DEEP INFERENCE, HERBRAND'S THEOREM AND EXPANSION PROOFS

BENJAMIN RALPH

Abstract. The reduction of undecidable first-order logic to decidable propositional logic via Herbrand's theorem has long been of interest to theoretical computer science, with the notion of a Herbrand proof motivating the definition of expansion proofs, a compositional formalism that expresses the strictly first-order content of a proof. Recently, a simple deep inference system for first-order logic, KSh2, has been constructed based around the notion of expansion proofs, as a starting point to developing a rich proof theory around this foundation. This abstract summarises the author's recent paper [9], with a slight change of focus due to the nature of the workshop.

A focus on the existential witnesses created in proofs has long been central to first-order proof theory. If one ignores all other information about a first-order proof except for the details of existential introduction rules, one still has an important kernel of the proof, in some sense the part of the proof that is inherently first-order, as opposed to merely propositional. Herbrand, in [6], innovated an approach to first-order proof theory that isolates this first-order content of the proof, and today the notion of a *Herbrand proof* is common, a proof-theoretic object that shows the carrying out of the following four steps, usually but not always in this order:

(1) Expansion of existential subformulae.
(2) Prenexification/elimination of universal quantifiers.
(3) Term assignment.
(4) Propositional tautology check.

An observation one can make is is that defining Herbrand proofs in a deep inference setting is easier and more natural than doing so in Gentzen-style systems (in particular the Sequent Calculus and Natural Deduction). This is because the steps (1), (2) and (3) as defined above are standard inference rules in first-order deep inference proof systems, and, while it is obviously possible to include them as *ad hoc* rules, they are not natural for Gentzen-style systems, especially carried out in this order.

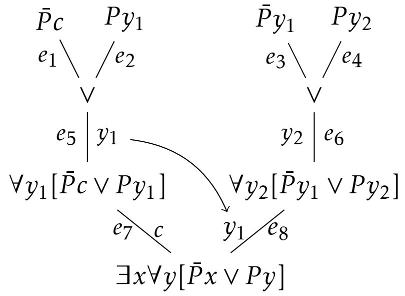
To put it another way: if we want to build a proof theory around Herbrand's theorem, in which the propositional and first-order content of a cut-free proof is separated in a natural way, then deep inference is a superior setting to the sequent calculus, in some concrete senses.

1. A Deep Inference proof system designed around Herbrand's Theorem

In [8], Miller generalises the concept of the Herbrand expansion to higher order logic, representing the witness information in a tree structure, and explicit transformations between these 'expansion proofs' and cut-free sequent proofs are provided. Below is an expansion proof E, which proves the "shallow formula" $Sh(E) =$

Date: April 13, 2018.

$\exists x \forall y \big(\bar{P}x \vee Py \big)$ by producing witnesses for the existential quantifiers and eliminating the universal quantifiers to produce the "deep formula" $Dp(E) = \big(\bar{P}c \vee Py_1\big) \vee \big(\bar{P}y_1 \vee Py_2\big)$, read off from the top line of the expansion proof.



What features would a proof system, PS, designed around Expansion Proofs, EP, have? Say we have a translation $\pi : EP \to PS$.

Firstly, we might want that composition of proofs in PS matches closely to composition of expansion proofs, that something close to functoriality of π holds:

$$\pi(E_1 \star_E E_2) \approx \pi(E_1) \star_F \pi(E_2)$$

For Gentzen-style systems this will prove difficult, as there is no natural way to compose two proofs by disjunction.

A second attractive feature would be that we could isolate a part of the proof system that is relevant to Herbrand's theorem, stating and proving it as a factorisation of proofs, where the first order content of the proof is isolated from the propositional content:

$$\pi(E) = \begin{array}{c} \pi^{Up}(E) \big\Vert Prop \\ Dp(E) \\ \pi^{Lo}(E) \big\Vert FO \\ Sh(E) \end{array}$$

2. KSh2

We can show that the following deep inference proof system, KSh2, satisfies both these features.

$$\mathsf{KSh2} = \mathsf{KS} + \boxed{\begin{array}{cc} \mathsf{r1}\!\downarrow \dfrac{\forall x[A \vee B]}{[\forall x A \vee B]} & \mathsf{h}\!\downarrow \dfrac{\exists x A \vee A\{x \Leftarrow t\}}{\exists x A} \\ \mathsf{r2}\!\downarrow \dfrac{\forall x(A \wedge B)}{(\forall x A \wedge B)} & \exists\mathsf{w}\!\downarrow \dfrac{\mathsf{f}}{\exists x A} \end{array}} + \boxed{\begin{array}{llll} \forall x A = \forall z A\{x \Leftarrow z\} & & \exists z A = \exists z A\{x \Leftarrow z\} & \\ \forall x \forall y A = \forall y \forall x A & & \exists x \exists y A = \exists y \exists x A & \\ \forall x \mathsf{t} = \mathsf{t} = \exists x \mathsf{t} & & \forall x \mathsf{f} = \mathsf{f} = \exists x \mathsf{f} & \end{array}}$$

Where z does not occur in A for the top two equalities.

We define a special class of KSh2 proofs, those in *Herbrand Normal Form* (HNF):

$$\begin{array}{c} Up(\phi) \big\Vert \mathsf{KS} \\ \forall \vec{x} H_\phi(A) \\ Lo(\phi) \big\Vert \{\mathsf{r1}\!\downarrow, \mathsf{r2}\!\downarrow, \mathsf{h}\!\downarrow\} \\ A \end{array}$$

where $H_\phi(A)$, the *Herbrand disjunction of A according to* ϕ, or just the *Herbrand disjunction of A*, contains no quantifiers.

We can prove two theorems tightly linking KSh2 proofs and expansion proofs:

Theorem 2.1. *If* ϕ *is an* KSh2 *proof of A in HNF, then we can construct an expansion proof* $E_\phi = \pi_1(\phi)$, *with* $Sh(E_\phi) = A$, *and* $Dp(E_\phi) = H_\phi(A)$.

Theorem 2.2. *If E is an expansion proof with* $Sh(E) = A$, *then we can construct an* KSh2 *proof* ϕ *of A in HNF, where* $H_\phi(A) = Dp(E)$.

This tight correspondence between KSh2 proofs in HNF and expansion proofs suggests KSh2 proof in HNF as a good candidate for canonical first-order proofs. Therefore, the translation between the two classes enables us to see Herbrand proofs as canonical first-order proofs.

3. Further Work

The translations between deep inference proofs and expansion proofs should be seen as a springboard for further investigations. One obvious next step is to extend KSh2 with cut, and prove cut elimination, so that completeness does not depend on the translation into KSh1 and Brünnler's result. Having done so, we can then make a proper comparison with the cut elimination procedures for expansion proofs described in [1, 5, 7]. Additionally, it would be interesting to try and situate this work in the context of recent work by Aler Tubella and Guglielmi [2, 3], in which they provide a general theory of normalisation for various different propositional logics. In their terminology, a Herbrand proof is close to the notion of a *decomposed* proof, which has two phases: the first contraction-free and the second consisting only of contractions. Extending the procedure, described in [4], to remove identity-cut cycles from SKS proofs to first-order systems is likely to be an important aspect of this research.

References

[1] A. Alcolei, P. Clairambault, M. Hyland, and G. Winskel. The true concurrency of Herbrand's theorem. Submitted, 2017.

[2] A. Aler Tubella. *A study of normalisation through subatomic logic*. Thesis, University of Bath, 2016.

[3] A. Aler Tubella and A. Guglielmi. Subatomic proof systems: Splittable systems. *arXiv preprint arXiv:1703.10258*, 2017. Accepted pending minor revisions by ACM Transactions on Computational Logic.

[4] A. Aler Tubella, A. Guglielmi, and B. Ralph. Removing cycles from proofs. In V. Goranko and M. Dam, editors, *26th EACSL Annual Conference on Computer Science Logic (CSL 2017)*, volume 82 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 9:1–9:17. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2017.

[5] W. Heijltjes. Classical proof forestry. *Annals of Pure and Applied Logic*, 161(11):1346–1366, 2010.

[6] J. Herbrand. *Logical writings*. Springer, 1971.

[7] R. McKinley. A sequent calculus demonstration of Herbrand's theorem. *arXiv preprint arXiv:1007.3414*, 2010.

[8] D. A. Miller. A compact representation of proofs. *Studia Logica*, 46(4):347–370, 1987.

[9] B. Ralph. A natural proof system for Herbrand's theorem. In *International Symposium on Logical Foundations of Computer Science*, pages 289–308. Springer, 2018.