About the unification type of topological logics

Çiğdem Gencer
Istanbul Aydın University
Istanbul, Turkey

Abstract

We introduce a new inference problem for topological logics, the unifiability problem. Our main result is that, within the context of the mereotopology of all regular closed polygons of the real plane, unifiable formulas always have finite complete sets of unifiers.

Keywords topological logics, unification problem.

1 Introduction

Topological logics (TLs) are formalisms for reasoning about topological relations between regions [7, 21–23]. Their languages are obtained from the language of Boolean algebras by the addition of predicates representing these relations. Interpreted over mereotopological spaces, the formulas of these languages describe configurations of concrete objects. Recently, the validity problem determined by different classes of mereotopological spaces has been intensively investigated [13, 14].

We introduce a new inference problem for TLs, the unifiability problem, which extends the validity problem by allowing one to replace variables by terms before testing for validity. There are different motivations for considering the unifiability problem in logics. In a semantically-presented logic, it can be defined as follows [3] : determine whether a given formula becomes valid after replacing its variables by appropriate expressions. An important question in unification theory is [8] : when a formula is unifiable, has it a minimal complete set of unifiers? When the answer is "yes", how large is this set?

There is a wide variety of situations where unifiability problems arise. Let us explain our motivation for considering them within the context of geographic information systems. Suppose the formula $\varphi(p_1,\ldots,p_m)$ describes a given geographic configuration of constant regions $p_1,\ldots,p_m$ and the formula $\psi(x_1,\ldots,x_n)$ represents a desirable geographic property of variable regions $x_1,\ldots,x_n$. It may happen that $\varphi(p_1,\ldots,p_m)\rightarrow\psi(x_1,\ldots,x_n)$ is not valid in the considered geographic environment. Hence, one may ask whether there are n-tuples $(a_1,\ldots,a_n)$ of terms such that $\varphi(p_1,\ldots,p_m)\rightarrow\psi(a_1,\ldots,a_n)$ is valid in this environment. Moreover, one may be interested to obtain the most general n-tuples $(a_1,\ldots,a_n)$ of terms such that $\varphi(p_1,\ldots,p_m)\rightarrow\psi(a_1,\ldots,a_n)$ is valid.

We adapt to the problem of unifiability with constants in TLs the line of reasoning developed by Balbiani and Gencer [6] within the simpler context of the problem of unifiability without constants in Boolean Region Connection Calculus. This adaptation is far from obvious. Our main result is that, within the context of the mereotopology of all regular closed polygons of the real plane, unifiable formulas always have finite complete sets of unifiers.

WiL'18, July, 2018, Oxford, UK
2018.

2 Syntax

It is now time to present the language we will work with.

Terms Let CON be a countable set of *constants* (p, q, etc) and VAR be a countable set of *variables* (x, y, etc). Let $(p_1,p_2,\ldots)$ be an enumeration of CON without repetitions and $(x_1,x_2,\ldots)$ be an enumeration of VAR without repetitions. An *atom* is either a constant, or a variable. The *Boolean terms* (a, b, etc) are defined by

- $a,b ::= p \mid x \mid 0 \mid a^\star \mid (a\cup b)$.

The other Boolean constructs for terms (for instance, 1 and $\cap$) are defined as usual. We adopt the standard rules for omission of the parentheses. Reading terms as regions, the constructs 0, * and $\cup$ should be regarded as the empty region, the complement operation and the union operation. As a result, the constructs 1 and $\cap$ should be regarded as the full region and the intersection operation. For all $m,n\geq 0$, let $TER_{m,n}$ be the set of all terms whose constants form a subset of $\{p_1,\ldots,p_m\}$ and whose variables form a subset of $\{x_1,\ldots,x_n\}$. Let TER be the set of all terms.

Formulas The *formulas* (φ, ψ, etc) are defined by

- $\varphi,\psi ::= C(a,b) \mid a\equiv b \mid \bot \mid \neg\varphi \mid (\varphi\vee\psi)$.

Here, a and b are terms whereas C is the predicate of contact and $\equiv$ is the predicate of equality. We use the notation $a\leq b$ for $a\cup b\equiv b$. For $C(a,b)$ and $a\equiv b$, we propose the readings "a is in contact with b" and "a is equal to b". The other connectives for formulas (for instance, $\top$ and $\wedge$) are defined as usual. We adopt the standard rules for omission of the parentheses. A formula is *equational* iff $\equiv$ is the only predicate possibly occurring in it. For all $m,n\geq 0$, let $FOR_{m,n}$ be the set of all formulas whose constants form a subset of $\{p_1,\ldots,p_m\}$ and whose variables form a subset of $\{x_1,\ldots,x_n\}$. Let FOR be the set of all formulas and FOR^{eq} be the set of all equational formulas. Note that FOR and FOR^{eq} are denoted $\mathcal{C}$ and $\mathcal{B}$ in [13, 14].

3 Semantics

The semantics can be given by interpreting terms and formulas in mereotopological spaces.

Topological spaces The best way to understand the meaning of C is by interpreting it in *topological spaces,* i.e. structures of the form (X, τ) where X is a nonempty set and τ is a set of subsets of X such that the following conditions hold : $\emptyset$ is in τ ; X is in τ ; if $\{A_i \ : \ i \in I\}$ is a finite subset of τ then $\bigcap\{A_i \ : \ i \in I\}$ is in τ ; if $\{A_i \ : \ i \in I\}$ is a subset of τ then $\bigcup\{A_i \ : \ i \in I\}$ is in τ. The subsets of X in τ are called *open sets* whereas their complements are called *closed sets.* In this paper, we will mainly interest with the topological space $\mathbb{R}^2$, i.e. the real plane $\mathbb{R}^2$ together with its ordinary topology.

Regular closed subsets Let (X, τ) be a topological space. Let Int_τ and Cl_τ denote the *interior operator* and the *closure operator* in (X, τ). A subset A of X is *regular closed* iff $Cl_\tau(Int_\tau(A)) = A$. Regular closed subsets of X will also be called *regions.* It is well-known that the set $RC(X, \tau)$ of all regular closed subsets of X forms a Boolean algebra $(RC(X, \tau), 0_X, \star_X, \cup_X)$ where for all $A, B \in RC(X, \tau)$: $0_X = \emptyset$; $A^{\star_X} = Cl_\tau(X \setminus A)$; $A \cup_X B = A \cup B$. As a result, for all $A, B \in RC(X, \tau)$, $1_X = X$ and $A \cap_X B = Cl_\tau(Int_\tau(A \cap B))$. Since regions are regular closed subsets of X, then two regions are *in contact* iff they have a nonempty intersection. For this reason, we define the relation $C^{(X,\tau)}$ on $RC(X, \tau)$ by $C^{(X,\tau)}(A, B)$ iff $A \cap B \neq \emptyset$.

Mereotopologies Let (X, τ) be a topological space. A *mereotopology* over (X, τ) is a Boolean subalgebra M of $RC(X, \tau)$ such that for all $P \in X$ and for all $A \in \tau$, if $P \in A$ then there exists $B \in M$ such that $P \in B$ and $B \subseteq A$. A *mereotopological space* over (X, τ) is a structure (X, τ, M) where M is a mereotopology over (X, τ). Over the topological space $\mathbb{R}^2$, several mereotopologies can be considered [16]. One can consider the mereotopology consisting of the set $RC(\mathbb{R}^2)$ of all regular closed subsets of $\mathbb{R}^2$. Nevertheless, as regions are supposed to be parts of the real plane occupied by concrete objects, it is clear that some of the regular closed subsets of $\mathbb{R}^2$ cannot count as regions. For this reason, one can consider the more concrete mereotopology consisting of the set $RCS(\mathbb{R}^2)$ of all regular closed semi-algebraic subsets of $\mathbb{R}^2$, i.e. those regular closed subsets of $\mathbb{R}^2$ definable by a first-order formula in the language of arithmetic interpreted over $\mathbb{R}$. The main property of this mereotopology is that any of its elements is a finite union of semi-algebraic cells, i.e. semi-algebraic subsets of $\mathbb{R}^2$ homeomorphic to a closed disc. But $RCS(\mathbb{R}^2)$ is not the only candidate for a region-based model of space. In this paper, we will consider the mereotopology consisting of the set $RCP(\mathbb{R}^2)$ of all regular closed polygons of $\mathbb{R}^2$, i.e. those regular closed subsets of $\mathbb{R}^2$ definable by a finite union of finite intersections of closed half-planes. Although this mereotopology may seem overly simple, its study from the perspective of the unifiability problem will turn out to be relatively interesting.

Models Let (X, τ, M) be a mereotopological space. A *valuation* on (X, τ, M) is a map associating with every atom a regular closed subset of X in M. Given a valuation $\mathcal{V}$ on (X, τ, M), we define :

- $\bar{\mathcal{V}}(p) = \mathcal{V}(p)$,
- $\bar{\mathcal{V}}(x) = \mathcal{V}(x)$,
- $\bar{\mathcal{V}}(0) = \emptyset$,
- $\bar{\mathcal{V}}(a^\star) = Cl_\tau(X \setminus \bar{\mathcal{V}}(a))$,
- $\bar{\mathcal{V}}(a \cup b) = \bar{\mathcal{V}}(a) \cup \bar{\mathcal{V}}(b)$.

As a result, $\bar{\mathcal{V}}(1) = X$ and $\bar{\mathcal{V}}(a \cap b) = Cl_\tau(Int_\tau(\bar{\mathcal{V}}(a) \cap \bar{\mathcal{V}}(b)))$. Thus, $\mathcal{V}$ interprets every term as a regular closed subset of X in M. A *model* on (X, τ, M) is a structure $\mathcal{M} = (X, \tau, M, \mathcal{V})$ where $\mathcal{V}$ is a valuation on (X, τ, M). The connectives $\perp$, $\neg$ and $\vee$ being classically interpreted, the *satisfiability* of $\varphi \in FOR$ in $\mathcal{M}$ (in symbols $\mathcal{M} \models \varphi$) is defined as follows :

- $\mathcal{M} \models C(a, b)$ iff $C^{(X,\tau)}(\bar{\mathcal{V}}(a), \bar{\mathcal{V}}(b))$,
- $\mathcal{M} \models a \equiv b$ iff $\bar{\mathcal{V}}(a) = \bar{\mathcal{V}}(b)$.

As a result, $\mathcal{M} \models a \leq b$ iff $\bar{\mathcal{V}}(a) \subseteq \bar{\mathcal{V}}(b)$.

Validity Let (X, τ, M) be a mereotopological space. A formula φ is *valid* in (X, τ, M) iff for all valuations $\mathcal{V}$ on (X, τ, M), $(X, \tau, M, \mathcal{V}) \models \varphi$. Let $\mathcal{C}$ be a class of mereotopological spaces. A formula φ is *$\mathcal{C}$-valid* iff for all mereotopological spaces (X, τ, M) in $\mathcal{C}$, φ is valid in (X, τ, M). The *$\mathcal{C}$-validity problem* consists in determining whether a given formula is $\mathcal{C}$-valid. Let $\mathcal{C}_{all}$ denote the class of all mereotopological spaces. The following formulas are $\mathcal{C}_{all}$-valid :

- $C(a, b) \wedge a \leq a' \to C(a', b)$,
- $C(a, b) \wedge b \leq b' \to C(a, b')$,
- $C(a \cup a', b) \to C(a, b) \vee C(a', b)$,
- $C(a, b \cup b') \to C(a, b) \vee C(a, b')$,
- $C(a, b) \to a \not\equiv 0 \wedge b \not\equiv 0$,
- $a \not\equiv 0 \to C(a, a)$,
- $C(a, b) \to C(b, a)$.

The validity problem is known to be **coNP**-complete in $\mathcal{C}_{all}$. As for the class $\mathcal{C}_{all}^{\mathbb{R}^2}$ of all mereotopological spaces over $\mathbb{R}^2$, the validity problem is known to be **PSPACE**-complete in it [13, 14]. In this paper, we will mainly be interested in the polygon-based mereotopological space $(\mathbb{R}^2, RCP(\mathbb{R}^2))$ over $\mathbb{R}^2$. As a result, from now on, when we write "valid", we mean "valid in the mereotopological space $(\mathbb{R}^2, RCP(\mathbb{R}^2))$".

Proposition 3.1. *For all $\varphi \in FOR^{eq}$, the following are equivalent :* (**1**) *φ is valid ;* (**2**) *for all finite Boolean algebras $\mathcal{B}$ and for all valuations $\mathcal{V}_\mathcal{B}$ on $\mathcal{B}$, $(\mathcal{B}, \mathcal{V}_\mathcal{B}) \models \varphi$;*

(**3**) *for all Boolean algebras* $\mathcal{B}$ *and for all valuations* $\mathcal{V}_{\mathcal{B}}$ *on* $\mathcal{B}$, $(\mathcal{B}, \mathcal{V}_{\mathcal{B}}) \models \varphi$.

Proof. Let $\varphi \in FOR^{eq}$.
($\mathbf{1} \Rightarrow \mathbf{2}$) This follows from the fact that every finite Boolean algebra is isomorphic to a finite Boolean subalgebra of $RCP(\mathbb{R}^2)$.
($\mathbf{2} \Rightarrow \mathbf{3}$) This follows from the fact that we consider here the quantifier-free fragment of the variety of Boolean algebras.
($\mathbf{3} \Rightarrow \mathbf{1}$) Obvious. □

4 Unification

The present section introduces the terminology about unification we will need. From now on, when we write "**CPL**", we mean "Classical Propositional Logic".

Substitutions A *substitution* is a function $\sigma : VAR \longrightarrow TER$ which moves at most finitely many variables. The *domain* of a substitution σ (in symbols $dom(\sigma)$) is the set of variables σ moves. Given a substitution σ, let $\bar{\sigma} : TER \cup FOR \longrightarrow TER \cup FOR$ be the endomorphism such that for all variables x, $\bar{\sigma}(x) = \sigma(x)$. The *composition* of the substitutions σ and τ is the substitution $\sigma \circ \tau$ such that for all $x \in VAR$, $(\sigma \circ \tau)(x) = \bar{\tau}(\sigma(x))$. For all $m, n \geq 0$, let $\Sigma_{m,n}$ be the set of all substitutions σ such that $dom(\sigma) \subseteq \{x_1, \ldots, x_n\}$ and for all positive integers $i \leq n$, $\sigma(x_i)$ is in $TER_{m,n}$. A substitution σ is *equivalent* to a substitution τ (in symbols $\sigma \simeq \tau$) iff for all variables x, $\sigma(x) \equiv \tau(x)$ is valid. Obviously, the relation $\simeq$ is reflexive, symmetric and transitive on the set of all substitutions. A substitution σ is *more general* than a substitution τ (in symbols $\sigma \preceq \tau$) iff there exists a substitution υ such that $\sigma \circ \upsilon \simeq \tau$. Obviously, the relation $\preceq$ is reflexive and transitive on the set of all substitutions. Moreover, it contains $\simeq$. A set of substitutions is *small* iff it contains finitely many non-pairwise equivalent substitutions modulo $\simeq$.

Proposition 4.1. *For all* $m, n \geq 0$, $\Sigma_{m,n}$ *is small.*

Proof. This follows from the fact that, considering terms as formulas in **CPL**, finitely many atoms define finitely many non-pairwise equivalent terms. □

Unifiable formulas A formula φ is *unifiable* iff there exists a substitution σ such that $\bar{\sigma}(\varphi)$ is valid. In that case, we say that σ is a *unifier* of φ. The *unifiability problem* (in symbols $\mathcal{UNIF}$) consists in determining whether a given formula is unifiable [3]. A set of unifiers of $\varphi \in FOR$ is *complete* iff for all unifiers σ of φ, there exists a unifier τ of φ in that set such that $\tau \preceq \sigma$. An important question in unification theory is [8] : when a formula is unifiable, has it a minimal complete set of unifiers? When the answer is "yes", how large is this set?

Unification types A unifiable formula φ is *finitary* iff there exists a finite complete set of unifiers of φ but there exists no with cardinality 1. A unifiable formula φ is *unitary* iff there exists a unifier σ of φ such that for all unifiers τ of φ, $\sigma \preceq \tau$. In that case, we say that σ is a *most general unifier* of φ.

Proposition 4.2. *For all unifiable* $\varphi \in FOR$, *the following are equivalent :* (**1**) φ *is either finitary, or unitary ;* (**2**) *there exists a small set* Σ *of substitutions such that for all unifiers* σ *of* φ, *there exists a unifier* τ *of* φ *in* Σ *such that* $\tau \preceq \sigma$.

Proof. By well-known properties of substitutions. □

Proposition 4.3. *Let* $\varphi \in FOR$, $n \geq 2$ *and* $\sigma_1, \ldots, \sigma_n$ *be substitutions. If the following hold then* φ *is finitary :* (**1**) *for all positive integers* $i \leq n$, σ_i *is a unifier of* φ *;* (**2**) *for all positive integers* $i, j \leq n$, *if* $i \neq j$ *then* $\sigma_i \not\preceq \sigma_j$ *;* (**3**) $\sigma_1, \ldots, \sigma_n$ *form a complete set of unifiers of* φ.

Proof. By well-known properties of substitutions. □

From now on, for all a in TER, when we write "a^0", we mean "$a^\star$" and when we write "a^1", we mean "a".

5 Examples

Some unifiable formulas are unitary. Typically, all formulas of the form $a_1 \equiv b_1 \wedge \ldots \wedge a_k \equiv b_k$: if such a formula is unifiable then it is unitary. Why? Simply because Boolean unifiability is unitary [15]. For some other formulas, if they are unifiable then they are finitary. Luckily, in many cases, this can be easily proved. For example, let us consider the formula

$$\varphi_{01} \quad := \quad x \equiv 0 \vee x \equiv 1.$$

Let σ_0 and σ_1 be the substitutions such that $\sigma_0(x) = 0$, $\sigma_1(x) = 1$ and for all variables y, if $x \neq y$ then $\sigma_0(y) = y$ and $\sigma_1(y) = y$.

Lemma 5.1.
- σ_0 *and* σ_1 *are unifiers of* φ_{01}
- *neither* $\sigma_0 \preceq \sigma_1$, *nor* $\sigma_1 \preceq \sigma_0$,
- σ_0 *and* σ_1 *form a complete set of unifiers of* φ_{01}.

Proof. For the sake of the contradiction, let τ be a unifier of φ_{01} such that neither $\sigma_0 \preceq \tau$, nor $\sigma_1 \preceq \tau$. Hence, neither $0 \equiv \tau(x)$ is valid, nor $1 \equiv \tau(x)$ is valid. Thus, by Proposition 3.1, there exists a finite Boolean algebra $\mathcal{B}_0$ and a valuation $\mathcal{V}_{\mathcal{B}_0}$ on $\mathcal{B}_0$ such that $\bar{\mathcal{V}}_{\mathcal{B}_0}(\tau(x)) \neq 0_{\mathcal{B}_0}$ and there exists a finite Boolean algebra $\mathcal{B}_1$ and a valuation $\mathcal{V}_{\mathcal{B}_1}$ on $\mathcal{B}_1$ such that $\bar{\mathcal{V}}_{\mathcal{B}_1}(\tau(x)) \neq 1_{\mathcal{B}_1}$. Let $\mathcal{B}$ be the product of $\mathcal{B}_0$ and $\mathcal{B}_1$ and $\mathcal{V}_{\mathcal{B}}$ be the valuation on $\mathcal{B}$ such that for all constants $q \in CON$, $\mathcal{V}_{\mathcal{B}}(q) = (\mathcal{V}_{\mathcal{B}_0}(q), \mathcal{V}_{\mathcal{B}_1}(q))$ and for all variables $y \in VAR$, $\mathcal{V}_{\mathcal{B}}(y) = (\mathcal{V}_{\mathcal{B}_0}(y), \mathcal{V}_{\mathcal{B}_1}(y))$. The reader may easily verify that for all terms a, $\bar{\mathcal{V}}_{\mathcal{B}}(a) = (\bar{\mathcal{V}}_{\mathcal{B}_0}(a), \bar{\mathcal{V}}_{\mathcal{B}_1}(a))$. Since $\bar{\mathcal{V}}_{\mathcal{B}_0}(\tau(x)) \neq$

$0_{\mathcal{B}_0}$ and $\bar{\mathcal{V}_{\mathcal{B}_1}}(\tau(x)) \neq 1_{\mathcal{B}_1}$, then $\bar{\mathcal{V}_{\mathcal{B}}}(\tau(x)) \neq 0_{\mathcal{B}}$ and $\bar{\mathcal{V}_{\mathcal{B}}}(\tau(x)) \neq 1_{\mathcal{B}}$. Consequently, $\tau(x) \equiv 0 \vee \tau(x) \equiv 1$ is not valid. Hence, τ is not a unifier of φ_{01} : a contradiction. ☐

Proposition 5.2. *φ_{01} is finitary.*

Proof. By Proposition 4.3 and Lemma 5.1. ☐

Unfortunately, there are unifiable formulas for which the proof that they are finitary can be more involved. For example, let us consider the formula

$$\varphi_{pq} \quad := \quad C(p,q) \to x \not\equiv 0 \wedge x \leq p \cup q.$$

Let σ_p and σ_q be the substitutions such that $\sigma_p(x) = p \cup (q \cap x)$, $\sigma_q(x) = q \cup (p \cap x)$ and for all variables y, if $x \neq y$ then $\sigma_p(y) = y$ and $\sigma_q(y) = y$.

Lemma 5.3.

- *σ_p and σ_q are unifiers of φ_{pq},*
- *if $p \neq q$ then neither $\sigma_p \preceq \sigma_q$, nor $\sigma_q \preceq \sigma_p$,*
- *if $p \neq q$ then σ_p and σ_q form a complete set of unifiers of φ_{pq}.*

Proof. Similiar to the proof of Lemma 5.1. ☐

Proposition 5.4. *If $p \neq q$ then φ_{pq} is finitary.*

Proof. By Proposition 4.3 and Lemma 5.3. ☐

6 Monomials

The purpose of this section is to introduce definitions and properties about terms. These definitions and properties are purely Boolean. They will be used later in Sections 7 and 8. Let $k, m, n \geq 0$ be such that $n \leq k$. An *m-vector* is a map $\vec{s}$ associating with every positive integer $i \leq m$ an element $\vec{s}(i)$ of $\{0,1\}$. A *(k,m,n)-correspondence* is a map f associating with every m-vector $\vec{s}$ a surjective function $f_{\vec{s}} : \{0,1\}^k \longrightarrow \{0,1\}^n$. An *$n$-monomial* is a term of the form $x_1^{\beta_1} \cap \ldots \cap x_n^{\beta_n}$ where $(\beta_1, \ldots, \beta_n) \in \{0,1\}^n$. For all m-vectors $\vec{s}$, considering a term a in $TER_{m,n}$ as a formula in **CPL**, let $mon_{\vec{s}}(n,a)$ be the set of all n-monomials $x_1^{\beta_1} \cap \ldots \cap x_n^{\beta_n}$ such that a is a tautological consequence of $p_1^{\vec{s}(1)} \cap \ldots \cap p_m^{\vec{s}(m)} \cap x_1^{\beta_1} \cap \ldots \cap x_n^{\beta_n}$.

Proposition 6.1. *Let $a \in TER_{m,n}$. Considered as formulas in **CPL**, the terms a and $\bigcup\{p_1^{\vec{s}(1)} \cap \ldots \cap p_m^{\vec{s}(m)} \cap x_1^{\alpha_1} \cap \ldots \cap x_n^{\alpha_n} : \vec{s}$ is an m-vector and $x_1^{\alpha_1} \cap \ldots \cap x_n^{\alpha_n} \in mon_{\vec{s}}(n,a)\}$ are equivalent.*

Proof. By well-known properties of **CPL**. ☐

For all positive integers $i \leq n$, let $\pi_i : \{0,1\}^n \longrightarrow \{0,1\}$ be the function such that for all $(\beta_1, \ldots, \beta_n) \in \{0,1\}^n$, $\pi_i(\beta_1, \ldots, \beta_n) = \beta_i$. Let f be a (k,m,n)-correspondence. For all m-vectors $\vec{s}$, for all $(\beta_1, \ldots, \beta_n) \in \{0,1\}^n$ and for all positive integers $i \leq n$, let $f_{\vec{s}}^{-1}(\beta_1, \ldots, \beta_n)$ be the set of all $(\alpha_1, \ldots, \alpha_k) \in \{0,1\}^k$ such that $f_{\vec{s}}(\alpha_1, \ldots, \alpha_k) = (\beta_1, \ldots, \beta_n)$, $\Delta_{\vec{s},i}$ be the set of all $(\alpha_1, \ldots, \alpha_k) \in \{0,1\}^k$ such that $\pi_i(f_{\vec{s}}(\alpha_1, \ldots, \alpha_k)) = 1$ and $c_{\vec{s},i}$ be the term $\bigcup\{x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k} : (\alpha_1, \ldots, \alpha_k) \in \Delta_{\vec{s},i}\}$. Remark that $\Delta_{\vec{s},i}$ and $c_{\vec{s},i}$ depend on f — more precisely, on $f_{\vec{s}}$ — too.

Proposition 6.2. *For all m-vectors $\vec{s}$ and for all $(\beta_1, \ldots, \beta_n) \in \{0,1\}^n$, considered as formulas in **CPL**, the terms $\bigcup\{x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k} : (\alpha_1, \ldots, \alpha_k) \in f_{\vec{s}}^{-1}(\beta_1, \ldots, \beta_n)\}$ and $c_{\vec{s},1}^{\beta_1} \cap \ldots \cap c_{\vec{s},n}^{\beta_n}$ are equivalent.*

Proof. By well-known properties of **CPL**. ☐

7 Tuples of terms

Let $k, m, n \geq 0$ be such that $n \leq k$. Let $(a_1, \ldots, a_n) \in TER_{m,k}^n$. For all m-vectors $\vec{s}$, we define on $\{0,1\}^k$ the equivalence relation $\sim_{(a_1,\ldots,a_n)}^{k,\vec{s}}$ by $(\alpha_1, \ldots, \alpha_k) \sim_{(a_1,\ldots,a_n)}^{k,\vec{s}} (\alpha'_1, \ldots, \alpha'_k)$ iff for all positive integers $i \leq n$, $x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k} \in mon_{\vec{s}}(k, a_i)$ iff $x_1^{\alpha'_1} \cap \ldots \cap x_k^{\alpha'_k} \in mon_{\vec{s}}(k, a_i)$.

Proposition 7.1. *For all m-vectors $\vec{s}$, $\sim_{(a_1,\ldots,a_n)}^{k,\vec{s}}$ has at most 2^n equivalence classes on $\{0,1\}^k$.*

Proof. By the definition of $\sim_{(a_1,\ldots,a_n)}^{k,\vec{s}}$. ☐

Proposition 7.2. *There exists a (k,m,n)-correspondence f such that for all m-vectors $\vec{s}$ and for all $(\alpha_1, \ldots, \alpha_k), (\alpha'_1, \ldots, \alpha'_k) \in \{0,1\}^k$, if $f_{\vec{s}}(\alpha_1, \ldots, \alpha_k) = f_{\vec{s}}(\alpha'_1, \ldots, \alpha'_k)$ then $(\alpha_1, \ldots, \alpha_k) \sim_{(a_1,\ldots,a_n)}^{k,\vec{s}} (\alpha'_1, \ldots, \alpha'_k)$.*

Proof. By Proposition 7.1. ☐

A (k,m,n)-correspondence f is *balanced* iff for all m-vectors $\vec{s}$ and for all $(\alpha_1, \ldots, \alpha_k), (\alpha'_1, \ldots, \alpha'_k) \in \{0,1\}^k$, if $f_{\vec{s}}(\alpha_1, \ldots, \alpha_k) = f_{\vec{s}}(\alpha'_1, \ldots, \alpha'_k)$ then $(\alpha_1, \ldots, \alpha_k) \sim_{(a_1,\ldots,a_n)}^{k,\vec{s}} (\alpha'_1, \ldots, \alpha'_k)$. By Proposition 7.2, let f be a balanced (k,m,n)-correspondence. For all m-vectors $\vec{s}$, by means of f — more precisely, of $f_{\vec{s}}$ —, we define the n-tuple $(b_{\vec{s},1}, \ldots, b_{\vec{s},n})$ of terms by setting for all positive integers $i \leq n$, $b_{\vec{s},i} = \bigcup\{x_1^{\beta_1} \cap \ldots \cap x_n^{\beta_n} : x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k} \in mon_{\vec{s}}(k, a_i)$ and $f_{\vec{s}}(\alpha_1, \ldots, \alpha_k) = (\beta_1, \ldots, \beta_n)\}$. An n-tuple $(b_1, \ldots, b_n) \in TER_{m,n}^n$ of terms is *properly obtained* from $(a_1, \ldots, a_n)$ iff for all positive integers $i \leq n$, $b_i = \bigcup\{p_1^{\vec{s}(1)} \cap \ldots \cap p_m^{\vec{s}(m)} \cap b_{\vec{s},i} : \vec{s}$ is an m-vector$\}$. For all m-vectors $\vec{s}$, for all $(\beta_1, \ldots, \beta_n) \in \{0,1\}^n$ and for all positive integers $i \leq n$, let $f_{\vec{s}}^{-1}(\beta_1, \ldots, \beta_n)$, $\Delta_{\vec{s},i}$ and $c_{\vec{s},i}$ be as in Section 6. A substitution υ is *properly obtained* from $(a_1, \ldots, a_n)$ iff for all variables y, if $y \notin \{x_1, \ldots, x_n\}$ then $\upsilon(y) = y$ and for all positive integers $i \leq n$, $\upsilon(x_i) = \bigcup\{p_1^{\vec{s}(1)} \cap \ldots \cap p_m^{\vec{s}(m)} \cap c_{\vec{s},i} : \vec{s}$ is an m-vector$\}$.

Proposition 7.3. *Let $(b_1, \ldots, b_n) \in TER_{m,n}^n$ and υ be a substitution. If $(b_1, \ldots, b_n)$ and υ are properly obtained from $(a_1, \ldots, a_n)$ then for all positive integers $i \leq n$,* *considered as formulas in* **CPL**, *the terms a_i and $\bar{\upsilon}(b_i)$ are equivalent.*

Proof. Suppose $(b_1, \ldots, b_n)$ and υ are properly obtained from $(a_1, \ldots, a_n)$. Let $i \leq n$ be a positive integer. Considered as formulas in **CPL**, the following terms are equivalent :

1. $\bigcup\{p_1^{\vec{s}(1)} \cap \ldots \cap p_m^{\vec{s}(m)} \cap \bar{\upsilon}(b_{\vec{s},i})\ :\ \vec{s}$ is an m-vector$\}$.
2. $\bigcup\{p_1^{\vec{s}(1)} \cap \ldots \cap p_m^{\vec{s}(m)} \cap \upsilon(x_1)^{\beta_1} \cap \ldots \cap \upsilon(x_n)^{\beta_n}\ :\ \vec{s}$ is an m-vector, $x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k} \in mon_{\vec{s}}(k, a_i)$ and $f_{\vec{s}}(\alpha_1, \ldots, \alpha_k) = (\beta_1, \ldots, \beta_n)\}$.
3. $\bigcup\{p_1^{\vec{s}(1)} \cap \ldots \cap p_m^{\vec{s}(m)} \cap c_{\vec{s},1}^{\beta_1} \cap \ldots \cap c_{\vec{s},n}^{\beta_n}\ :\ \vec{s}$ is an m-vector, $x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k} \in mon_{\vec{s}}(k, a_i)$ and $f_{\vec{s}}(\alpha_1, \ldots, \alpha_k) = (\beta_1, \ldots, \beta_n)\}$.
4. $\bigcup\{p_1^{\vec{s}(1)} \cap \ldots \cap p_m^{\vec{s}(m)} \cap x_1^{\alpha_1'} \cap \ldots \cap x_k^{\alpha_k'}\ :\ \vec{s}$ is an m-vector, $x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k} \in mon_{\vec{s}}(k, a_i)$, $f_{\vec{s}}(\alpha_1, \ldots, \alpha_k) = (\beta_1, \ldots, \beta_n)$ and $(\alpha_1', \ldots, \alpha_k') \in f_{\vec{s}}^{-1}(\beta_1, \ldots, \beta_n)\}$.
5. $\bigcup\{p_1^{\vec{s}(1)} \cap \ldots \cap p_m^{\vec{s}(m)} \cap x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k}\ :\ \vec{s}$ is an m-vector and $x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k} \in mon_{\vec{s}}(k, a_i)\}$.

The equivalence between 1 and 2 is a consequence of the definition of $(b_{\vec{s},1}, \ldots, b_{\vec{s},n})$. The equivalence between 2 and 3 is a consequence of the definition of υ. The equivalence between 3 and 4 is a consequence of Proposition 6.2. The equivalence between 4 and 5 is a consequence of the definitions of $mon_{\vec{s}}$ and $\sim_{(a_1,\ldots,a_n)}^{k,\vec{s}}$ and the fact that f is balanced. The equivalence between $\bar{\upsilon}(b_i)$ and 1 is a consequence of the definition of $(b_1, \ldots, b_n)$. The equivalence between 5 and a_i is a consequence of Proposition 6.1. $\square$

Proposition 7.4. *Let σ be the substitution such that for all variables y, if $y \notin \{x_1, \ldots, x_n\}$ then $\sigma(y) = y$ and for all positive integers $i \leq n$, $\sigma(x_i) = a_i$. Let $(b_1, \ldots, b_n) \in TER_{m,n}^n$ and τ be the substitution such that for all variables y, if $y \notin \{x_1, \ldots, x_n\}$ then $\tau(y) = y$ and for all positive integers $i \leq n$, $\tau(x_i) = b_i$. Let υ be a substitution. If $(b_1, \ldots, b_n)$ and υ are properly obtained from $(a_1, \ldots, a_n)$ then $\tau \circ \upsilon \simeq \sigma$.*

Proof. Suppose $(b_1, \ldots, b_n)$ and υ are properly obtained from $(a_1, \ldots, a_n)$. Hence, by Proposition 7.3, for all positive integers $i \leq n$, considered as formulas in **CPL**, the terms a_i and $\bar{\upsilon}(b_i)$ are equivalent. Thus, for all positive integers $i \leq n$, $\bar{\upsilon}(\tau(x_i)) \equiv \sigma(x_i)$ is valid. Consequently, $\tau \circ \upsilon \simeq \sigma$. $\square$

Proposition 7.5. *Let $(b_1, \ldots, b_n) \in TER_{m,n}^n$. If $(b_1, \ldots, b_n)$ is properly obtained from $(a_1, \ldots, a_n)$ then for all valuations $\mathcal{V}$ on $RCP(\mathbb{R}^2)$, there exists a valuation $\mathcal{V}'$ on $RCP(\mathbb{R}^2)$ such that for all positive integers $i \leq n$, $\bar{\mathcal{V}}(b_i) = \bar{\mathcal{V}}'(a_i)$.*

Proof. Suppose $(b_1, \ldots, b_n)$ is properly obtained from $(a_1, \ldots, a_n)$. Let $\mathcal{V}$ be a valuation on $RCP(\mathbb{R}^2)$. Let $\mathcal{V}'$ be a valuation on $RCP(\mathbb{R}^2)$ such that for all positive integers $i \leq m$, $\mathcal{V}'(p_i) = \mathcal{V}(p_i)$ and for all m-vectors $\vec{s}$ and for all $(\beta_1, \ldots, \beta_n) \in \{0,1\}^n$, $\bigcup\{\bar{\mathcal{V}}'(p_1^{\vec{s}(1)} \cap \ldots \cap p_m^{\vec{s}(m)} \cap x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k})\ :\ x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k}$ is a k-monomial such that $f_{\vec{s}}(\alpha_1, \ldots, \alpha_k) = (\beta_1, \ldots, \beta_n)\} = \bar{\mathcal{V}}(p_1^{\vec{s}(1)} \cap \ldots \cap p_m^{\vec{s}(m)} \cap x_1^{\beta_1} \cap \ldots \cap x_n^{\beta_n})$. For all positive integers $i \leq n$, the following subsets of $\mathbb{R}^2$ are equal :

1. $\bigcup\{\bar{\mathcal{V}}'(p_1^{\vec{s}(1)} \cap \ldots \cap p_m^{\vec{s}(m)} \cap x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k})\ :\ \vec{s}$ is an m-vector and $x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k} \in mon_{\vec{s}}(k, a_i)\}$.
2. $\bar{\mathcal{V}}(\bigcup\{p_1^{\vec{s}(1)} \cap \ldots \cap p_m^{\vec{s}(m)} \cap x_1^{\beta_1} \cap \ldots \cap x_n^{\beta_n}\ :\ \vec{s}$ is an m-vector, $x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k} \in mon_{\vec{s}}(k, a_i)$ and $f_{\vec{s}}(\alpha_1, \ldots, \alpha_k) = (\beta_1, \ldots, \beta_n)\})$.
3. $\bar{\mathcal{V}}(\bigcup\{p_1^{\vec{s}(1)} \cap \ldots \cap p_m^{\vec{s}(m)} \cap b_{\vec{s},i}\ :\ \vec{s}$ is an m-vector$\})$.

The equality between 1 and 2 is a consequence of the definitions of $mon_{\vec{s}}$, $\sim_{(a_1,\ldots,a_n)}^{k,\vec{s}}$ and $\mathcal{V}'$ and the fact that f is balanced. The equality between 2 and 3 is a consequence of the definition of $(b_{\vec{s},1}, \ldots, b_{\vec{s},n})$. The equality between $\bar{\mathcal{V}}'(a_i)$ and 1 is a consequence of Proposition 6.1. The equality between 3 and $\bar{\mathcal{V}}(b_i)$ is a consequence of the definition of $(b_1, \ldots, b_n)$. $\square$

Proposition 7.6. *Let σ be the substitution such that for all variables y, if $y \notin \{x_1, \ldots, x_n\}$ then $\sigma(y) = y$ and for all positive integers $i \leq n$, $\sigma(x_i) = a_i$. Let $\varphi \in FOR$. Let $(b_1, \ldots, b_n) \in TER_{m,n}^n$ and τ be the substitution such that for all variables y, if $y \notin \{x_1, \ldots, x_n\}$ then $\tau(y) = y$ and for all positive integers $i \leq n$, $\tau(x_i) = b_i$. If $(b_1, \ldots, b_n)$ is properly obtained from $(a_1, \ldots, a_n)$ then σ is a unifier of φ only if τ is a unifier of φ.*

Proof. Suppose $(b_1, \ldots, b_n)$ is properly obtained from $(a_1, \ldots, a_n)$ and τ is not a unifier of φ. Let $\mathcal{V}$ be a valuation on $RCP(\mathbb{R}^2)$ such that $(RCP(\mathbb{R}^2), \mathcal{V}) \not\models \bar{\tau}(\varphi)$. Hence, by Proposition 7.5, let $\mathcal{V}'$ be a valuation on $RCP(\mathbb{R}^2)$ such that for all positive integers $i \leq n$, $\bar{\mathcal{V}}(b_i) = \bar{\mathcal{V}}'(a_i)$. Since $(RCP(\mathbb{R}^2), \mathcal{V}) \not\models \bar{\tau}(\varphi)$, then $(RCP(\mathbb{R}^2), \mathcal{V}') \not\models \bar{\sigma}(\varphi)$. Thus, σ is not a unifier of φ. $\square$

8 Unification type

Now, we are ready to prove the main results of this paper.

Proposition 8.1. *Let $\varphi \in FOR$. Let $m, n \geq 0$ be such that φ's constants form a subset of $\{p_1, \ldots, p_m\}$ and φ's variables form a subset of $\{x_1, \ldots, x_n\}$. For all unifiers σ of φ, there exists a unifier τ of φ in $\Sigma_{m,n}$ such that $\tau \preceq \sigma$.*

Proof. Let σ be a unifier of φ. Without loss of generality, we can assume that for all constants q, if $q \notin \{p_1, \ldots, p_m\}$ then for all positive integers $i \leq n$, q does not occur in $\sigma(x_i)$ and for all variables y, if $y \notin \{x_1, \ldots, x_n\}$ then $\sigma(y) = y$. Let $k \geq 0$ and $(a_1, \ldots, a_n) \in$

$TER^n_{m,k}$ be such that $n \leq k$ and for all positive integers $i \leq n$, $\sigma(x_i) = a_i$. For all m-vectors $\vec{s}$, let $\sim^{k,\vec{s}}_{(a_1,\ldots,a_n)}$ be as in Section 7. By Proposition 7.2, let f be a balanced (k, m, n)-correspondence. For all m-vectors $\vec{s}$, for all $(\beta_1, \ldots, \beta_n) \in \{0,1\}^n$ and for all positive integers $i \leq n$, let $f^{-1}_{\vec{s}}(\beta_1, \ldots, \beta_n)$, $\Delta_{\vec{s},i}$ and $c_{\vec{s},i}$ be as in Section 6. Let $(b_1, \ldots, b_n) \in TER^n_{m,n}$ be an n-tuple of terms properly obtained from $(a_1, \ldots, a_n)$. Let τ be the substitution such that for all variables y, if $y \notin \{x_1, \ldots, x_n\}$ then $\tau(y) = y$ and for all positive integers $i \leq n$, $\tau(x_i) = b_i$. Remark that τ is in $\Sigma_{m,n}$. Moreover, by Proposition 7.6, τ is a unifier of φ. Let υ be a substitution properly obtained from $(a_1, \ldots, a_n)$. By Proposition 7.4, $\tau \circ \upsilon \simeq \sigma$. Hence, $\tau \preceq \sigma$. □

By Proposition 8.1, one can reduce the unifiability problem in TLs to the validity problem. Since the validity problem in TLs is decidable [13, 14], then the unifiability problem is decidable too, its exact complexity being still unknown. As a consequence of Proposition 8.1, we obtain the following

Proposition 8.2. *Let $\varphi \in FOR$. If φ is unifiable then φ is either finitary, or unitary.*

Proof. By Propositions 4.1, 4.2 and 8.1. □

9 Conclusion

In this paper, we have adapted to the problem of unifiability with constants in TLs the line of reasoning developed by Balbiani and Gencer [6] within the simpler context of the problem of unifiability without constants in Boolean Region Connection Calculus. We anticipate a number of further investigations. Firstly, about the choice of the mereotopological space $RCP(\mathbb{R}^2)$. It remains to see whether the line of reasoning developed in this paper will still apply to $RC(\mathbb{R}^2)$ and $RCS(\mathbb{R}^2)$. What happens if we consider mereotopological spaces over the topological space $\mathbb{R}^n$, i.e. the real space $\mathbb{R}^n$ of dimension n together with its ordinary topology, when $n \geq 3$? Secondly, about the computability of the unifiability problem in TLs. By Proposition 8.1, this problem is decidable. Nevertheless, its exact complexity is still unknown. In this respect, we believe that arguments developed in [1] could be used. Thirdly, about adding to the language the predicate of connectedness or the predicate of internal connectedness considered in [13, 14, 21]. The line of reasoning developed in this paper will still apply to these extended languages. Nevertheless, in that case, as proved in [13, 14], the validity problem becomes undecidable.

Acknowledgments

Special acknowledgement is heartly granted to Philippe Balbiani and Tinko Tinchev who contributed to the development of the work we present today.

References

[1] Baader, F. : *On the complexity of Boolean unification.* Information Processing Letters **67** (1998) 215–220.

[2] Baader, F., Borgwardt, S., Morawska, B. : *Extending unification in $\mathcal{EL}$ towards general TBoxes.* In: Principles of Knowledge Representation and Reasoning. AAAI Press (2012) 568–572.

[3] Baader, F., Ghilardi, S. : *Unification in modal and description logics.* Logic Journal of the IGPL **19** (2011) 705–730.

[4] Baader, F., Morawska, B. : *Unification in the description logic $\mathcal{EL}$.* In: Rewriting Techniques and Applications. Springer (2009) 350–364.

[5] Balbiani, P., Gencer, Ç. : *Admissibility and unifiability in contact logics.* In: Logic, Language, and Computation. Springer (2015) 44–60.

[6] Balbiani, P., Gencer, Ç. : *Finitariness of elementary unification in Boolean Region Connection Calculus.* In: Frontiers of Combining Systems. Springer (2017) 281–297.

[7] Balbiani, P., Tinchev, T., Vakarelov, D. : *Modal logics for region-based theories of space.* Fundamenta Informaticæ **81** (2007) 29–82.

[8] Dzik, W. : *Unification Types in Logic.* Wydawnicto Uniwersytetu Slaskiego (2007).

[9] Gencer, Ç. : *Description of modal logics inheriting admissible rules for $K4$.* Logic Journal of the IGPL **10** (2002) 401–411.

[10] Gencer, Ç., de Jongh, D. : *Unifiability in extensions of $K4$.* Logic Journal of the IGPL **17** (2009) 159–172.

[11] Ghilardi, S. : *Best solving modal equations.* Annals of Pure and Applied Logic **102** (2000) 183–198.

[12] Iemhoff, R. : *On the admissible rules of intuitionistic propositional logic.* Journal of Symbolic Computation **66** (2001) 281–294.

[13] Kontchakov, R., Nenov, Y., Pratt-Hartmann, I., Zakharyaschev, M. : *Topological logics with connectedness over Euclidean spaces.* ACM Transactions on Computational Logic **14** (2013) DOI: 10.1145/2480759.2480765.

[14] Kontchakov, R., Pratt-Hartmann, I., Zakharyaschev, M. : *Spatial reasoning with $RCC8$ and connectedness constraints in Euclidean spaces.* Artificial Intelligence **217** (2014) 43–75.

[15] Martin, U., Nipkow, T. : *Boolean unification — the story so far.* Journal of Symbolic Computation **7** (1989) 275–293.

[16] Pratt-Hartmann, I. : *First-order mereotopology.* In: Handbook of Spatial Logics. Springer (2007) 13–97.

[17] Rybakov, V. : *A criterion for admissibility of rules in the model system $S4$ and the intuitionistic logic.* Algebra and Logic **23** (1984) 369–384.

[18] Rybakov, V. : *Admissibility of Logical Inference Rules.* Elsevier (1997).

[19] Rybakov, V., Gencer, Ç., Oner, T. : *Description of modal logics inheriting admissible rules for $S4$.* Logic Journal of the IGPL **7** (1999) 655–664.

[20] Rybakov, V., Terziler, M., Gencer, Ç. : *On self-admissible quasi-characterizing inference rules.* Studia Logica **65** (2000) 417–428.

[21] Tinchev, T., Vakarelov, D. : *Logics of space with connectedness predicates: complete axiomatizations.* In: *Advances in Modal Logic.* College Publications (2010) 434–453.

[22] Vakarelov, D. : *Region-based theory of space: algebras of regions, representation theory, and logics.* In: *Mathematical Problems from Applied Logic. Logics for the XXIst Century. II.* Springer (2007) 267–348.

[23] Wolter, F., Zakharyaschev, M. : *Spatio-temporal representation and reasoning based on RCC-8.* In: Proceedings of the Seventh International Conference on Principles of Knowledge Representation and Reasoning. Morgan Kaufmann (2000) 3–14.