On Dual-Rail Based MaxSAT Solving

M. L. Bonet[1], S. Buss[2], A. Ignatiev[3], J. Marques-Silva[3], and A. Morgado[3]

[1] Computer Science Department, Universidad Politécnica de Cataluña, Barcelona, Spain
[2] Department of Mathematics, University of California, San Diego, USA
[3] LASIGE, Faculdade de Ciências, Universidade de Lisboa, Portugal

Abstract. This work overviews recent results on the dual-rail based MaxSAT solving, including polynomial upper bounds on the refutation of PHP and 2PHP formulae with core-guided MaxSAT solvers and MaxSAT resolution as well as their relative efficiency compared to general resolution and cutting planes.

Introduction. The performance improvements observed in MaxSAT solvers in recent years [9] have motivated an effort to devise effective ways of encoding decision and optimization problems as MaxSAT. One recent line of work has been to encode such problems as Horn MaxSAT, using the so-called dual-rail encoding [5, 10]. A side result was the observation that such approach enabled polynomial size refutations of the Pigenhole Principle (PHP) [8]. This result motivated additional insights, that enabled proving that dual-rail MaxSAT solving is stronger than resolution or clause learning [3], but leaving a number of results still open. This paper overviews the recent work on the dual-rail MaxSAT proof system.

Dual-Rail MaxSAT Encoding. The proposed ideas heavily rely on the variant [3, 8] of the dual-rail encoding (DRE) [5, 10]. Let $\mathcal{F}$ be a CNF formula on the set of N variables $X = \{x_1 \ldots, x_N\}$. Given $\mathcal{F}$, the dual-rail MaxSAT encoding [3, 8] creates a (Horn) MaxSAT problem $\langle \mathcal{S}, \mathcal{H} \rangle$, where $\mathcal{H}$ is the set of hard clauses and $\mathcal{S}$ is the set of soft clauses s.t. $|\mathcal{S}| = 2N$. The following holds.

Theorem 1. *$\mathcal{F}$ is satisfiable iff there exists an assignment that satisfies $\mathcal{H}$ and at least N clauses in $\mathcal{S}$.*

Upper Bound for (Doubled) Pigeonhole Principle. The DRE and Theorem 1 enable refuting the renowned pigeonhole principle (PHP) in polynomial time by core-guided MaxSAT [9] or MaxSAT resolution [4]. Recall that PHP states that if $m + 1$ pigeons are distributed by m holes, then at least one hole contains more than one pigeon.[1] As proved in [8], core-guided MaxSAT algorithms are able to refute PHP in polynomial time by applying *only* unit propagation if working with its DRE. Recall that (see Theorem 1) refuting a dual-rail encoded formula requires obtaining a given number of falsified (empty, resp.) clauses with core-guided MaxSAT (MaxSAT resolution, resp.). More concretely, the unit propagation steps to obtain the desired lower bound on the number of falsified soft clauses can be done in $\mathcal{O}(m^3)$ time [8]. As also shown in [8], a MaxSAT

[1] Propositional formulations of PHP_m^{m+1} are well-known [6] and encode the negation of the principle, asking for an assignment such that the $m + 1$ pigeons are placed into m holes. It is well-known that resolution has an exponential lower bound for PHP [1, 7, 11] but also that MaxSAT resolution [4] requires an exponentially large proof to refute PHP.

resolution refutation following these unit propagation steps can be constructed resulting in the following proposition.

Theorem 2. *There exist polynomial size MaxSAT resolution refutations for the dual-rail encoding of* PHP_m^{m+1}.

This result can be extended [3] to the case of the doubled pigeonhole principle (2PHP), also called the "two pigeons per hole principle" [2], which states that if $2m+1$ pigeons are mapped to m holes, then some hole contains at least three pigeons.

Theorem 3. *There are polynomial size MaxSAT resolution refutations of the dual-rail encoding of the* $2\mathrm{PHP}_m^{2m+1}$ *clauses.*

An open question remains whether or not a polynomial upper bound can be obtained for dual-rail MaxSAT refutations of the generalized pigeonhole principle, i.e. for kPHP with $k > 2$.

Simulation Results. Measuring the efficiency of the dual-rail based MaxSAT compared to general resolution and cutting planes, a few statements can be made [3].

Theorem 4. *Dual-rail based MaxSAT resolution polynomially simulates tree-like and general resolution if weighted dual-rail encoding is allowed.*

Theorem 2 and Theorem 4 imply that dual-rail MaxSAT resolution is strictly stronger than resolution. On the other hand, and as shown in [3], it is not stronger than cutting planes:

Theorem 5. *The dual-rail MaxSAT resolution does not p-simulate* CP *or even* $\mathrm{CP}^\star$.

A few open questions remain. First, it is still open whether or not CP can p-simulate dual-rail MaxSAT resolution. Second, the relative efficiency of core-guided MaxSAT compared to the common proof systems is unknown. And finally, it is important to understand the role of the dual-rail encoding and whether or not it can be modified or generalized allowing for construction of a more powerful proof system.

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