a computable function.

There are classes K of graphs with *unbounded* tree-depth and with p-$\mathrm{MC}(K,\mathrm{FO})\in$ para-AC^0, e.g., the class K of all complete graphs. However, we prove that if K is closed under taking subgraphs, then p-$\mathrm{MC}(K,\mathrm{FO})\in$ para-AC^0 if and only if K has bounded tree-depth (see Theorem 7.1). Interestingly, in [12] it was shown that under the same closure condition the logic MSO collapses to FO if and only if K has bounded tree-depth.

We study the parameterized model-checking problem for a set of FO-sentences of bounded tree-depth. The tree-depth of a sentence is tightly connected to its quantifier rank (see Theorem 8.2).

Organization of this paper. In Section 2 we fix some notations and introduce the logics and parameterized classes relevant to the paper. Section 3 contains a proof of the equivalence between (ii) and (iii) mentioned in the Abstract. As a first step towards a proof of Courcelle's Theorem for tree-depth (proven in Section 5 and Section 6), in Section 4 we show that FO has generalized quantifier elimination on any class of rooted labelled trees of bounded tree-depth. Section 7 we deal with classes of graphs closed under taking subgraphs and Section 8 is devoted to the tree-depth of FO-sentences.

Due to space limitations we only sketch some proofs or refer to the full version of the paper.

2. Preliminaries

***First-order logic* FO *and monadic second-order logic* MSO.** A *vocabulary* τ is a finite set of relation symbols and of constants (i.e., constant symbols). Each relation symbol has an *arity*. A vocabulary is *relational* if it does not contain constants. A *structure* $\mathcal{A}$ of vocabulary τ, or τ-*structure*, consists of a nonempty finite set A, called the *universe* of $\mathcal{A}$, of an interpretation $R^{\mathcal{A}}\subseteq A^r$ of each r-ary relation symbol $R\in\tau$, and of an element $c^{\mathcal{A}}$ in A for every constant $c\in\tau$. A τ-structure $\mathcal{B}$ is a *substructure* of a τ-structure $\mathcal{A}$ if $B\subseteq A$, $R^{\mathcal{B}}\subseteq R^{\mathcal{A}}$ for $R\in\tau$, and $c^{\mathcal{B}}=c^{\mathcal{A}}$ for $c\in\tau$. It is an *induced substructure* if in addition $R^{\mathcal{B}}=R^{\mathcal{A}}\cap B^r$ for r-ary R. If τ and τ' are vocabularies with $\tau'\subseteq\tau$ and $\mathcal{A}$ is a τ-structure, then $\mathcal{A}\upharpoonright\tau'$ denotes the τ'-*reduct* of $\mathcal{A}$, that is, the τ'-structure obtained from $\mathcal{A}$ by forgetting the interpretation of the symbols in $\tau\setminus\tau'$. The τ-structure $\mathcal{A}$ is an *expansion* of the τ'-structure $\mathcal{B}$ if $\mathcal{A}\upharpoonright\tau'=\mathcal{B}$. If τ contains a binary relation symbol $<$ and in $\mathcal{A}$ the relation $<^{\mathcal{A}}$ is an ordering of the universe, then $\mathcal{A}$ is an *ordered structure*.

Formulas φ of *first-order logic* FO of vocabulary τ are built up from *atomic formulas* $t_1=t_2$ and $Rt_1\dots t_r$ (where $R\in\tau$ is of arity r and $t_1,t_2,\dots,t_r$ are variables from $x_1,x_2,\dots$ or constants in τ) using the boolean connectives $\neg$, $\wedge$, and $\vee$ and the existential $\exists$ and universal $\forall$ quantifiers. By the notation $\varphi(\bar{x})$ with $\bar{x}=x_1,\dots,x_e$ we indicate that the variables free in φ are among $x_1,\dots,x_e$.

In addition to the individual variables of FO, formulas of *monadic second-order logic* MSO may also contain *set variables*. We use lower case letters (usually x,y,z) to denote individual variables and uppercase letters (usually X,Y,Z) to denote set variables. To obtain MSO the syntax of FO is enhanced by new atomic formulas of the form Xy and quantification is also allowed over set variables.

For a vocabulary τ we denote by $\mathrm{FO}[\tau]$ (by $\mathrm{MSO}[\tau]$) the set of formulas of FO (of MSO) of vocabulary τ. A formula φ is a *sentence* if it has neither free individual variables nor free set variables. The quantifier rank of φ is defined inductively:

$$\mathrm{qr}(\varphi):=0 \ \text{ if } \varphi \text{ is atomic}, \qquad \mathrm{qr}(\neg\varphi):=\mathrm{qr}(\varphi),$$
$$\mathrm{qr}(\varphi_1\wedge\varphi_2)=\mathrm{qr}(\varphi_1\vee\varphi_2):=\max\{\mathrm{qr}(\psi_1),\mathrm{qr}(\psi_2)\},$$
$$\mathrm{qr}(\exists x\varphi)=\mathrm{qr}(\forall x\varphi)=\mathrm{qr}(\exists X\varphi)=\mathrm{qr}(\forall X\varphi)=1+\mathrm{qr}(\varphi).$$

Parameterized problems and slicewise definability. For $n\in\mathbb{N}$ let $[n]:=\{0,1,\dots,n-1\}$. We denote the cardinality of a set A by $|A|$. Let $<^{[n]}$ be the natural ordering on $[n]$. Clearly, if $\mathcal{A}$ is any ordered structure, then $(A,<^{\mathcal{A}})$ is isomorphic to $([|A|],<^{[|A|]})$ and the isomorphism is unique. For ternary relation symbols $+$ and $\times$ we consider the ternary relations $+^{[n]}$ and $\times^{[n]}$ on $[n]$ that are the relations of addition and multiplication of $\mathbb{N}$ restricted to $[n]$. That is, $+^{[n]}:=\{(a,b,c)\in[n]^3 \mid c=a+b\}$ and $\times^{[n]}:=\{(a,b,c)\in[n]^3 \mid c=a\cdot b\}$. Finally, for every $m\in\mathbb{N}$ let $N(m):=\{\overline{\ell}\mid \ell<m\}$ be a set of "numerical" constant symbols and set

$$\overline{\ell}^{[n]}:=\ell,\ \text{ if } \ell<n \qquad \text{and} \qquad \overline{\ell}^{[n]}:=n-1,\ \text{ if } \ell\ge n.$$

Assume that τ contains $<$, $+$, and $\times$ but no numerical constants. A $\tau\cup N(m)$-structure $\mathcal{A}$ *has built-in* $<,+,\times,N(m)$ if its $\{<,+,\times,N(m)\}$-reduct is isomorphic to $([n],<^{[n]},+^{[n]},\times^{[n]},(\overline{\ell}^{[n]})_{\ell<m})$. If $m=0$, we briefly say that $\mathcal{A}$ *has built-in arithmetic*. We denote by $\mathrm{ARI}[\tau]$ the class of τ-structures with built-in arithmetic. If $\mathcal{A}\in\mathrm{ARI}[\tau]$ and $m\in\mathbb{N}$, we denote by $\mathcal{A}_{N(m)}$ its unique expansion to a $\tau\cup N(m)$-structure with built-in $<,+,\times,N(m)$. So, if we consider the class $\mathrm{ARI}[\tau]$, then τ will be a vocabulary containing $<$, $+$, and $\times$ but no numerical constants.

A vocabulary is *without arithmetical symbols* if it neither contains $<,+,\times$ nor numerical constants.

A *parameterized problem of vocabulary* τ is a subclass Q of $\mathrm{ARI}[\tau]\times\mathbb{N}$, where for each $k\in\mathbb{N}$ the class $Q_k:=\{\mathcal{A}\in\mathrm{ARI}[\tau]\mid(\mathcal{A},k)\in Q\}$ is closed under isomorphism (cf. [10]).

As the generalized quantifier elimination refers to formulas and not only to sentences we have to consider model-checking problems for formulas. This suggests an extension of the previous definition of parameterized problem. In our applications the vocabulary τ_k of the following definition will mostly have the form $\tau\cup\{c_1,\dots,c_k\}$ for a fixed relational vocabulary τ.

Definition 2.1. Let $(\tau_k)_{k\in\mathbb{N}}$ be a computable sequence of vocabularies. A *parameterized problem of type* $(\tau_k)_{k\in\mathbb{N}}$ is a class Q with

$$Q\subseteq\bigcup_{k\in\mathbb{N}}\mathrm{ARI}[\tau_k]\times\{k\},$$

where for each $k\in\mathbb{N}$ the class $Q_k:=\{\mathcal{A}\in\mathrm{ARI}[\tau_k]\mid(\mathcal{A},k)\in Q\}$ of τ_k-structures is closed under isomorphism. The class Q_k is the *kth slice of* Q. Every pair $(\mathcal{A},k)$ in $\mathrm{ARI}[\tau_k]\times\{k\}$ is an *instance* of Q, $\mathcal{A}$ its *input* and k its *parameter*.

The parameterized complexity class para-AC^0 is the class of parameterized problems that are in dlogtime-uniform AC^0 after a precomputation. In the Introduction we already indicated what kind of family of circuits exist for problems in para-AC^0. We do not need to repeat it, as we use the following characterization of para-AC^0 to show that certain problems belong to para-AC^0.

Theorem 2.2 ([9]). *Let $(\tau_k)_{k\in\mathbb{N}}$ be a computable sequence of vocabularies of bounded arity (that is, there is an $r \in \mathbb{N}$ such that all relation symbols in $\bigcup_{k\in\mathbb{N}} \tau_k$ have arity $\leq r$). Then for every parameterized problem Q of type $(\tau_k)_{k\in\mathbb{N}}$,*

$$Q \in \text{para-AC}^0 \iff Q \in \text{XFO}_{\text{qr}}.$$

We recall the definition of $\mathrm{XFO}_{\mathrm{qr}}$. For $q \in \mathbb{N}$ we denote by FO_q the set of sentences of FO of quantifier rank $\leq q$.

Definition 2.3. A parameterized problem Q of type $(\tau_k)_{k\in\mathbb{N}}$ is *slicewise definable in* FO *with bounded quantifier rank*, briefly $Q \in \mathrm{XFO}_{\mathrm{qr}}$, if there is a $q \in \mathbb{N}$ and computable functions $h : \mathbb{N} \to \mathbb{N}$ and $f : \mathbb{N} \to \mathrm{FO}_q[\tau_k \cup N(h(k))]$ such that for all instances $(\mathcal{A}, k)$ of Q,

$$\mathcal{A} \in Q_k \iff \mathcal{A}_{N(h(k))} \models f(k).$$

That is, if $m_k := h(k)$ and $f(k) := \varphi_k$, then

$$\mathcal{A} \in Q_k \iff \mathcal{A}_{N(m_k)} \models \varphi_k.$$

Then Q *is slicewise definable in* FO_q and we write $Q \in \mathrm{XFO}_q$.

3. Model-checking and generalized quantifier elimination

In this section we prove the equivalence of the statements, denoted by (ii) and (iii) in the Abstract, that is, we prove:

Theorem 3.1. *Let τ be a vocabulary without arithmetical symbols. For a class K of τ-structures the following are equivalent:*

(i) FO *has an effective generalized quantifier elimination on K.*
(ii) p-$\mathrm{MC}(K, \mathrm{FO}) \in$ para-AC^0.

To define the concepts involved in this theorem, we need a notion of union of two structures with built-in arithmetic.

Definition 3.2. Assume $\mathcal{A} \in \mathrm{ARI}[\tau]$ and $\mathcal{A}' \in \mathrm{ARI}[\tau']$ satisfy

$$A \cap A' = \emptyset \quad \text{and} \quad \tau \cap \tau' = \{<, +, \times\}.$$

Let U be a new unary relation symbol. We set $\tau \uplus \tau' := \tau \cup \tau' \cup \{U\}$. Then $\mathcal{A} \uplus \mathcal{A}'$ is the structure $\mathcal{B} \in \mathrm{ARI}(\tau \uplus \tau')$ with

- $B := A \cup A'$; $\quad U^{\mathcal{B}} = A'$;
- $R^{\mathcal{B}} := R^{\mathcal{A}}$ for $R \in \tau \setminus \{<, +, \times\}$ and $R^{\mathcal{B}} := R^{\mathcal{A}'}$ for $R \in \tau' \setminus \{<, +, \times\}$;
- $c^{\mathcal{B}} := c^{\mathcal{A}}$ for $c \in \tau$ and $c^{\mathcal{B}} := c^{\mathcal{A}'}$ for $c \in \tau'$;
- $<^{\mathcal{B}} := <^{\mathcal{A}} \cup <^{\mathcal{A}'} \cup \{(a, a') \mid a \in A \text{ and } a' \in A'\}$, that is, the ordering $<^{\mathcal{B}}$ extends the orderings $<^{\mathcal{A}}$ and $<^{\mathcal{A}'}$, and in $<^{\mathcal{B}}$ every element of A precedes every element of A'.

If $A \cap A' \neq \emptyset$, then we pass to isomorphic structures with disjoint universes before defining $\mathcal{A} \uplus \mathcal{A}'$.

Let τ be a vocabulary without arithmetical symbols. If for a class K of τ-structures and a set Φ of formulas of vocabulary τ (containing only free individual variables) we claim that the *parameterized model-checking problem* p-$\mathrm{MC}(K, \Phi)$

Input: $\mathcal{A} \in K$, $\varphi(x_1, \ldots, x_e) \in \Phi$, and $a_1, \ldots, a_e \in A$.
Parameter: $k \in \mathbb{N}$.
Problem: Decide if $k = |\varphi|$ and $\mathcal{A} \models \varphi(\bar{a})$.

is in para-AC^0 or is not in para-AC^0, we mean that this holds for the problem

Input: $\mathcal{A} \in \mathrm{ARI}[\tau \cup \{<, +, \times\}]$ with $\mathcal{A} \upharpoonright \tau \in K$, $\varphi(x_1, \ldots, x_e) \in \Phi$, and $a_1, \ldots, a_e \in A$.
Parameter: $k \in \mathbb{N}$.
Problem: Decide if $k = |\varphi|$ and $\mathcal{A} \models \varphi(\bar{a})$.

How do we interpret the input of this problem as a structure? Any string $x \in \{0, 1\}^*$ of length n can be identified with the $\{<, +, \times, One\}$-structure $Str(x) := ([n], <^{[n]}, +^{[n]}, \times^{[n]}, One^x)$. Here $i \in [n]$ is in One^x, the interpretation of the unary relation symbol One, if and only if the ith bit of x is a '1'.

We can view formulas as strings over $\{0, 1\}$. We take fresh constants (i.e., constants not in τ) $c_1, c_2, \ldots$ and set $\tau_e := \tau \cup \{c_1, \ldots, c_e\}$. We identify the input with the structure $(\mathcal{A}, \bar{a}) \uplus Str(\varphi(\bar{x}))$.

For easier presentation we tacitly assume that every FO-*formula of length $\leq k$ contains no variable x_ℓ with $\ell > k$.* (3)

Now we define the notion of quantifier elimination relevant to Theorem 3.1.

Definition 3.3. Let K be a class of τ-structures, where τ is a vocabulary without arithmetical symbols. Then FO *has generalized quantifier elimination on K* if there is an $\ell \in \mathbb{N}$ and a function $h : \mathbb{N} \to \mathbb{N}$ such that for all $\mathrm{FO}[\tau]$-formulas $\varphi(\bar{x})$ there is an $\mathrm{FO}[\tau \cup \{<, +, \times\} \cup N(h(|\varphi|))]$-formula $\varphi^*(\bar{x})$ of quantifier rank at most ℓ such that for all $(\mathcal{A}, <, +, \times) \in \mathrm{ARI}[\tau \cup \{<, +, \times\}]$ with $\mathcal{A} \in K$ and $\bar{a}$ in $\mathcal{A}$ we have

$$\mathcal{A} \models \varphi(\bar{a}) \iff (\mathcal{A}, <, +, \times)_{N(h(|\varphi|))} \models \varphi^*(\bar{a}),$$

or equivalently,

$$(\mathcal{A}, <, +, \times)_{N(h(|\varphi|))} \models \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \varphi^*(\bar{x})). \tag{4}$$

If there is a computable mapping $\varphi \mapsto \varphi^*$, then FO *has an effective generalized quantifier elimination on K*.

Loosely speaking if FO has generalized quantifier elimination on K, then for some $\ell \in \mathbb{N}$ every FO-formula φ is equivalent in structures of K to an FO-formula of quantifier rank at most ℓ if we use built-in arithmetic and constants for an initial segment of natural numbers, whose length depends only on the length of φ.

Often we will make use of the following observation.

Remark 3.4. In structures of the form $\mathcal{A}_{N(m)}$ the sentence $\overline{m-1} = \overline{m-2}$ expresses that A has less than m elements. Therefore for every $\mathcal{A} \in \mathrm{ARI}[\tau]$ with $|A| < m$ and $a_1, \ldots, a_e$ in A there is a quantifier free $\mathrm{FO}[\tau \cup N(m)]$-formula $\varphi_{\mathcal{A}_{N(m)}, \bar{a}}(x_1, \ldots, x_e)$ such that for all structures $\mathcal{B} \in \mathrm{ARI}[\tau]$ and $\bar{b} \in B^e$ we have

$$\mathcal{B}_{N(m)} \models \varphi_{\mathcal{A}_{N(m)}, \bar{a}}(\bar{b}) \iff (\mathcal{A}, \bar{a}) \cong (\mathcal{B}, \bar{b}).$$

Hence, in Definition 3.3, it suffices to require that the equivalence between φ and φ^* in (4) holds for $\mathcal{A} \in K$ with $|A| \geq m$. In fact, then for the following formula $\widehat{\varphi^*}(\bar{x})$

$$(\overline{m-1} \neq \overline{m-2} \wedge \varphi^*(\bar{x})) \vee \bigvee_{\substack{\mathcal{A} \in K,\ |A| < m,\ \bar{a} \in A^e \\ \mathcal{A} \models \varphi(\bar{a})}} \varphi_{\mathcal{A}_{N(m)}, \bar{a}}(\bar{x})$$

we have $\mathrm{qr}(\widehat{\varphi^*}) = \mathrm{qr}(\varphi^*)$ and for all $\mathcal{A} \in K$,

$$(\mathcal{A}, <, +, \times)_{N(\max\{h(\mathrm{qr}(\varphi)), m\})} \models \forall \bar{x}(\varphi \leftrightarrow \widehat{\varphi^*}).$$

A similar remark applies to the notion of slicewise definability in FO with bounded quantifier rank (see [10, Prop. 8]).

In the proof of Theorem 3.1 we need the notion of an FO-interpretation. Let τ and τ' be vocabularies with $\tau := \{R_1, \dots, R_m, c_1, \dots, c_n\}$ and r_i-ary R_i. An FO-*interpretation I of τ in τ' of width w* consists of FO$[\tau']$ formulas

$$\begin{aligned}\varphi_{\mathrm{uni}}(\bar{x}), \quad & \varphi_{R_1}(\bar{x}_1, \dots, \bar{x}_{r_1}), \dots, \varphi_{R_m}(\bar{x}_1, \dots, \bar{x}_{r_m}), \\ & \varphi_{c_1}(\bar{y}), \dots, \varphi_{c_n}(\bar{y}),\end{aligned} \tag{5}$$

where all tuples $\bar{x}, \bar{x}_1, \dots, \bar{y}$ have length w. Let $\mathcal{A}$ be a τ'-structure with $\mathcal{A} \models \exists\bar{x}\varphi_{\mathrm{uni}}(\bar{x})$ and

$$\mathcal{A} \models \exists\bar{y}\big(\varphi_{c_i}(\bar{y}) \wedge \forall\bar{z}(\varphi_{c_i}(\bar{z}) \to \bar{y} = \bar{z})\big) \quad \text{for } i = 1, \dots, n,$$

where $\bar{y} = \bar{z}$ for $\bar{y} = y_1, \dots, y_w$ and $\bar{z} = z_1, \dots, z_w$ denotes the formula $y_1 = z_1 \wedge \dots \wedge y_w = z_w$. The interpretation I induces the τ-structure $\mathcal{A}^I$ with universe $A^I := \{\bar{a} \in A^w \mid \mathcal{A} \models \varphi_{\mathrm{uni}}(\bar{a})\}$, with $R_i^{\mathcal{A}^I} := \{(\bar{a}_1, \dots, \bar{a}_{r_i}) \in (A^I)^{r_i} \mid \mathcal{A} \models \varphi_{R_i}(\bar{a}_1, \dots, \bar{a}_{r_i})\}$, and $c_i^{\mathcal{A}^I} := \bar{a}$ if $\mathcal{A} \models \varphi_{c_i}(\bar{a})$. For every FO$[\tau]$-sentence φ there is an FO$[\tau']$-sentence φ^I such that for all τ'-structures $\mathcal{A}$ we have

$$\mathcal{A}^I \models \varphi \iff \mathcal{A} \models \varphi^I. \tag{6}$$

Furthermore, for an FO-interpretation of width 1 there is for every FO$[\tau]$-formula $\varphi(x_1, \dots, x_e)$ an FO$[\tau']$-formula $\varphi^I(x_1, \dots, x_e)$ such that for all τ'-structures $\mathcal{A}$ and for arbitrary $a_1, \dots, a_e \in A^I$,

$$\mathcal{A}^I \models \varphi(\bar{a}) \iff \mathcal{A} \models \varphi^I(\bar{a}). \tag{7}$$

Let the interpretation I of width w be given by (5). The *quantifier rank q_I of I* is defined by

$$q_I := \max\{\mathrm{qr}(\varphi_{\mathrm{uni}}), \mathrm{qr}(\varphi_{R_1}), \dots, \mathrm{qr}(\varphi_{R_m}), \mathrm{qr}(\varphi_{c_1}), \dots, \mathrm{qr}(\varphi_{c_n})\}.$$

Then

$$\mathrm{qr}(\varphi^I) \le w \cdot \mathrm{qr}(\varphi) + q_I. \tag{8}$$

Lemma 3.5. *Let τ and τ' be vocabularies with $\tau \cap \tau' = \{<, +, \times\}$. Let $\mathcal{B} \in$ ARI$[\tau']$ and $k' \ge k := |B|$. There is an interpretation $I_{\mathcal{B}}$ of width 2 such that for every $\mathcal{A} \in$ ARI$[\tau]$ with $|A| \ge k'$ we have*

$$\big(\mathcal{A}_{N(k')}\big)^{I_{\mathcal{B}}} \cong \big(\mathcal{A} \uplus \mathcal{B}\big)_{N(k')}.$$

The quantifier rank $q_{I_{\mathcal{B}}}$ of $I_{\mathcal{B}}$ doesn't depend on τ, τ', $\mathcal{B}$, k, and k'.

Sketch of proof: As formula defining the universe of the interpreted structure we choose $\varphi_{\mathrm{uni}}(x, x') := (x = \bar{0} \vee (x = \bar{1} \wedge x' \le \overline{k-1}))$. Thus the set $\{(\bar{0}^{\mathcal{A}_{N(k')}}, a) \mid a \in A\} \cup \{(\bar{1}^{\mathcal{A}_{N(k')}}, \bar{i}^{\mathcal{A}_{N(k')}}) \mid i \in [k]\}$ is the universe of $(\mathcal{A}_{N(k')})^{I_{\mathcal{B}}}$. We define the formulas $\psi_U(x, x')$ and $\psi_<(x, x', y, y')$ (cf. Definition 3.2) by

$$\psi_U := x = \bar{1} \quad \text{and} \quad \psi_< := x < y \vee (x = y \wedge x' < y').$$

We look at addition: Let $a, b \in A$ be such that $a +^{\mathcal{A}} b$ does not exist in $\mathcal{A}$. Let v be the $<^{\mathcal{A}}$-greatest element of A, $a +^{\mathcal{A}} u = v$, and $(u +^{\mathcal{A}} (\bar{m})^{\mathcal{A}}) + \bar{1}^{\mathcal{A}} = b$ for some $m \in [k]$, then we have $(\bar{0}, a) + (\bar{0}, b) = (\bar{1}, \bar{m})$ in $(\mathcal{A}_{N(k')})^{I_{\mathcal{B}}}$. For elements $(\bar{0}^{\mathcal{A}_{N(k')}}, a)$ with $a \in A$ and $(\bar{1}^{\mathcal{A}_{N(k')}}, \bar{i}^{\mathcal{A}_{N(k')}})$ with $i \in [k]$ their sum exists iff for some $m \in [k]$ we have $a = \bar{m}^{\mathcal{A}_{N(k')}}$ and $m + i \in [k]$. Then $(\bar{1}^{\mathcal{A}_{N(k')}}, \bar{m}^{\mathcal{A}_{N(k')}})$ is their sum. Now it is easy to write down an FO-formula $\psi_+(\bar{x}, \bar{y}, \bar{z})$ defining addition. The remaining cases are treated similarly. □

Proof of Theorem 3.1: Let τ and K be as in the statement of Theorem 3.1. For fresh constants $c_1, c_2, \dots$ we set $\tau_k := \tau \cup \{c_1, \dots, c_k\}$.

(i) ⇒ (ii): By Theorem 2.2 it suffices to show that p-MC(K, FO) is slicewise definable with bounded quantifier rank. So we have to find a $q \in \mathbb{N}$ and computable functions $k \mapsto m_k$ and $k \mapsto \varphi_k$, where φ_k is an FO$_q[\tau_k \cup \{<+, \times\} \cup N(m_k)]$-sentence, such that for all

$$(\mathcal{A}, <, +, \times) \in \mathrm{ARI}[\tau \cup \{<, +, \times\}]$$

with $\mathcal{A} \in K$, all FO$[\tau]$-formula $\chi(x_1, \dots, x_e)$, all $\bar{a} \in A^e$, and all $k \in \mathbb{N}$,

$$(\mathcal{A} \models \chi(\bar{a}) \text{ and } |\chi| = k) \iff ((\mathcal{A}, \bar{a}, <, +, \times) \uplus Str(\chi))_{N(m_k)} \models \varphi_k.$$

Fix $k \in \mathbb{N}$. Let $\chi_1(x_1, \dots, x_e), \dots, \chi_s(x_1, \dots, x_e)$ be all FO$[\tau]$-formulas of length k (by (3) we know that we can choose $e \le k$). By (i) there is an $\ell \in \mathbb{N}$ (not depending on k), a function $h : \mathbb{N} \to \mathbb{N}$ and FO$_\ell[\tau \cup \{< +, \times\}]$-formulas $\chi_1^*(\bar{x}), \dots, \chi_s^*(\bar{x})$ equivalent to $\chi_1, \dots, \chi_s$ in all $(\mathcal{A}, <, +, \times)_{N(h(k))}$ with $\mathcal{A} \in K$. Then for $\bar{a} \in A^e$,

$$\begin{aligned}\big(\mathcal{A} \models \chi(\bar{a}) \text{ and } |\chi| = k\big) \iff & ((\mathcal{A}, \bar{a}, <, +, \times) \uplus Str(\chi))_{N(h(k))} \models \\ & \bigvee_{i=1}^{s} (\text{“}\chi = \chi_i\text{"} \wedge (\chi_i^*)^{\neg U}(c_1, \dots, c_e)).\end{aligned}$$

Here (compare Definition 3.2) $(\chi_i^*)^{\neg U}(c_1, \dots, c_e)$ denotes the sentence obtained from $\chi_i^*(c_1, \dots, c_e)$ by relativizing all quantifiers to the complement of U, i.e., to the universe of $\mathcal{A}$. Furthermore, "$\chi = \chi_i$" is an abbreviation for

$$\begin{aligned}\exists x\Big(Ux \wedge \forall y(y < x \to \neg Uy) \wedge & \bigwedge_{\substack{j \in [k] \text{ and the} \\ j\text{th bit of } \chi_i \text{ is } 1}} \exists z(x + \bar{j} = z \wedge One\, z) \\ \wedge \bigwedge_{\substack{j \in [k] \text{ and the} \\ j\text{th bit of } \chi_i \text{ is } 0}} & \exists z(x + \bar{j} = z \wedge \neg One\, z) \wedge \neg\exists y\, x + \bar{k} = y\Big).\end{aligned}$$

Thus, $\mathrm{qr}(\bigvee_{i=1}^{s}(\text{“}\chi = \chi_i\text{"} \wedge (\chi_i^*)^{\neg U}(c_1, \dots, c_e)) = \max\{2, \ell\}$.

(ii) ⇒(i): By (ii) there is a $q \in \mathbb{N}$, computable functions $k \mapsto m_k$ and $k \mapsto \varphi_k$, where each φ_k is an FO$_q[\tau_k \cup \{<, +, \times\} \cup N(m_k)]$-sentence, such that for all $(\mathcal{A}, <, +, \times) \in$ ARI$[\tau \cup \{<, +, \times\}]$ with $\mathcal{A} \in K$, every FO$[\tau]$-formula $\chi(x_1, \dots, x_e)$, and all $\bar{a} \in A^e$,

$$\begin{aligned}(\mathcal{A} \models \chi(\bar{a}) \text{ and } |\chi| = k) \iff & \\ ((\mathcal{A}, \bar{a}, <, +, \times) \uplus Str(\chi))_{N(m_k)} \models \varphi_k.\end{aligned} \tag{9}$$

We have to show that there is an $\ell \in \mathbb{N}$ and a function $h : \mathbb{N} \to \mathbb{N}$ such that for every FO$[\tau]$-formula $\psi(x_1, \dots, x_e)$ there is an FO$_\ell[\tau \cup \{<, +, \times\} \cup N(h(|\psi|))]$-formula $\psi^*(x_1, \dots, x_e)$ such that for $(\mathcal{A}, <, +, \times) \in$ ARI$[\tau \cup \{<, +, \times\}]$ with $\mathcal{A} \in K$ and all $\bar{a} \in A^e$,

$$\mathcal{A} \models \psi(\bar{a}) \iff \big(\mathcal{A}, <, +, \times\big)_{N(h(|\psi|))} \models \psi^*(\bar{a}).$$

Let $k := |\psi|$; by (3), w.l.o.g. we can assume that $e = k$. Then, by (9),

$$\mathcal{A} \models \psi(\bar{a}) \iff (\mathcal{A}, \bar{a}, <, +, \times) \uplus Str(\psi))_{N(m_k)} \models \varphi_k. \tag{10}$$

By Remark 3.4 we may restrict ourselves to structures $\mathcal{A} \in K$ with $|A| \ge m_k$ and furthermore we may assume that $m_k \ge k$. Now we apply Lemma 3.5. For the interpretation $I_{Str(\psi)}$ we have $((\mathcal{A}, \bar{a}, <, +, \times))_{N(m_k)})^{I_{Str(\psi)}} = ((\mathcal{A}, \bar{a}, <, +, \times) \uplus Str(\psi))_{N(m_k)}$. Hence, by (7),

$$\begin{aligned}((\mathcal{A}, \bar{a}, <, +, \times) \uplus Str(\psi))_{N(m_k)} \models \varphi_k \iff & \\ (\mathcal{A}, \bar{a}, <, +, \times)_{N(m_k)} \models (\varphi_k)^{I_{Str(\psi)}}.\end{aligned} \tag{11}$$

By (10), and (11) we get

$$\mathcal{A} \models \psi(\bar{a}) \iff \big(\mathcal{A}, \bar{a}, <, +, \times\big)_{N(m_k)} \models (\varphi_k)^{I_{Str(\psi)}}.$$

We set $\psi^* := \big(\varphi_k\big)^{I_{Str(\psi)}} \frac{x_1, \dots, x_e}{c_1, \dots, c_e}$, the formula obtained from $\big(\varphi_k\big)^{I_{Str(\psi)}}$ by replacing the constants $c_1, \dots, c_e$ by the variables $x_1, \dots, x_e$ (and renaming quantified variables if necessary). Then

$$\big(\mathcal{A}, <, +, \times\big)_{N(m_k)} \models \forall\bar{x}\big(\psi(\bar{x}) \leftrightarrow \psi^*(\bar{x})\big).$$

The set $\big\{\psi^* \bigm| \psi \in \mathrm{FO}[\tau]\big\}$ is of bounded quantifier rank by (8) and Lemma 3.5. □

4. Quantifier elimination for trees of bounded depth

As a step towards a proof of Courcelle's Theorem for tree-depth we prove that FO has an effective generalized quantifier elimination on every class of rooted labelled trees of bounded depth.

We view rooted trees with s labels as $\tau_s := \{P, L_1, \dots, L_s\}$-structures $\mathcal{T} = \big(T, P^{\mathcal{T}}, L_1^{\mathcal{T}}, \dots, L_s^{\mathcal{T}}\big)$. Here P is binary and $P^{\mathcal{T}}$ is the *parent-child relation* of the tree. The root of the tree can be defined by the formula $root(x) := \forall y \neg Pyx$. The unary relations $L_1^{\mathcal{T}}, \dots, L_s^{\mathcal{T}}$ are the *labels*. The *depth of* $\mathcal{T}$ is the maximum length of a path from the root to a leaf. For $s, d \in \mathbb{N}$ we denote by $\textsc{Tree}[s,d]$ the class of rooted trees with s labels and of depth $\leq d$. Here we show that $\textsc{Tree}[s,d]$ has an effective generalized quantifier elimination.

Applying the Ehrenfeucht-Fraïssé method. We now recall results obtained by the technique of Ehrenfeucht and Fraïssé (e.g. see [12, 15, 18]) which we use in order to prove the generalized quantifier elimination result just mentioned. Recall that $\tau_s := \{P, L_1, \dots, L_s\}$.

Fact 4.1. *There is a computable function that given $k \geq 1$ computes finitely many* $\mathrm{FO}[\tau_s]$*-sentences*

$$\rho_1^k, \dots, \rho_{t_k}^k$$

of quantifier rank at most k such that every τ_s-structure $\mathcal{A}$ satisfies exactly one ρ_i^k, denoted by $typ_k(\mathcal{A})$ and called the k-type of $\mathcal{A}$. *If two structures have the same k-type, then they satisfy the same* $\mathrm{FO}[\tau_s]$*-sentences of quantifier rank at most k.*

The next result is a Feferman-Vaught type application of the characterization due to Ehrenfeucht and Fraïssé of when two structures have the same k-type. For a rooted labelled tree $\mathcal{T}$ and a node $a \in T$ let *the subtree $\mathcal{T}_a$ of $\mathcal{T}$ rooted at a* be the induced substructure of $\mathcal{T}$ whose universe T_a consists of a and all its descendants.

Proposition 4.2. *Let $\mathcal{T}$ be a rooted tree with s labels. Let a_0 be its root and $(a_j)_{j\in J}$ the family of its children. For every k-type ρ_i^k, we briefly say for every k-type ρ, we set*

$$n(\rho) := |\{j \in J \mid typ_k(\mathcal{T}_{a_j}) = \rho\}|.$$

Then $typ_k(\mathcal{T})$ depends only on $(\min\{k, n(\rho)\})_{\rho\ k\text{-type}}$ and the sequence (of answers to) $(L_1^{\mathcal{T}} a_0?, \dots, L_s^{\mathcal{T}} a_0?)$. Here the answer to $L_j^{\mathcal{T}} a_0?$ is 1 if $L_j^{\mathcal{T}} a_0$ and 0 otherwise.

For $m_1, \dots, m_{t_k} \leq k$ (recall that t_k is the number of k-types in Fact 4.1) and $w \in \{0,1\}^s$ we set

$$H(m_1, \dots, m_{t_k}, w) = j \tag{12}$$

if in Prop. 4.2 we have $typ_k(\mathcal{T}) = \rho_j^k$ assuming that for $1 \leq i \leq t_k$ we have $\min\{k, n(\rho_i^k)\} = m_i$ and $(L_1^{\mathcal{T}} a_0?, \dots, L_s^{\mathcal{T}} a_0?) = w$. Note that $H(m_1, \dots, m_{t_k}, w)$ is only defined for tuples $(m_1, \dots, m_{t_k}, w)$ with the property that $m_j = 0$ if ρ_j^k has no (finite) model, which is a rooted tree with s labels (compare Remark 4.4).

For $d \in \mathbb{N}$ we set

$$h(d,k) := \sum_{i=0}^{d} (t_k \cdot k)^i \leq (t_k \cdot k)^{d+1}. \tag{13}$$

Corollary 4.3. *Let $\mathcal{T} \in \textsc{Tree}[s,d]$. Then there is an induced subtree $\mathcal{T}'$ of $\mathcal{T}$ with the same root and of the same k-type as $\mathcal{T}$ and with at most $h(d,k)$ nodes.*

Idea of proof: The claim is proved by a straightforward induction on the depth d of the tree using the preceding proposition. □

Remark 4.4. By the previous corollary every $\mathrm{FO}[\tau_s]$-sentence of quantifier rank at most k, satisfiable in a tree of $\textsc{Tree}[s,d]$, is already satisfiable in a tree of $\textsc{Tree}[s,d]$ with at most $h(d,k)$ nodes. Clearly we can check whether a sentence φ is satisfiable in a tree with at most $h(d,k)$ nodes. Hence the function H (defined in (12)) is computable and its domain is decidable. Similarly, we can decide whether an $\mathrm{FO}[\tau_s]$-sentence implies another one in $\textsc{Tree}[s,d]$.

Applying the color-coding method. The following result allows to express in FO the existence of k elements with an FO-property by a number of quantifiers independent of k. The result, a main tool in the proof of the claimed generalized quantifier elimination, was shown in [10] using the color-coding technique of Alon et al. [1].

Lemma 4.5. *There is an $n_0 \in \mathbb{N}$ and an algorithm that assigns to every $k \in \mathbb{N}$ and every $\mathrm{FO}[\tau]$-formula $\varphi(\bar{x}, y)$, where τ is a vocabulary containing $<, +, \times$, an $\mathrm{FO}[\tau \cup N(k^2)]$-formula $\chi_\varphi^k(\bar{x})$ such that for every $\mathcal{A} \in \mathrm{ARI}[\tau]$ with $|A| \geq \max\{2^{k^2}, n_0\}$ we have*

$$\text{for all } \bar{a} \in A\ \big(\mathcal{A}_{N(k^2)} \models \chi_\varphi^k(\bar{a}) \iff \text{there are pairwise distinct } b_0, \dots, b_{k-1} \in A \text{ with } \mathcal{A} \models \varphi(\bar{a}, b_i) \text{ for } i \in [k]\big), \tag{14}$$

or more succinctly: $\mathcal{A}_{N(k^2)} \models \forall \bar{x}(\chi_\varphi^k(\bar{x}) \leftrightarrow \exists^{\geq k} y \varphi(\bar{x}, y))$.
Furthermore, $\mathrm{qr}\big(\chi_\varphi^k(\bar{x})\big) = \max\big\{12, \mathrm{qr}\big(\varphi(\bar{x}, y)\big) + 3\big\}$.

The following proposition shows that FO has an effective generalized quantifier elimination *for sentences* on the class $\textsc{Tree}[s,d]$. There we only consider sufficiently large trees $\mathcal{T} \in \textsc{Tree}[s,d]$. The result for all trees in $\textsc{Tree}[s,d]$ is then obtained by applying Remark 3.4. Let n_0 be as in the preceding lemma.

Proposition 4.6. *Let $d \in \mathbb{N}$. Set $\ell := 2$ for $d = 0$ and $\ell := 13 + 3(d-1)$ for $d \geq 1$. Then there is an algorithm which for every $s \in \mathbb{N}$ assigns to every $\mathrm{FO}[\tau_s]$-sentence φ of quantifier rank $k \geq 2$ an $\mathrm{FO}_\ell[\tau_s \cup \{<,+,\times\} \cup N(k^2)]$-sentence φ^* such that for every $(\mathcal{T}, <, +, \times) \in \mathrm{ARI}[\tau_s \cup \{<,+,\times\}]$ with $\mathcal{T} \in \textsc{Tree}[s,d]$ and $|T| \geq \max\{2^{k^2}, n_0\}$ we have*

$$(\mathcal{T}, <, +, \times)_{N(k^2)} \models (\varphi \leftrightarrow \varphi^*). \tag{15}$$

Proof : Recall that $root(x) := \forall y \neg Pyx$ defines the root. For trees in $\textsc{Tree}[s,d]$ we express in FO that "x is a leaf" by $leaf(x) := \forall y \neg Pxy$. For every $i \in \mathbb{N}$ we define $depth_i(x)$ ("i is the depth of x") by induction on $i \geq 0$:

$$depth_0(x) := root(x), \qquad depth_{i+1}(x) := \exists y(Pyx \wedge depth_i(y)).$$

Now let φ be an $\mathrm{FO}[\tau_s]$-sentence of quantifier rank k. As every $\mathrm{FO}[\tau_s]$-sentence of quantifier rank at most k is equivalent to a disjunction of some k-types, i.e., of some of the sentences in $\rho_1^k, \dots, \rho_{t_k}^k$, we may assume that φ is one of the ρ's. For $d' \leq d$ and every k-type ρ_j^k (with $1 \leq j \leq t_k$) we define an $\mathrm{FO}[\tau_s \cup \{<,+,\times\} \cup N(k^2)]$-formula $\psi_{d',j}^k(x)$ of quantifier rank $\leq \ell - 1$ such that for every $(\mathcal{T}, <, +, \times) \in \mathrm{ARI}[\tau_s \cup \{<,+,\times\}]$ with $\mathcal{T} \in \textsc{Tree}[s,d]$ and every $a \in T$ of depth d' we have

$$(\mathcal{T}, <, +, \times)_{N(k^2)} \models \psi_{d',j}^k(a) \iff \mathcal{T}_a \models \rho_j^k. \tag{16}$$

Then, as $\mathcal{T}_{a_0} = \mathcal{T}$ for the root a_0 of $\mathcal{T}$, we get

$$\begin{aligned}(\mathcal{T}, <, +, \times)_{N(k^2)} &\models \exists x(root(x) \wedge \psi_{0,j}^k(x)) \\ &\iff (\mathcal{T}, <, +, \times)_{N(k^2)} \models \psi_{0,j}^k(a_0) \\ &\iff \mathcal{T} \models \rho_j^k.\end{aligned}$$

Thus, for $\rho = \rho_j^k$ we set

$$\rho^* := \exists x(root(x) \wedge \psi_{0,j}^k(x)) \qquad (17)$$

and have $(\mathcal{T}, <, +, \times)_{N(k^2)} \models (\rho \leftrightarrow \rho^*)$, that is, the equivalence (15) for $\varphi = \rho$.

In the definition of $\psi_{d',j}^k(x)$ (with the property (16)) we have to consider the two cases "x is a leaf" and "x is not a leaf" separately. So $\psi_{d',j}^k(x)$ has the form:

$$\psi_{d',j}^k(x) := depth_{d'}(x) \wedge \Big(leaf\text{-}\psi_{d',j}^k(x) \vee non\text{-}leaf\text{-}\psi_{d',j}^k(x)\Big). \quad (18)$$

If $\mathcal{T}_a \models \rho_j^k$ for some leaf a, then in particular, $\rho_j^k \models \forall x \forall y\, x = y \wedge \forall x \neg Pxx$. If $\rho_j^k \models \neg(\forall x \forall y\, x = y \wedge \forall x \neg Pxx)$, we set $leaf\text{-}\psi_{d',j}^k(x) := \neg x = x$. Otherwise we set (recall that $\tau_s := \{P, L_1, \dots, L_s\}$)

$$leaf\text{-}\psi_{d',j}^k(x) := leaf(x) \wedge \bigwedge_{\substack{1 \le s' \le s \\ \rho_j^k \models \exists y L_{s'} y}} L_{s'} x \wedge \bigwedge_{\substack{1 \le s' \le s \\ \rho_j^k \models \neg \exists y L_{s'} y}} \neg L_{s'} x.$$

Then for $leaf\text{-}\psi_{d',j}^k(x)$ instead of $\psi_{d',j}^k$ the equivalence (16) holds for leaves a of depth d'. Note that $\text{qr}(leaf\text{-}\psi_{d',j}^k(x)) = 1$. Now we look at nonleaves of depth d'. In particular, then $d' + 1 \le d$. We take as $non\text{-}leaf\text{-}\psi_{d',j}^k(x)$ the formula (here if $w \in \{0,1\}^s$, then $w = (w_1, \dots, w_s)$; for the definition of the function H compare (12))

$$\neg leaf(x) \wedge \bigvee_{\substack{m_1,\dots,m_{t_k} \le k,\ w \in \{0,1\}^s \\ H(m_1,\dots,m_{t_k}, w) = j}} \Big(\bigwedge_{\substack{1 \le i \le t_k \\ m_i < k}} \exists^{=m_i} y (Pxy \wedge \psi_{d'+1,i}^k(y))$$

$$\wedge \bigwedge_{\substack{1 \le i \le t_k \\ m_i = k}} \exists^{\ge m_i} y (Pxy \wedge \psi_{d'+1,i}^k(y)) \wedge \bigwedge_{\substack{1 \le s' \le s \\ w_{s'} = 1}} L_{s'} x \wedge \bigwedge_{\substack{1 \le s' \le s \\ w_{s'} = 0}} \neg L_{s'} x \Big).$$

The quantifiers $\exists^{=m_i} y$ and $\exists^{\ge m_i} y$ may be expressed by FO-formulas of quantifier rank independent of m_i by Lemma 4.5; more precisely, $\text{qr}(non\text{-}leaf\text{-}\psi_{d',j}^k(x)) = \max\{12, \max(\{\text{qr}(\psi_{d'+1,i}^k(x)) \mid 1 \le i \le t_k\}) + 3\}$. Now, using (17) and (18) a simple induction shows that $\text{qr}(\rho^*) \le 13 + 3(d-1)$. □

We use the previous result to get the generalized quantifier elimination for FO-*formulas*.

Proposition 4.7. *Let $d \in \mathbb{N}$. Set $\ell := 2$ for $d = 0$ and $\ell := 13 + 3(d-1)$ for $d \ge 1$. Then there is an algorithm which for every $s, n \in \mathbb{N}$ assigns to every $\text{FO}[\tau_s]$-formula $\varphi(x_1, \dots, x_n)$ of quantifier rank $k \ge 2$ an $\text{FO}_\ell[\tau_s \cup \{<, +, \times\} \cup N((k+2)^2)]$-formula $\varphi^*(x_1, \dots, x_n)$ such that for every $(\mathcal{T}, <, +, \times) \in \text{ARI}[\tau_s \cup \{<, +, \times\}]$ with $\mathcal{T} \in \text{TREE}[s, d]$ and $|T| \ge \max\{2^{(k+2)^2}, n_0\}$,*

$$(\mathcal{T}, <, +, \times)_{N(k^2)} \models \forall x_1 \dots \forall x_n (\varphi(x_1, \dots, x_n) \leftrightarrow \varphi^*(x_1, \dots, x_n)).$$

Idea of proof: We replace the variable x_i by a unary relation symbol S_i playing the role of the singleton $\{x_i\}$. and applies Prop. 4.6. □

5. An analogue of Courcelle's theorem for tree-depth

Courcelle's Theorem [11] states that the model-checking problem for sentences of monadic second-order logic MSO on a class of structures of bounded tree-width, parameterized by the length of the MSO-sentence, is fixed-parameter tractable. We show that the problem lies in para-AC^0 if we restrict to classes of structures of bounded tree-depth. Already in [13] it was shown that for every fixed MSO-sentence φ the question whether structures from a class of bounded tree-depth have the property φ can be solved by a dlogtime-uniform AC^0 class of circuits[1]. Before we present the precise statement of our result, we recall the notion of tree-depth.

A forest $\mathcal{F}$ consisting of rooted trees is a *forest for a graph* $\mathcal{G}$ if

- the set of nodes of $\mathcal{F}$ is the set G of vertices of $\mathcal{G}$;
- every edge of $\mathcal{G}$ connects a pair of nodes of $\mathcal{F}$ that have an ancestor-descendant relationship to each other in $\mathcal{F}$.

The *tree-depth* $\text{td}(\mathcal{G})$ *of* $\mathcal{G}$ is defined as 1 plus the minimum depth of a forest for $\mathcal{G}$. The *depth of* $\mathcal{F}$ is the maximum length of a path from a root to a leaf.

The *tree-depth* $\text{td}(\mathcal{A})$ *of a τ-structure* $\mathcal{A}$ is the tree-depth of the Gaifman graph $\mathcal{G}(\mathcal{A})$ of $\mathcal{A}$. Here $\mathcal{G}(\mathcal{A})$ has vertex set A and there is an edge between distinct $a, b \in A$ if for some $R \in \tau$ and some $(a_1, \dots, a_r) \in R^{\mathcal{A}}$ we have $a, b \in \{a_1, \dots, a_r\}$.

We denote by $\text{STR}[\tau, d]$ the class of all τ-structures of tree-depth $\le d$. For a class Φ of formulas we let Φ^{sent} be the class of sentences in Φ. We show the following analogue of Courcelle's Theorem.

Theorem 5.1. *Let $d \ge 1$ and let τ be a vocabulary without arithmetical symbols. Then the problem p-$\text{MC}(\text{STR}[\tau, d], \text{MSO}^{\text{sent}})$ is in para-AC^0. Moreover p-$\text{MC}(\text{STR}[\tau, d], \text{MSO}^{\text{sent}}) \in \text{XFO}_{O(d)}$.*

Elberfeld et al. [12] proved that on every class of graphs of bounded tree-depth FO and monadic second-order logic MSO (and even guarded second-order logic) have the same expressive power. This result can easily be generalized to the class $\text{STR}[\tau, d]$. Moreover there is an effective procedure assigning to an MSO-sentence an equivalent FO-sentence (we present a proof of the effectivity in the full version):

Theorem 5.2 ([12]). *For every $d \ge 1$ and every vocabulary τ there is an effective procedure assigning to every* MSO*-sentence an* FO*-sentence equivalent to it on the class* $\text{STR}[\tau, d]$.

This result allows to reduce p-$\text{MC}(\text{STR}[\tau, d], \text{MSO}^{\text{sent}})$ to the problem p-$\text{MC}(\text{STR}[\tau, d], \text{FO}^{\text{sent}})$. We first introduce the corresponding notion of para-FO reduction (which corresponds to the notion of para-AC^0 reduction [10]).

Definition 5.3. Let $Q \subseteq \text{ARI}[\tau] \times \mathbb{N}$ and $Q' \subseteq \text{ARI}[\tau'] \times \mathbb{N}$ be parameterized problems. A mapping $\mathcal{R} : \text{ARI}[\tau] \times \mathbb{N} \to \text{ARI}[\tau'] \times \mathbb{N}$ is a para-FO *reduction from Q to Q'* if for some vocabulary τ_1 with $\tau \cap \tau_1 = \{<, +, \times\}$ there are computable functions $red : \mathbb{N} \to \text{ARI}[\tau_1]$ and $g : \mathbb{N} \to \mathbb{N}$ and an FO-interpretation I such that for every instance $(\mathcal{A}, k)$ of Q and for $(\mathcal{A}', k') := \mathcal{R}(\mathcal{A}, k)$ we have

(i) $(\mathcal{A}, k) \in Q \iff (\mathcal{A}', k') \in Q'$;
(ii) $\mathcal{A}' \uplus Str(k') = \big(\mathcal{A} \uplus red(k)\big)^I$; (iii) $k' \le g(k)$.

Here $Str(k') = Str(1 \dots 1)$ for the string $1 \dots 1$ of k' ones.

If there is such an $\mathcal{R}$ we write $Q \le_{\text{para-FO}} Q'$. And $Q \equiv_{\text{para-FO}} Q'$ means that $Q \le_{\text{para-FO}} Q'$ and $Q' \le_{\text{para-FO}} Q$.

Lemma 5.4. *If $Q \le_{\text{para-FO}} Q'$ and $Q' \in$ para-AC^0, then $Q \in$ para-AC^0.*

Proof : As $Q' \in$ para-AC^0, for some $q \in \mathbb{N}$ there are computable functions $k' \mapsto m_{k'}$ and $k' \mapsto \varphi_{k'}$, where all $\varphi_{k'}$ are $\text{FO}_q[\tau' \cup N(m_{k'})]$-sentences, such that for every instance $(\mathcal{A}', k')$ we have

$$(\mathcal{A}', k') \in Q \iff \mathcal{A}'_{N(m_{k'})} \models \varphi_{k'}. \qquad (19)$$

We may assume that the function $k' \mapsto m_{k'}$ is increasing. Let $\mathcal{R}$ be a para-FO reduction from Q to Q' and let red, g, and I be as in the

[1] However, the depth of the circuits depended on the MSO-sentence.

Def. 5.3. Fix $k \in \mathbb{N}$. We let r_k be the maximum of $m_{g(k)}$ and of the cardinality of the universe of $red(k)$. For the FO-interpretation $I_{red(k)}$ (defined in Lemma 3.5) we have $(\mathcal{A}_{N(r_k)})^{I_{red(k)}} = (\mathcal{A} \uplus red(k))_{N(r_k)}$. for every $\mathcal{A} \in \mathrm{ARI}[\tau]$ with $|A| \geq r_k$. By a trivial modification of the interpretation I of (ii) in Def. 5.3 we get an interpretation I_k of the same quantifier rank as I such that

$$\Big(\big(\mathcal{A}_{N(r_k)}\big)^{I_{red(k)}}\Big)^{I_k} = \big(\mathcal{A}' \uplus Str(k')\big)_{N(r_k)}. \tag{20}$$

Now we have

$$(\mathcal{A}, k) \in Q \iff (\mathcal{A}', k') \in Q' \qquad \text{(by (i) in Def. 5.3)}$$

$$\iff \mathcal{A}'_{N(r_k)} \models \varphi_{k'} \quad \big(\text{by (19) as } r_k \geq m_{g(k)} \text{ and } m_{g(k)} \geq m_{k'} \text{ by (iii) in Def. 5.3}\big)$$

$$\iff \big((\mathcal{A}' \uplus Str(k')\big)_{N(r_k)} \models \bigvee_{\ell \leq g(k)} \big(\text{“}k' = \ell\text{”} \wedge (\varphi_\ell)^{\neg U}\big)$$

(here $(\varphi_\ell)^{\neg U}$ is obtained from φ_ℓ by relativizing the quantifiers to $\neg U$, i.e., to the universe A')

$$\iff \big((\mathcal{A}_{N(r_k)})^{I_{red(k)}}\big)^{I_k} \models \bigvee_{\ell \leq g(k)} (\text{“}k' = \ell'' \wedge (\varphi_\ell)^{\neg U}) \text{ (by (20))}$$

$$\iff \mathcal{A}_{N(r_k)} \models \big(\big(\bigvee_{\ell \leq g(k)} (\text{“}k' = \ell'' \wedge \varphi_\ell)\big)^{I_k}\big)^{I_{red(k)}} \text{ (by (6))}.$$

As the quantifier ranks of $I_{red(k)}$ and of I_k are independent of k, the problem Q is slicewise definable with bounded quantifier rank. □

Proposition 5.5. *For d and τ as in Theorem 5.1,*

$$p\text{-MC}(\mathrm{STR}[\tau, d], \mathrm{MSO}^{\mathrm{sent}}) \leq_{\text{para-FO}} p\text{-MC}(\mathrm{STR}[\tau, d], \mathrm{FO}^{\mathrm{sent}}).$$

Proof : Let $((\mathcal{A}, \chi), k)$, i.e., $(\mathcal{A} \uplus Str(\chi), k)$ be an instance of the problem p-MC(STR$[\tau, d]$, MSO$^{\mathrm{sent}}$). Let $\chi_1, \dots, \chi_s$ be all MSO$[\tau]$-sentences of length k. By Theorem 5.2 we can compute FO$[\tau]$-sentences $\chi_1^*, \dots, \chi_s^*$ equivalent to $\chi_1, \dots, \chi_s$ on STR$[\tau, d]$. Denoting by $x^\frown y$ the concatenation of the strings x and y we set

$$red(k) := \big(Str(\chi_1)^\frown Str(\chi_1^*)^\frown \cdots ^\frown Str(\chi_s)^\frown Str(\chi_s^*), S, S^*\big),$$

where S and S^* are unary relations indicating where a substring representing a χ_j and a substring representing a χ_j^* start. We define the FO-interpretation I for the instance $\big((\mathcal{A}, \chi), k\big)$ such that

$$\big((\mathcal{A} \uplus Str(\chi)) \uplus red(k)\big)^I := \big(\mathcal{A} \uplus Str(\chi_i^*)\big) \uplus Str(k_i)$$

if $\chi = \chi_i$ and $k_i = |\chi_i^*|$ for some i with $1 \leq i \leq s$, and such that

$$\big((\mathcal{A} \uplus Str(\chi)) \uplus red(k)\big)^I := \big(\mathcal{A} \uplus Str(\exists x \neg x = x)\big) \uplus Str(1)$$

otherwise. Setting $\mathcal{R}\big((\mathcal{A}, \chi_i), k\big) := \big((\mathcal{A}, \chi_i^*), k_i\big)$ and $\mathcal{R}\big((\mathcal{A}, \chi), k\big) := \big((\mathcal{A}, \exists x \neg x = x), 1\big)$ if $|\chi| \neq k$, and $g(k) := \max\{k_1, \dots, k_s\}$, we have the desired para-FO reduction (compare Definition 5.3). □

We show that in order to get Theorem 5.1 it suffices to prove:

Theorem 5.6. *Let τ be a vocabulary without arithmetical symbols and $d \geq 1$. Then first-order logic has an effective generalized quantifier elimination on the class* STR$[\tau, d]$. *In the terminology of the Definition 3.3 we have $\ell = O(d)$ for $K :=$* STR$[\tau, d]$.

Proof of Theorem 5.1: FO has an effective generalized quantifier elimination on STR$[\tau, d]$ by Thm. 5.6. Thus, p-MC(STR$[\tau, d]$, FO) $\in$ para-AC^0 by Thm. 3.1. As p-MC(STR$[\tau, d]$, MSO$^{\mathrm{sent}}$) is para-FO reducible to p-MC(STR$[\tau, d]$, FO$^{\mathrm{sent}}$), the problem p-MC(STR$[\tau, d]$, MSO$^{\mathrm{sent}}$) is in para-AC^0, too (by Lemma 5.4). □

The next section is devoted to a proof of Theorem 5.6.

6. FO has generalized quantifier elimination on any class of structures of bounded tree-depth

In this section we prove Theorem 5.6, that is, FO has an effective generalized quantifier elimination on the class STR$[\tau, d]$, the class of τ-structures of tree-depth $\leq d$. Here $d \in \mathbb{N}$ and τ is a vocabulary without arithmetical symbols. The two main steps of the proof are:

– There is an FO$[\tau \cup \{<\}]$-formula $parent(x, y)$ defining in all ordered $\tau \cup \{<\}$-structures $(\mathcal{A}, <)$ with $\mathcal{A} \in$ STR$[\tau, d]$ the "parent-child"-relation of a forest witnessing the tree-depth of $\mathcal{A}$ (Prop. 6.5).

– For some $s \in \mathbb{N}$ the class of structures $(\mathcal{A}, <, +, \times) \in \mathrm{ARI}\big[\tau \cup \{<, +, \times\}\big]$ with $\mathcal{A} \in$ STR$[\tau, d]$ and the class of structures $(\mathcal{T}, <, +, \times) \in \mathrm{ARI}\big[\tau_s \cup \{<, +, \times\}\big]$ with $\mathcal{T} \in$ TREE$[\tau_s, d]$ are mutually FO-interpretable.

Then we obtain the desired result by applying the generalized quantifier elimination for FO on TREE$[\tau_s, d]$ proven in Prop. 4.7.

We start with the first step. As the Gaifman graph of a structure $\mathcal{A}$ (see page 6) is FO-definable in $\mathcal{A}$ we can restrict ourselves to the class GRAPH$[d]$ of graphs of tree-depth $\leq d$. We make use of the following well-known facts:

Fact 6.1 ([17])**.** *Let $\mathcal{G}$ be a graph and let $\mathcal{G}_i$ for $i \in I$ be the subgraphs induced on the connected components of $\mathcal{G}$. Then*

$$\mathrm{td}(\mathcal{G}) := \begin{cases} 1 & \text{if } |G| = 1 \\ \max\{\mathrm{td}(\mathcal{G}_i) \mid i \in I\} & \text{if } |I| > 1 \\ 1 + \min\{\mathrm{td}(\mathcal{G} \setminus \{u\}) \mid u \in G\} & \text{otherwise.} \end{cases}$$

Corollary 6.2. *Let $\mathcal{G}$ be a graph and $d \geq 2$.*

(i) *$\mathcal{G}$ has tree-depth 1 if and only if every vertex in $\mathcal{G}$ is isolated.*
(ii) *$\mathcal{G}$ has tree-depth $\leq d$ if and only if every subgraph induced on a connected component of $\mathcal{G}$ has tree-depth $\leq d$.*
(iii) *A connected $\mathcal{G}$ has tree-depth $\leq d$ if and only if there is a vertex $u \in G$ such that every subgraph induced on a connected component of $\mathcal{G} \setminus \{u\}$ has tree-depth $\leq d - 1$.*

Fact 6.3 ([17])**.** *All paths in any graph in* GRAPH$[d]$ *have length at most $2^d - 2$.*

For $i \in \mathbb{N}$ let $path_{\leq i}(x, y)$ be an FO-formula of quantifier rank $O(\log i)$ expressing in graphs that there is a path of length at most i from x to y. By the previous fact, for $d \in \mathbb{N}$ we can express in graphs of tree-depth $\leq d$ that there is a path from x to y by the FO-formula $path_{\leq 2^d - 2}(x, y)$. Moreover, for a sequence of variables $\bar{z} = z_0, \dots, z_{\ell-1}$ we can define an FO-formula $path^\ell_{\leq i}(x, y, \bar{z})$ with $\mathrm{qr}(path^\ell_{\leq i}(x, y, \bar{z})) = \mathrm{qr}(path_{\leq i}(x, y)) = O(\log i)$. such that for every $u, v, \bar{w} \in G$,

$$\mathcal{G} \models path^\ell_{\leq i}(u, v, \bar{w}) \iff \text{there is a path of length at most } i \text{ from } u \text{ to } v \text{ in } \mathcal{G} \setminus \{w_0, \dots, w_{\ell-1}\}.$$

In particular, this implies that u and v belong to $G \setminus \{w_0, \dots, w_{\ell-1}\}$.

Proposition 6.4. *Let $d \geq 1$. The class* GRAPH$[d]$ *is* FO*-axiomatizable; more precisely, there is an* FO$[\{E\}]$*-sentence $treedepth(d)$ of quantifier rank $O(d)$ such that for all graphs $\mathcal{G}$,*

$$\mathcal{G} \models treedepth(d) \iff \mathcal{G} \in \mathrm{GRAPH}[d]. \tag{21}$$

Proof : First for every $\ell > 0$ and $\bar{z} = z_0, \dots, z_{\ell-1}$ we define

$$\Delta^\ell_s(\bar{z}) := \forall x \forall y \big(\neg path^\ell_{\leq 2^s - 2}(x, y, \bar{z}) \rightarrow \neg path^\ell_{\leq 2^s - 1}(x, y, \bar{z})\big).$$

Then $\mathrm{qr}(\Delta^\ell_s) = O(s)$. More importantly, if $\mathcal{G} \models \Delta^\ell_s(\bar{w})$ for a graph $\mathcal{G}$ and $\bar{w} = w_0, \dots, w_{\ell-1} \in G$, then for all vertices u and v of

$\mathcal{H} := \mathcal{G} \setminus \{w_0, \ldots, w_{\ell-1}\}$,

there is a path from u to v in $\mathcal{H}$ iff $\mathcal{G} \models path^{\ell}_{\leq 2^s-2}(u, v, \bar{w})$. (22)

Observe that by Fact 6.3 if $\mathcal{G} \in \text{GRAPH}[s]$, then $\mathcal{G} \models \Delta^{\ell}_s(\bar{w})$ for every $\bar{w} = w_0, \ldots, w_{\ell-1} \in G$. We define by induction on d the formulas

$$treedepth^{\ell}_{\leq d}(\bar{z}) \quad \text{and} \quad ctreedepth^{\ell}_{\leq d}(x, \bar{z})$$

such that for every graph $\mathcal{G}$ and $\bar{w} = w_0, \ldots, w_{\ell-1} \in G$:

– $\mathcal{G} \models treedepth^{\ell}_{\leq d}(\bar{w})$ if and only if $\mathcal{G} \setminus \{w_0, \ldots, w_{\ell-1}\}$ has tree-depth $\leq d$.

– For $u \in G \setminus \{w_0, \ldots, w_{\ell-1}\}$, we have $\mathcal{G} \models ctreedepth^{\ell}_{\leq d}(u, \bar{w})$ if and only if the subgraph of $\mathcal{G} \setminus \{w_0, \ldots, w_{\ell-1}\}$ induced on the connected component containing u has tree-depth $\leq d$.

Let (the formula expresses that x is isolated in $\mathcal{G} \setminus \{z_0, \ldots, z_{\ell-1}\}$)

$$ctreedepth^{\ell}_{\leq 1}(x, \bar{z}) := \forall y (Exy \to \bigvee_{i \in [\ell]} y = z_i)$$

and for $d \geq 2$ (compare (iii) in Corollary 6.2),

$$ctreedepth^{\ell}_{\leq d}(x, \bar{z}) := \Delta^{\ell}_d(\bar{z}) \wedge \exists y \big(path^{\ell}_{\leq 2^d-2}(x, y, \bar{z}) \wedge \forall x' (path^{\ell}_{\leq 2^d-2}(y, x', \bar{z}) \to ctreedepth^{\ell+1}_{\leq d-1}(x', y\bar{z}))\big).$$

If $\mathcal{G} \models \Delta^{\ell}_d(\bar{w})$, then $path^{\ell}_{\leq 2^d-2}(x, y, \bar{w})$ and $path^{\ell}_{\leq 2^d-2}(y, x', \bar{w})$ express that x, y, and x' are in the same connected component of $\mathcal{G} \setminus \{w_0, \ldots, w_{\ell-1}\}$ (by (22)). Next we let

$$treedepth^{\ell}_{\leq d}(\bar{z}) := \forall x \ \big((\bigvee_{i \in [\ell]} x = z_i) \vee ctreedepth^{\ell}_{\leq d}(x, \bar{z})\big).$$

Finally, we set $treedepth(d) := treedepth^{0}_{\leq d}$. □

Fact 6.1 (and Cor. 6.2) state that a graph $\mathcal{G}$ can be decomposed recursively into graphs of strictly decreasing tree-depth by eliminating vertices. This process yields a forest $\mathcal{F}$ for $\mathcal{G}$. If $\text{td}(\mathcal{G}) = d$, then every tree in $\mathcal{F}$ has depth $\leq d-1$ and at least one has depth $d-1$.

Let $d \in \mathbb{N}$. If we have an ordered graph $(\mathcal{G}, <^{\mathcal{G}})$, then in the third line of the recursive definition of Fact 6.1 we remove the $<^{\mathcal{G}}$-least u with the required property. Using the formulas introduced in the previous proof, we can then inductively $\text{FO}[\{E, <\}]$-define the roots of the trees of the forest, their children, their grandchildren, ... By induction on d we can get an $\text{FO}[\{E, <\}]$-formula that in ordered graphs $(\mathcal{G}, <^{\mathcal{G}})$ with $\mathcal{G} \in \text{GRAPH}[d]$ expresses that x is the parent of y (and y is a child of x) in the forest. We replace the forest by a tree by adding a new root whose children consist of the roots of the trees in the forest. We call this tree the *canonical tree* $\mathcal{T}(\mathcal{G})$ *of* $\mathcal{G}$ (more precisely, of $(\mathcal{G}, <^{\mathcal{G}})$). The tree witnesses the tree-depth of $\mathcal{G}$ in the sense that $\mathcal{T}(\mathcal{G})$ has depth d if $\text{td}(\mathcal{G}) = d$. So we have seen:

Proposition 6.5. *There is an* $\text{FO}[\{E, <\}]$*-formula* $parent(x, y)$ *of quantifier rank* $O(d)$ *defining in all ordered graphs* $(\mathcal{G}, <^{\mathcal{G}})$ *with* $\mathcal{G} \in \text{GRAPH}[d]$*, where* $d \geq 1$*, the "parent-child"-relation of the canonical tree of* $(\mathcal{G}, <^{\mathcal{G}})$*.* [2]

By replacing in the formulas $depth_i(x)$ (introduced in the proof of Proposition 4.6) atomic formulas of the form Pxy by $parent(x, y)$ we obtain a formula, which we again denote by $depth_i(x)$, and that expresses in ordered graphs $(\mathcal{G}, <^{\mathcal{G}})$ with $\mathcal{G} \in \text{GRAPH}[d]$ that x has depth i in the canonical tree of $(\mathcal{G}, <^{\mathcal{G}})$. We also can express that "x is an ancestor of y" and that "x is the ancestor of y of depth i", say as follows:

– $ancestor(x, y) := x = y \vee \bigvee_{i=1}^{d} \exists x_1 \ldots \exists x_i (x = x_1 \wedge parent(x_i, y) \wedge \bigwedge_{j=1}^{i-1} parent(x_j, x_{j+1}))$

– $ancestor_i(x, y) := ancestor(x, y) \wedge depth_i(x)$.

We can FO-define in $(\mathcal{G}, <^{\mathcal{G}})$ a labelling $L^{\mathcal{T}(\mathcal{G})}_1, \ldots, L^{\mathcal{T}(\mathcal{G})}_d$ on $\mathcal{T}(\mathcal{G})$ coding the edge relation $E^{\mathcal{G}}$: A vertex $u \in G$ gets the label L_i if in $\mathcal{T}(\mathcal{G})$ its depth is greater than i and $(u, v) \in E^{\mathcal{G}}$ for its ancestor v on level i. More formally, $L^{\mathcal{T}(\mathcal{G})}_i$ is the set

$$\{u \in G \mid (\mathcal{G} <^{\mathcal{G}}) \models \bigvee_{i<j\leq d} depth_j(u) \wedge \exists y (ancestor_i(y, u) \wedge Euy)\}.$$

On the other hand, in $(\mathcal{T}(\mathcal{G}), L^{\mathcal{T}(\mathcal{G})}_1, \ldots, L^{\mathcal{T}(\mathcal{G})}_d)$ we can FO-define the graph $\mathcal{G}$: Its universe consists of the nodes of $\mathcal{T}(\mathcal{G})$ distinct from the root and $E^{\mathcal{G}}$ consists of the pairs (u, v) of this universe with $(\mathcal{T}(\mathcal{G}), L^{\mathcal{T}(\mathcal{G})}_1, \ldots, L^{\mathcal{T}(\mathcal{G})}_d) \models \psi(u, v)$, where

$$\psi(x, y) := \bigvee_{1\leq i<j\leq d} \big((depth_j(x) \wedge L_i x \wedge ancestor_i(y, x)) \vee (depth_j(y) \wedge L_i y \wedge ancestor_i(x, y))\big).$$

Summing up, (implicitly) we have defined FO-interpretations I_1 and J_1 such that for all ordered graph $(\mathcal{G}, <^{\mathcal{G}})$ with $\mathcal{G} \in \text{GRAPH}[d]$,

$$(\mathcal{G}, <^{\mathcal{G}})^{I_1} \in \text{TREE}[d, d] \quad \text{and} \quad ((\mathcal{G}, <^{\mathcal{G}})^{I_1})^{J_1} = \mathcal{G}. \quad (23)$$

Moreover, we can FO-define in $\big(\mathcal{G}, <^{\mathcal{G}}\big)$ an ordering $<^{\mathcal{T}(\mathcal{G})}$ on $T(\mathcal{G})$ by putting the root of $\mathcal{T}(\mathcal{G})$ (the only node not in G) at the end of $<^{\mathcal{G}}$. On the other hand, in $(\mathcal{T}(\mathcal{G}), L^{\mathcal{T}(\mathcal{G})}_1, \ldots, L^{\mathcal{T}(\mathcal{G})}_d, <^{\mathcal{T}(\mathcal{G})})$ we can FO-define $<^{\mathcal{G}}$ from $<^{\mathcal{T}(\mathcal{G})}$ by "forgetting its last element." We denote the corresponding extensions of I_1 and J_1 by I'_1 and J'_1, respectively.

If we add built-in addition and multiplication the interpretations I'_1 and J'_1 can be extended to interpretations I_2 and J_2 so that for $(\mathcal{G}, <, +, \times) \in \text{ARI}[\{E, <, +, \times\}]$ with $\mathcal{G} \in \text{GRAPH}[d]$ we have

$$(\mathcal{G}, <, +, \times)^{I_2} \in \text{ARI}[\tau_d \cup \{<, +, \times\}] \quad \text{and} \quad \big((\mathcal{G}, <, +, \times)^{I_2}\big)^{J_2} = (\mathcal{G}, <, +, \times). \quad (24)$$

For $(\mathcal{G}, <, +, \times) \in \text{ARI}[\{E, <, +, \times\}]$ with $\mathcal{G} \in \text{GRAPH}[d]$, an $\text{FO}[\{E\}]$-formula $\varphi(x_1, \ldots, x_e)$ and $u_1, \ldots, u_e \in G$ we get

$\mathcal{G} \models \varphi(\bar{u}) \iff \big((\mathcal{G}, <)^{I_1}\big)^{J_1} \models \varphi(\bar{u})$ (by (23))

$\iff (\mathcal{G}, <)^{I_1} \models \varphi^{J_1}(\bar{u})$ (by (7))

$\iff (\mathcal{G}, <, +, \times)^{I_2} \models (\varphi^{J_1})^*(\bar{u})$ (as I_2 extends I_1 and by Prop 4.7; the formula φ^{J_1} does not contain $<$ by (23))

$\iff \big(\mathcal{G}, <, +, \times\big) \models \big((\varphi^{J_1})^*\big)^{I_2}(\bar{u})$ (by (7)).

By Proposition 4.7 and Proposition 6.5 the set of formulas $(\varphi^{J_1})^*$, where φ ranges over the $\text{FO}[\{E\}]$-formulas, has quantifier rank bounded by $O(d)$. Thus the same holds for the set $\{((\varphi^{J_1})^*)^{I_2} \mid \varphi \in \text{FO}[\{E\}]\}$. Thus, $\varphi \mapsto \big((\varphi^{J_1})^*\big)^{I_2}$ is the desired effective generalized quantifier elimination procedure on $\text{GRAPH}[d]$.

Due to space limitations we do not present the interpretations corresponding to I_1 and J_1 for arbitrary τ-structures which yield the effective generalized quantifier elimination on $\text{STR}[\tau, d]$

[2] As the root of $\mathcal{T}(\mathcal{G})$ is not in G we should introduce an FO-interpretation of width 2 to be formally correct (see page 4 for the definition of FO-interpretation). We omit this as we believe that it helps to grasp the main idea better.

7. Classes of graphs closed under taking subgraphs

In this section we present two characterizations of the classes of graphs of bounded tree-depth closed under taking subgraphs. Namely:

Theorem 7.1. *Let K be a class of graphs closed under taking subgraphs. Then the following statements are equivalent:*

(i) *K has bounded tree-depth.*
(ii) FO *has an effective generalized quantifier elimination on K.*
(iii) p-MC$(K, \text{FO}) \in$ para-AC^0.

Theorem 3.1 shows the equivalence between (ii) and (iii). If K has bounded tree-depth, then $K \subseteq \text{STR}[\{E\}, d]$ for some d. Thus the implication (i) $\Rightarrow$ (ii) follows from Theorem 5.6. So we do not need the assumption "K is closed under taking subgraphs" for (ii) $\Leftrightarrow$ (iii) and (i) $\Rightarrow$ (ii). The remaining implication (iii) $\Rightarrow$ (i) is a simple consequence of the following fact and of the following theorem.

Fact 7.2 ([17]). *A class K of graphs has bounded tree-depth if and only if there is an upper bound on the lengths of paths in K.*

Let PATH denote the class of graphs that are paths and SUB-PATH be its closure under taking subgraphs.

Theorem 7.3. p-MC(SUB-PATH, FO) $\notin$ para-AC^0.

Proof of (iii) $\Rightarrow$ (i) in Theorem 7.1: If (i) doesn't hold, i.e., if K doesn't have bounded tree-depth, then, by Fact 7.2, there is no bound on the lengths of paths in graphs of K. Therefore K contains the class SUB-PATH as K is closed under taking subgraphs. Hence, p-MC$(K, \text{FO}) \notin$ para-AC^0 by the previous theorem. □

Proof of Theorem 7.3: For $n, k \in \mathbb{N}$ let the graph $\mathcal{G}(n,k)$ consist of $k+1$ disjoint layers $V_0, \ldots, V_k$ of sets of vertices, with each V_i containing exactly n vertices, and with edges appearing only between adjacent layers such that the induced subgraph on $V_i \cup V_{i+1}$ is always a perfect bipartite matching. Let BIP be the class of all $\{E, S, T\}$-structures $(\mathcal{G}(n,k), \{s\}, \{t\})$ with $n, k \geq 3$ and $s \in V_0$ and $t \in V_k$ (so S and T are unary relation symbols). Let p-STCONN(BIP) be the problem

Input: $(\mathcal{G}, \{s\}, \{t\}) \in$ BIP.
Parameter: $k \in \mathbb{N}$.
Problem: Decide whether there is a path of length at most k from s to t.

By a result due to Beame et al [6, Section 3], p-STCONN(BIP) is not in para-AC^0. Thus it suffices to show that there is an FO-reduction from p-STCONN(BIP) to p-MC(SUB-PATH, FO).

So let $(\mathcal{G}, \{s\}, \{t\}) \in$ BIP be an input of p-STCONN(BIP) and $k \in \mathbb{N}$ a parameter. We know that $\mathcal{G} = \mathcal{G}(n, \ell)$ for some $n, \ell \geq 3$. We add to $\mathcal{G}$ two vertices s' and t' and the edges $\{s, s'\}$ and $\{t, t'\}$, thus obtaining a graph $\mathcal{H}$, which is the disjoint union of paths and hence in SUB-PATH. Now $((\mathcal{G}, \{s\}, \{t\}), k) \mapsto ((\mathcal{H}, \varphi_k), k')$ is the desired reduction, where $k' := |\varphi_k|$ and the FO-sentence φ_k expresses that there is an ℓ with $3 \leq \ell \leq k$ and there are vertices u and v of degree one such that there is a path from u to v of length exactly ℓ and there is a path of length exactly $\ell + 2$.

Let us mention in passing that p-STCONN(BIP) $\notin$ para-AC^0 implies that the following problem $Q(L)$ is notin para-AC^0 for $L = \text{FO}$.

Input: A graph $\mathcal{G}$ and an L-sentence φ.
Parameter: k.
Problem: Decide whether $k = \text{td}(\mathcal{G}) + |\varphi|$ and $\mathcal{G} \models \varphi$.

The corresponding problem, where the tree-depth td$(\mathcal{G})$ of $\mathcal{G}$ is replaced by the tree-width of $\mathcal{G}$, is fixed-parameter tractable, even for MSO, i.e., $Q(\text{MSO}) \in$ FPT. The result for $Q(\text{MSO})$ is a (strong) version of Courcelle's Theorem.

As mentioned in the Introduction, in [3] it is shown that $Q(\text{MSO})$ can be solved by a uniform family of circuits of depth bounded by $f(\text{td}(\mathcal{G}) + |\varphi|)$ and size bounded by $f(\text{td}(\mathcal{G}) + |\varphi|) \cdot |G|^{O(1)}$, where f is a computable function. Translating our proofs and results into the language of circuits, they yield a uniform family of circuits solving $Q(\text{MSO})$ of depth bounded by $O(\text{td}(\mathcal{G}))$ and size bounded by $O(\text{td}(\mathcal{G})) \cdot f(|\varphi|) \cdot |G|^{O(1)}$ for some computable function f.

In general the implication (iii) $\Rightarrow$ (i) in Theorem 7.1 does not hold without the assumption "K is closed under taking substructures." Indeed, by Fact 7.2 the class COMP of all complete graphs and the class PATH have unbounded tree-depth. However, by standard techniques it is easy to show that the model-checking problems p-MC(COMP, FO) and p-MC(PATH, FO) are in para-AC^0.

8. The model-checking problem for classes of FO-sentences

The model-checking for a class of Σ_1-sentences of bounded tree-width is fixed-parameter tractable (cf. [16]). With the color-coding technique the result is extended to any class of Σ_1-sentences of modified bounded tree-width in [14]. We introduce the notion of tree-depth of an FO-sentence and prove that the model-checking problem for any class of FO-sentences of bounded tree-depth is in para-AC^0. We get this result by showing that the notions of tree-depth and of quantifier rank are the two sides of the same coin.

Finally we explain the close connection between the complexity of a parameterized problem slicewise definable by FO-sentences and the parameterized model-checking problem for these sentences.

In contrast to the preceding sections, here we consider the parameterized model-checking problem for sentences in a vocabulary possibly containing relations in $\{<, +, \times\}$ or numerical constants.

Let $N(\infty) := \{\bar{i} \mid i < \infty\}$. For an FO-formula φ we let $N(\varphi)$ be $N(m)$ for the least $j \in \mathbb{N}$ such that the numerical constants in φ are in $N(j)$. Let τ be a relational vocabulary possibly containing (some of) the relation symbols in $\{<, +, \times\}$ and $L \subseteq \text{FO}[\tau \cup \{<, +, \times\} \cup N(\infty)]$. Then the *parameterized model-checking problem* p-MC(L^{sent}) *for sentences in L* is the problem

Input: $\mathcal{A} \in \text{ARI}[\tau \cup \{<, +, \times\}]$ and $\varphi \in L^{\text{sent}}$.
Parameter: $k \in \mathbb{N}$.
Problem: Decide if $k = |\varphi|$ and $\mathcal{A}_{C(\varphi)} \models \varphi$.

By space limitations we omit the simple proof of:

Proposition 8.1. *If $L \subseteq \text{FO}[\tau \cup \{<, +, \times\} \cup N(\infty)]$ has bounded quantifier rank, then* p-MC$(L) \in$ para-AC^0.

Tree-depth of first-order sentences. We define the tree-depth of an FO-sentence φ with the property that all quantifiers in φ bind distinct variables (note that every FO-formula is logically equivalent to an FO-formula of the same quantifier rank with this additional property). The graph $\mathcal{G}(\varphi)$ has the set var(φ) of all variables in φ as universe. There is an edge between distinct $x, y \in \text{var}(\varphi)$ if φ has an atomic subformula in which both, x and y, occur.

For example, for the sentence

$$\psi := \exists x_1 \ldots \exists x_n \forall y \bigvee_{i=1}^{n} E y x_i \tag{25}$$

there is a tree of depth 1 (a star with root y) for $\mathcal{G}(\psi)$ (in fact, td$(\mathcal{G}(\psi)) = 2$). However, one easily proves that ψ is not equivalent to a sentence of quantifier rank $\leq n$.

To relate the quantifier rank with the tree-depth, for a sentence φ we introduce a partial ordering $<_\varphi$ on $\mathrm{var}(\varphi)$ and the notion of a good forest for $\mathcal{G}(\varphi)$. Let x, y be two variables in φ. Then $x <_\varphi y$ if y is quantified in the scope of x. For example, $x <_\varphi y$ if φ is of the form $\ldots \exists x(\ldots \forall y \ldots) \ldots$. A forest $\mathcal{F}$ for $\mathcal{G}(\varphi)$ is *good* if for all variables $x, y \in \mathrm{var}(\varphi)$ with $x <_\varphi y$ the variable y is not an ancestor of x in $\mathcal{F}$. For a quantifier-free sentence φ we set $\mathrm{td}(\varphi) = 0$.

It is easy to see that $\mathrm{td}(\psi) = \mathrm{qr}(\psi) = n + 1$ for the ψ in (25).

As $\mathrm{qr}(\varphi)$ is the maximum number of nested quantifiers in φ, one easily gets that every FO-sentence φ (where every two quantifiers bind distinct variables) of quantifier rank $\le q$ has tree-depth $\le q$. In the full version of the paper we show the following converse:

Theorem 8.2. *If an* FO*-sentence φ with the property that all quantifiers bind distinct variables has tree-depth $\le d$, then φ is logically equivalent to a sentence of quantifier rank $\le d$.*

Hence, by Proposition 8.1:

Corollary 8.3. *If* $L \subseteq \mathrm{FO}^{\mathrm{sent}}[\tau \cup \{<, +, \times\} \cup N(\infty)]$ *has bounded tree-depth, then* $p\text{-MC}(L) \in \text{para-AC}^0$.

For a Σ_1-sentence, i.e, for a sentence of the form $\exists x_1 \ldots \exists x_s \chi$ with quantifier free χ, the preceding result has the following corollary, which is, at least implicitly, in [8].

Corollary 8.4. *Every Σ_1-sentence φ, where all quantifiers bind distinct variables, is logically equivalent to a sentence of quantifier rank* $\le \mathrm{td}(\mathcal{G}(\varphi))$.

Proof : Let $\varphi := \exists x_1 \ldots \exists x_s \chi$. For every forest $\mathcal{F}$ for $\mathcal{G}(\varphi)$ there is a permutation π of $\{1, \ldots, n\}$ such that $\mathcal{F}$ is good for $\varphi_\pi := \exists x_{\pi(1)} \ldots \exists x_{\pi(x_s)} \chi$. Now the result follows from the preceding theorem, as φ and φ_π are logically equivalent. □

The Σ_1-sentence $\varphi := \exists x_1 \ldots \exists x_n \exists y \bigwedge_{i=1}^n Eyx_i$ has tree-depth $n + 1$. Permuting the quantifiers in φ we get the logically equivalent sentence $\exists y \exists x_1 \ldots \exists x_n \bigwedge_{i=1}^n Eyx_i$ of tree-depth 2 (choose as good forest a star with root y). And indeed, as claimed by the corollary, φ is logically equivalent to a sentence of quantifier rank 2, e.g., to $\exists y \bigwedge_{i=1}^n \exists x_i Eyx_i$

Model-checking and slicewise* FO*-definability. Let CFO be the extension of FO with counting quantifiers $\exists^{\ge i}$ for $i \in \mathbb{N}$ and specify that such a quantifier adds 1 to the quantifier rank. With the color-coding technique we have shown that FO has generalized quantifier elimination on K if every FO-sentence is equivalent in K to an CFO-sentence of bounded quantifier rank (with built-in arithmetic and constants for an initial segment). So far this is the only general method known to us to prove the generalized quantifier elimination. In the following Remark 8.5 we describe the relationship between parameterized model-checking problems and slicewise FO-definable parameterized problems. Summarizing we will see a close connection between "generalized quantifier elimination," "membership of $p\text{-MC}(-,-) \in \text{para-AC}^0$," and "slicewise definability with bounded quantifier rank." Perhaps it explains why nearly all nontrivial proofs showing membership of a parameterized problem in para-AC^0 use the color-coding technique.

Remark 8.5. Let τ be a vocabulary with $\{<, +, \times\} \subseteq \tau$.

(a) Let $Q \subsetneq \mathrm{ARI}[\tau] \times \mathbb{N}$ be a parameterized problem. Assume that the kth slice of Q is definable by the FO-sentence $\varphi_k \in \mathrm{FO}[\tau \cup N(\infty)]$, that the enumeration $(\varphi_k)_{k \in \mathbb{N}}$ is computable, and that the set $L := \{\varphi_\ell \mid \ell \in \mathbb{N}\}$ is decidable. Assume that $\varphi_\ell \ne \varphi_{\ell'}$ for $\ell \ne \ell'$ (pass from φ_ℓ to $\varphi_\ell \wedge \bar{i} = \bar{i}$ for a suitable $i \in \mathbb{N}$). Then

$$p\text{-MC}(L) \equiv_{\text{para-FO}} Q.$$

In fact note that

$$\begin{aligned} (\mathcal{A}, k) \in Q \quad &\iff \quad \mathcal{A}_{N(\varphi_k)} \models \varphi_k \\ &\iff \quad ((\mathcal{A}, \varphi_k), |\varphi_k|) \in p\text{-MC}(L) \\ &\iff \quad (\mathcal{A} \uplus \mathit{Str}(\varphi_k), |\varphi_k|) \in p\text{-MC}(L). \end{aligned}$$

For $Q \le_{\text{para-FO}} p\text{-MC}(L)$ we take as *red* in Def. 5.3 the function $k \mapsto \mathit{Str}(\varphi_k) \uplus \mathit{Str}(|\varphi_k|)$.

$p\text{-MC}(L) \le_{\text{para-FO}} Q$: As $Q \subsetneq \mathrm{ARI}[\tau] \times \mathbb{N}$, there is an $(\mathcal{A}_0, k_0) \notin Q$. Let $((\mathcal{A}, \varphi), k)$ be an instance of $p\text{-MC}(L)$. For $k \in \mathbb{N}$ let $\varphi_{\ell_1}, \ldots, \varphi_{\ell_s}$ be all FO-sentences φ_ℓ with $|\varphi_\ell| = k$. We set $\mathit{red}(k) := (\mathit{Str}(\varphi_{\ell_1}^\frown \ell_1^\frown \ldots ^\frown \varphi_{\ell_s}^\frown \ell_s), S, P) \uplus (\mathcal{A}_0 \uplus \mathit{Str}(k_0))$ (here $x^\frown y$ denotes the concatenation of the strings x and y), where S (and P) are unary relations indicating where a substring representing a φ_{ℓ_i} (and a parameter ℓ_i) starts. We define the FO-interpretation I such that for an instance

$$\big((\mathcal{A} \uplus \mathit{Str}(\varphi)) \uplus \mathit{red}(k)\big)^I := \begin{cases} \mathcal{A} \uplus \mathit{Str}(\ell_i), & \text{if } \varphi = \varphi_{\ell_i} \text{ for some } \ell_i \\ \mathcal{A}_0 \uplus \mathit{Str}(k_0), & \text{otherwise.} \end{cases}$$

(b) Conversely, let $L \subseteq \mathrm{FO}[\tau \cup N(\infty)]$ be an infinite decidable set of sentences. Let $(\varphi_k)_{k \in \mathbb{N}}$ be a computable enumeration without repetitions of this set. Define $Q \subseteq \mathrm{ARI}[\tau] \times \mathbb{N}$ by

$$(\mathcal{A}, k) \in Q \iff \mathcal{A}_{N(\varphi_k)} \models \varphi_k.$$

Assume that $Q \subsetneq \mathrm{ARI}[\tau] \times \mathbb{N}$. Then, by (a) we have

$$p\text{-MC}(L) \equiv_{\text{para-FO}} Q.$$

References

[1] N. Alon, R. Yuster, and U. Zwick. Color-coding. *Journal of the ACM*, 42(4):844–856, 1995.

[2] M. Bannach, C. Stockhusen, and T. Tantau. Fast parallel fixed-parameter algorithms via color coding. In *Proceedings of IPEC 2015*, pages 224–235, 2015.

[3] M. Bannach and T. Tantau. Parallel multivariate meta-theorems. In *IPEC 2016*, pages 4:1–4:17, 2017.

[4] M. Bannach and T. Tantau. Computing hitting set kernels by AC^0-circuits. In *Proceedings of STACS 2018*, pages 9:1–9:14, 2018.

[5] D. A. Mix Barrington, N. Immerman, and H. Straubing. On uniformity within NC^1. *Journal of Computer and System Sciences*, 41(3):274–306, 1990.

[6] P. Beame, R. Impagliazzo, and T. Pitassi. Improved depth lower bounds for small distance connectivity. *Computational Complexity*, 7(4):325–345, 1998.

[7] C. C. Chang and H. J. Keisler. *Model Theory*. Studies in Logic and the Foundations of Mathematics. Elsevier, 3rd edition, 1990.

[8] H. Chen and M. Müller. The fine classification of conjunctive queries and parameterized logarithmic space. *TOCT*, 7(2):7:1–7:27, 2015.

[9] Y. Chen and J. Flum. Some lower bounds in parameterized AC^0. In *Proceedings of MFCS 2016*, pages 27:1–27:14, 2016.

[10] Y. Chen, J. Flum, and X. Huang. Slicewise definability in first-order logic with bounded quantifier rank. In *Proceedings of CSL 2017*, pages 19:1–19:16, 2017.

[11] B. Courcelle. Graph rewriting: An algebraic and logic approach. In *Handbook of Theoretical Computer Science, Volume B:*, pages 193–242. 1990.

[12] M. Elberfeld, M. Grohe, and T. Tantau. Where first-order and monadic second-order logic coincide. *ACM Trans. Comput. Log.*, 17(4):25:1–25:18, 2016.

[13] M. Elberfeld, A. Jakoby, and T. Tantau. Algorithmic meta theorems for circuit classes of constant and logarithmic depth. In *STACS 2012*, pages 66–77, 2012.

[14] J. Flum and M. Grohe. Fixed-parameter tractability, definability, and model-checking. *SIAM Journal on Computing*, 31(1):113–145, 2001.

[15] J. Gajarský and P. Hlinený. Kernelizing MSO properties of trees of fixed height, and some consequences. *Logical Methods in Computer Science*, 11(1), 2015.

[16] P. G. Kolaitis and M. Y. Vardi. Conjunctive-query containment and constraint satisfaction. In *Proceedings of the 17th Symposium on Principles of Database Systems*, pages 205–213, 1998.

[17] J. Nesetril and P. Ossona de Mendez. Tree-depth, subgraph coloring and homomorphism bounds. *Eur. J. Comb.*, 27(6):1022–1041, 2006.

[18] A. Sankaran. A finitary analogue of the downward Löwenheim-Skolem property. In *Proceedings of CSL 2017*, pages 37:1–37:21, 2017.