Non-well-founded proof system for Transitive Closure Logic

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Abstract

Transitive closure logic is obtained by a modest addition to first-order logic that affords enormous expressive power. Most importantly, it provides a uniform way of capturing finitary inductive definitions. Thus, particular induction principles do not need to be added to the logic; instead, all induction schemes are available within a single, unified language. We here present a non-well-founded proof system for transitive closure logic which is complete for the standard semantics. This system captures implicit induction, and its soundness is underpinned by the principle of *infinite descent*. In search for effectiveness, we also consider its subsystem of regular, i.e. cyclic, proofs.

Transitive closure (TC) logic has been identified as a promising candidate for a minimal, 'most general' system for inductive reasoning, which is also very suitable for automation [1, 7, 8]. TC adds to first order logic a single operator for forming binary relations: specifically, the transitive closures of arbitrary formulas. This simple addition affords enormous expressive power: namely it provides a uniform way of capturing inductive principles. If an induction scheme is expressed by a formula φ, then the elements of the inductive collection it defines are those 'reachable' from the base elements x via the iteration of the induction scheme. That is, those y's for which (x, y) is in the transitive closure of φ. Thus, bespoke induction principles do not need to be added to, or embedded within, the logic; instead, all induction schemes are available within a single, unified language.

Although the expressiveness of TC logic renders any effective proof system for it incomplete for the standard semantics, a natural, effective proof system which is sound for TC logic was shown to be complete with respect to a generalized Henkin-semantics [9]. Here, following similar developments in other formalizations for fixed point logics and inductive reasoning (e.g. [4, 5, 6, 12, 14]), we present an *infinitary* proof theory for TC logic which is cut-free complete with respect to the standard semantics. The soundness of such infinitary proof theories is underpinned by the principle of *infinite descent*: proofs are permitted to be infinite (i.e. non-well-founded) trees, but subject to the restriction that every infinite path in the proof admits some infinite descent. In the context of formalized induction, we can use formulas interpreted by the elements of inductive collections for witnessing the descent. For this reason, such theories are considered systems of implicit induction, as opposed to those which employ explicit rules for applying induction principles. While a full infinitary proof theory is clearly not effective, such a system can be obtained by restricting consideration to only the *regular* infinite proofs. These are precisely those proofs that can be finitely represented as (possibly cyclic) graphs.

These infinitary proof theories generally subsume systems of explicit induction in expressive power, but also offer a number of advantages. Most notably, they can ameliorate the primary challenge for inductive reasoning: finding an induction *invariant*. In explicit induction systems, this must be provided *a priori*, and is often much stronger than the goal one is ultimately interested in proving. However, in implicit systems the inductive arguments and hypotheses are encoded in the cycles of a proof, so cyclic proof systems seem better for automation. As pointed out in [3], "for proof search, a cyclic proof system can find an induction formula in a more efficient way … since [it] does not have to choose fixed induction formulas in advance."

In the setting of TC logic, we observe some further benefits over more traditional formal systems of inductive definitions and their infinitary proof theories (cf. LKID [6, 11]). As previously mentioned, TC (together with a pairing function) has all inductive definitions immediately 'available' within the language of the logic. As with inductive hypotheses, one does not need to 'know' in advance which induction schemes will be required. Moreover, the use of a single transitive closure operator provides a uniform treatment of all induction schemes. That is, instead of having a proof system parameterized by a set of inductive predicates and rules for them (as is the case in LKID), TC offers a single proof system with a single rule scheme for induction. This has immediate benefits in developing the metatheory: the proofs of completeness w.r.t. standard semantics and adequacy (e.g. subsumption of explicit induction) for the infinitary system are simpler and more straightforward. Furthermore, it allows a simple syntactic criterion to define a cyclic subsystem that is complete for Henkin semantics. This suggests the possibility of more focussed proof-search strategies, further enhancing the potential for automation. TC logic is more expressive in other ways too. The transitive closure operator may be applied to *any* formula, thus one is not restricted to induction principles corresponding only to *monotone* generation schemes (as in, e.g., [4, 6]).

We show that the explicit and cyclic TC systems are equivalent under arithmetic, as is the case for LKID [3, 13]. However, there are cases in which the cyclic system for LKID is strictly more expressive than the explicit induction system [2]. To obtain a similar result for TC, the fact that all induction schemes are available 'at once' poses a serious challenge. For one, the construction used in [2] does not serve to show this result holds for TC. If this strong inequivalence indeed holds also for TC, it must be witnessed by a more subtle and complex counter-example. Conversely, it may be that the explicit and cyclic systems do coincide for TC. In either case, this points towards fundamental aspects that require further investigation.

For a full technical report of this extended abstract see [10].

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