Inner Models of Univalence

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Abstract

We present a simple inner model construction for dependent type theory, which preserves univalence.

CCS Concepts • **Software and its engineering → General programming languages**; • **Social and professional topics →** *History of programming languages*;

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Introduction

A part of my talk will be about the meta-theory of dependent type theory extended with the univalence axiom [6] and, in particular, some independence or consistency results about this formal system. Two examples of such results are the following. First, countable choice cannot be proved in type theory with univalence. (Countable choice can be stated in the form $(\Pi(n : N)\ \|A\|) \to \|\Pi(n : N)A\|$, where $\|T\|$ is the propositional truncation [5] of T.) Second, the fact that univalence is consistent with Brouwer's fan theorem.

The goal of this note is to present one step in building such models, which can be expressed purely syntactically and can be seen as a simple case of the translation described in [3]. This also can be seen as a type theoretic version of the inner model construction in set theory.

1 Type theoretic inner model

We consider a model of type theory with a cumulative sequence of universe U_n.

We assume given an operation $G(A)$ on types together with elements of types

$$(\Pi(x : A)G(B)) \to G(\Pi(x : A)B)$$
$$G(A) \to (\Pi(x : A)G(B)) \to G(\Sigma(x : A)B)$$
$$G(A) \to \Pi(x\ y : A)G(\mathsf{Id}\ A\ x\ y)$$

as well as elements of $G(\Sigma(X : \mathsf{U}_n)G(X))$ for all n.

Besides these closure conditions, we also assume that each $G(A)$ is a h-proposition [5, 6].

We can now define an internal translation which provides a new model of type theory. This is a purely syntactical process. We define, where pf denotes the proof that the first component satisfies G (for a type A, $[\![A]\!]$ will be the first component $[A].1$ of $[A]$ and $[A].2$ will be a proof of $G([\![A]\!])$).

$$\begin{array}{lcl}
[x] & = & x \\
[M\ N] & = & [M]\ [N] \\
[\lambda(x : A)\ M] & = & \lambda(x : [\![A]\!])[M] \\
[M.1] & = & [M].1 \\
[M.2] & = & [M].2 \\
[M, N] & = & [M], [N] \\
\\
[\Pi(x : A)\ B] & = & (\Pi(x : [\![A]\!])[\![B]\!], pf) \\
[\Sigma(x : A)\ B] & = & (\Sigma(x : [\![A]\!])[\![B]\!], pf) \\
[\mathsf{Id}\ A\ M\ N] & = & (\mathsf{Id}\ [\![A]\!]\ [M]\ [N], pf) \\
[\mathsf{U}_n] & = & (\Sigma(X : \mathsf{U}_n)G(X), pf) \\
\\
[\![A]\!] & = & [A].1
\end{array}$$

We then have that if $x_1 : A_1, \ldots, x_n : A_n \vdash M : A$ then $x_1 : [\![A_1]\!], \ldots, x_n : [\![A_n]\!] \vdash [M] : [\![A]\!]$.

Theorem 1.1. *The internal model of type theory described by the translation above satisfies univalence if the underlying model satisfies univalence.*

Proof. Univalence for U_n can be expressed by the type [2]

$$\Pi(A : \mathsf{U}_n)\mathsf{isProp}(\Sigma(X : \mathsf{U}_n)\mathsf{Equiv}\ A\ X)$$

where

$$\mathsf{isContr}\ A = \Sigma(a : A)\Pi(x : A)\mathsf{Id}\ A\ a\ x$$
$$\mathsf{isProp}\ A = \Pi(x\ y : A)\mathsf{isContr}\ (\mathsf{Id}\ A\ x\ y)$$
$$\mathsf{isEquiv}\ A\ X\ f = \Pi(x : X)\mathsf{isContr}(\Sigma(a : A)\mathsf{Id}\ X\ (f\ a)\ x)$$
$$\mathsf{Equiv}\ A\ X = \Sigma(f : A \to X)\mathsf{isEquiv}\ A\ X\ f$$

The translation of the statement of univalence is then

$$\Pi(A : [\![\mathsf{U}_n]\!])\mathsf{isProp}(\Sigma(X : [\![\mathsf{U}_n]\!])\mathsf{Equiv}\ A.1\ X.1)$$

and this is provable given that $G(A)$ is a h-proposition. □

2 Examples

Assume that we are given a family $L_a\ (a : A)$ of left exact modalities [4]. If we define $G(X)$ as $\Pi(a : A)\mathsf{isEquiv}\ \eta_a^X$ where $\eta_a^X : X \to L_a\ X$ is the unit of L_a, then the operation $G(X)$ admits all the required operations[4] listed in the previous section.

It is possible to build non trivial examples of such a family of left exact modalities on suitable presheaf models of cubical type theory, models that satisfy the univalence axiom [1]. The type A represents the type of "coverings" while $L_a\ X$ represents the type of "descent data" for the covering a. The fact that η_a^X is an equivalence expresses then that any descent data can be glued together in a unique way.

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