Refining Properties of Filter Models: Sensibility, Approximability and Reducibility

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April 15, 2018

Abstract

In this paper, we study the tedious link between the properties of sensibility and approximability of models of untyped λ-calculus. Approximability is known to be a slightly, but strictly stronger property that sensibility. However, we will see that so far, each and every (filter) model that have been proven sensible are in fact approximable. We explain this result as a weakness of the sole known approach of sensibility: the Tait reducibility candidates and its realizability variants.

In fact, we will reduce the approximability of a filter model D for the λ-calculus to the sensibility of D but for an extension of the λ-calculus that we call λ-calculus with D-tests. Then we show that traditional proofs of sensibility of D for the λ-calculus are smoothly extendable for this λ-calculus with D-tests.

Introduction

Sensibility. It is the ability, for a model, to distinguish non terminating programs from meaningful ones by collapsing the interpretations of the formers (Def. 3). Through Curry-Howard isomorphism, it also corresponds to the consistence of the internal theory of the model. This shows the importance in understanding sensibility, but also the undecidability of such a property.

Such profound but undecidable results are often targets for classification into a hierarchy of subclasses, serving as grinding stone for proof techniques. Here we take an unorthodox approach consisting in classifying sensible models by using as discriminator a slightly stronger property called "approximability". To our surprise, we found out that available methods to prove sensibility (reducibility) where not powerful enough to distinguish sensibility from approximability.

Approximability. The approximation theorem (Def. 5) is an important concept when considering denotational models of the head reduction. In order to study head reduction, λ-calculists systematically use Böhm trees, which are basically normal forms of a degenerated λ-calculus using an error symbol (Def. 4). Such objects are able to approximate terms, the same way as partial evaluations approximate the notion of evaluation. A model is approximable if the interpretation of a term is the limit of its finite Böhm approximants; i.e., infinite behaviors are, in the model, limits of finite ones.

This notion has been extensively studied [1, Section III.17.3] and this article presents a new sufficient condition for approximability, the *weak positivity* by far encompassing any previous results on approximability (of filter models). As a property on models, approximability is supposed to be strictly stronger than sensibility. Indeed, approximability implies that the interpretation of any diverging terms (and only those) are collapsed into the interpretation of the error symbol Ω. This inclusion is supposed to be strict as, for example, approximable models are not able to distinguish the Turing fixpoint from the Church fixpoint. In fact, there is a continuity of sensible but non-approximable λ-theories, it is surprising that we are not able to model any of those.

Reducibility. In this title, "Reducibility" refers to Tait reducibility methods [23] and its modern extensions (including realisability). These methods used to prove structural properties of type systems and models, such as sensibility and approximability but also more practical properties [24]. For type systems, it consists of constructing saturated sets of terms with the wanted property by induction on types, and then in proving that every typable term has been included. For denotational models, the method is more subtle due to the structure not being inductive : one must find a fixpoint to be able to apply the method, but the fixpoint does not need to be computable or constructive.

In Section 2, we use the sensibility and the approximability as a grinding stone to perform yet a new dissection of those reducibility/realisability methods. We try to be as general as possible until the last moment in order to get the the coarsest possible characterization, but also in order to point over the specific weaknesses of the method. We will discuss in the conclusion and along the paper why we were not able to fill the gap between approximability and sensibility. In particular, we insist on the link between this obstacle and the difficulty to perform fixpoint on non-monotonous functions.

Filter Models. Introduced in the 80s using the notion of type as the elementary brick for their construction, filter models [12] (Def.1) are extracted from a type theory with simple types enlarged by intersection types and subtyping. Formally, the interpretation of a λ-term is the filter generated by the set of its types. Variations on the intersection type theory induce different filter models. The resulting class essentially corresponds to the class of Scott complete lattices.

Filter models (and domains) form one of the classes of models of untyped λ-calculus that have been the more broadly studied, but properties such as sensibility and approximability are yet to be understood perfectly. In particular, a simple bibliographical analysis show that that the theoretically huge gap between sensible and approximable models have never been filed by any model. The best advancements toward this direction are covered by the third part of "Lambda-calculus with types" [1].

λ-calculi with tests. In order to exhibit the link between sensibility and approximability, we are using λ-calculi with tests. These are syntactic extensions of the untyped λ-calculus with operators defining types of the underlying intersection type system. The approximability of a filter model D is equivalent to the sensibility of the same model D for the λ-calculus with D-tests $\Lambda_{\tau,D}$ (with respect to a notion of head convergence). This theorem brings together the notions of sensibility and approximability in a novel way. Originally inspired from Wadsworth's labeled $\lambda\bot$-calculus [25] and Girard experiments [17, 14], λ-calculi with D-tests are syntactic extensions of the λ-calculus with operators defining compact elements of the given models. Expressing the model in the syntax allows perform inductions directly on the reduction steps, rather than on the construction of Böhm trees.

1 Preliminaries

1.1 Filter Models

We introduce here the main semantical object of this article: distributive extensional filter models (DEFiM).

The models consists of a set D of "types" (or compact elements), and two operations: the intersection $\wedge$ (characterizing the induced order) and the functional arrow $\rightarrow$ (characterizing the reflexive embedding). Moreover, we will consider extensionality, which means that the η-conversion is viable, it is enabled by (and is equivalent to) the existence of a specific function $\mathbf{ext}_D : D \rightarrow \mathcal{P}_f(D{\times}D)$.

Definition 1 ([12]). *A* filter model *is a triple* $(D, \wedge, \rightarrow)$ *where:*

- $D = (|D|, \wedge)$ *is a pointed meet-semilattice, with* ω *and* $\geq_D$ *denoting top element and the order*
- $\rightarrow$ *is a binary operation on D such that for any finite sequence* $(\alpha_i, \beta_i) \in (D \times D)^n$*:*

$$\gamma{\rightarrow}\delta \geq_D \bigwedge_i \alpha_i{\rightarrow}\beta_i \qquad \Leftrightarrow \qquad \delta \geq_D \bigwedge_{\{i | \gamma \leq \alpha_i\}} \beta_i,$$

in particular, $\gamma{\rightarrow}\delta = \omega$ *iff* $\delta = \omega$.

A filter model is extensional *whenever there is a function* $\mathbf{ext}_D : D{\rightarrow}\mathcal{P}_f(D \times D)$ *that associates to each* $\alpha \in D$ *a finite subset* $\mathbf{ext}_D(\alpha) \subseteq D \times D$ *such that:*

$$\alpha = \bigwedge_{(\beta,\gamma) \in \mathbf{ext}_D(\alpha)} \beta{\rightarrow}\gamma$$

Unfortunately, the choice of the function $\mathbf{ext}_D$ *is generally not unique or even canonical. In order remove any influence from this choice, we restrict our study to* distributive *filter models. A filter model D is distributive whenever any* $\alpha \geq \beta \wedge \gamma$ *is accessible in the sens that there exists a decomposition* $\alpha = \beta' \wedge \gamma'$ *such that* $\beta' \geq_D \beta$ *and* $\gamma' \geq_D \gamma$.

For short, we call *DEFiM* the distributive extensional filter models.

Creating a DEFiM from scratch is often heavy, as they have to satisfy complex rules even forcing the model to be an infinite object. Fortunately, there is a way to automatically infer the required properties from a smaller (often finite) core object. This core object is a partial DEFiM which is a basically a subset of a DEFiM that can be completed into a proper DEFIM.

Example 2. *1.* Scott's D_∞ *[22] is the completion of*

$$|D| := \{\omega, *\}, \qquad \omega \wedge * := *, \qquad \omega{\rightarrow}* := * \qquad \mathbf{ext}_D(*) := \{(\omega, *)\}.$$

Notice, that $*{\rightarrow}*$ *is undefined in D so that we need the completion.*

2. Z_∞ *is the completion of*

$$|Z| := \{\underline{n} \mid n \geq 0\}, \quad n \wedge \omega := n, \quad \omega{\rightarrow}\underline{n{+}1} := \underline{n} \quad \mathbf{ext}_D(n) := \{(\omega, \underline{n{+}1})\}.$$

3. U_∞ *is the completion of*

$$|U| := \{\underline{n} \mid n \geq 0\}, \quad n \wedge \omega := n, \quad \underline{n{+}1}{\rightarrow}\underline{n{+}1} := \underline{n} \quad \mathbf{ext}_D(n) := \{(\underline{n{+}1}, \underline{n{+}1})\}.$$

Definition 3 (Sensibility). *A filter model D is sensible for the untyped* λ*-calculus if diverging terms corresponds exactly to those of empty interpretation:*

$$M{\Downarrow^{\mathrm{h}}} \quad \Leftrightarrow \quad [\![M]\!]^{\vec{x}} \neq \emptyset\ .$$

1.2 Böhm Approximants

The Böhm approximants (or finite Böhm trees) are the normal forms of a λ-calculus extended with a constant[1] Ω and an additional reduction $\rightarrow_\Omega$.

Definition 4. *Let $M \in \lambda$.*

1. *The* direct approximant of *M, written $\boldsymbol{ap}(M)$, is the λ-term defined as:*
   - $\boldsymbol{ap}(M) := \Omega$ *if* $M = \lambda x_1 \dots x_k.(\lambda y.M')NM_1 \cdots M_k$,
   - $\boldsymbol{ap}(M) := \lambda x_1 \dots x_n.x_i \boldsymbol{ap}(M_1) \cdots \boldsymbol{ap}(M_k)$ *if* $M = \lambda x_1 \dots x_n.x_i M_1 \cdots M_k$,
2. *The* set of finite approximants of *M is defined by* $\boldsymbol{B}(M) := \{\boldsymbol{ap}(M') \mid M \rightarrow_h^* M'\}$.

Definition 5. *A filter model is approximable iff the interpretation of any term $M \in \Lambda$ is the sup of its approximants:* $[\![M]\!]^{\vec{x}} = \bigcup [\![N]\!]^{\vec{x}}$ *over all $N \in \boldsymbol{B}(M)$.*

1.3 Collapsing Sensibility and Approximability for Tests

The original idea of using *tests* to recover full abstraction (via a theorem of definability) is due to Bucciarelli *et al.* [10]. In [5, 7], the author caried a precise study of variants of Bucciarelli *et al.*'s calculus adapted to Krivine's models. Here we extend a bit his definition to get all DEFiMs.

Directly dependent on a given DEFiM D, the λ-calculus with D-tests $\Lambda_{\tau,D}$ is, to some extent, an internal calculus for D. In fact, we will see that, for D to be fully abstract for $\Lambda_{\tau,D}$, it is sufficient to be sensible (Th. 6). Notice that in the notation $\Lambda_{\tau,D}$, τ stands for tests and D if the considered DEFiM.

Proposition 1. *Any DEFiM D is a model for its own test extension (the λ-calculus with D-tests), in the sens that the interpretation is contextual and invariant under reduction.*

Theorem 6 (full abstraction)**.** *For any DEFiM D, if D is sensible for $\Lambda_{\tau,D}$, then D is inequationaly fully abstract for the observational preorder of $\Lambda_{\tau,D}$:*

$$[\![M]\!] \subseteq [\![N]\!] \quad \Leftrightarrow \quad \forall C \in \boldsymbol{T}_{\tau,D}^{(\!|\cdot|\!)}, C(\!|M|\!)\Downarrow^{\mathrm{h}} \Rightarrow C(\!|N|\!)\Downarrow^{\mathrm{h}}.$$

Once we have said that sensibility and full abstraction are equivalent properties for test, it should not surprise the reader to learn that approximability is also equivalent to those properties. Indeed, approximability usually corresponds to the adequation of the Böhm-tree's equality, which is a property between sensibility and full abstraction. However, the situation is a bit mere subtle: if the properties of sensibility and full abstraction for $\Lambda_{\tau,D}$ strongly refer to tests mechanisms, the property of approximability is defined independently from tests. This really means that D-tests will behave well exactly whenever D is approximable.

Theorem 7. *Any extensional filter model D, is approximable if and only if it is sensible for D-tests.*

Proof. Both implications are considered separately. □

[1] In other context, the constant Ω has been replaced by $\perp$.

2 Sufficient Condition for the Sensibility of Tests

Using standard (but technical) realisability methods, we can prove the sensibility of calculi with D-tests for the associated model D provided a positiveness condition. Due to the shortness of the abstract, we cannot present the proof, but it follows exactly the proof of sensibility of the λ-caluculus for the same model.

Definition 8. *A (partial) DEFiM D is* stratified positive *(SP for short) if there exist*

- *a valuation $\mathcal{V}$, called polarity, from $D - \{\omega\}$ in the Booleans $\{\mathbb{t}, \mathbb{f}\}$,*
- *a well founded and total preorder $\leq$ in D with ω as a bottom,*

such that for all $\gamma \in D$ and all $(\alpha,\beta) \in \mathbf{ext}_D(\gamma)$:

$$\gamma \geq \beta, \qquad \gamma \simeq \beta \;\Rightarrow\; \mathcal{V}(\gamma) = \mathcal{V}(\beta), \qquad \gamma \geq \alpha, \qquad \gamma \simeq \alpha \;\Rightarrow\; \mathcal{V}(\gamma) \neq \mathcal{V}(\alpha),$$

(where $\simeq := (\leq \cap \geq)$ is the equivalence relation induced by the preorder) and such that:

$$\alpha \wedge \beta \leq \gamma \text{ for } \gamma = \alpha \text{ or for } \gamma = \beta \qquad\qquad (\alpha \wedge \beta) \prec \alpha \Rightarrow (\alpha \wedge \beta) = \beta$$

Moreover, we also require that the polarity is coherent with the intersections on $\simeq$-equivalence classes:

$$\alpha \simeq \beta \quad \Rightarrow \quad \mathcal{V}(\alpha \wedge \beta) = \mathcal{V}(\alpha) \wedge \mathcal{V}(\beta).$$

This condition can be seen as a stratification given by $\preceq$, where the quotient $D_{/\simeq}$ represents the different levels of the stratification, each level endowed with a positive polarity $\mathcal{V}$. This stratification improves the condition of [3] that only considers completions of positive partial DEFiM. This condition is the invariant by completion, which simplify the proof of stratified positivity of DEFiMs of Example 2 (save for P_∞).

Example 9.

- *D_∞ is SP: The stratified positivity is given by $\mathcal{V}(*) = \mathbb{f}$ and $\omega \prec *$.*
- *Z_∞ is SP: Idem, we set $\mathcal{V}(\underline{2n}) = \mathbb{t}$, $\mathcal{V}(\underline{2n+1}) = \mathbb{f}$ and $\omega \prec \underline{m} \simeq \underline{n}$ for all m and n.*
- *U_∞ is not SP: Since $\underline{n} = \underline{n{+}1}{\to}\underline{n{+}1}$, we must have $\underline{n} \succ \underline{n{+}1}$, which creates a non well-founded chain.*

Theorem 10. *Any stratified positive DEFiM D is sensible for $\Lambda_{\tau,D}$ and approximable.*

Unfortunately, those standard realizability proofs does not allows for a full characterization of sensible models due to the need of exhibiting a fixpoint for the realizability candidates. Here are few conjectures for which such approach is inefficient:

Conjecture 1. *Let D a filter model satisfying all the conditions of stratified positiveness except for the well foundedness of the preorder $\preceq$. If Conjecture* **??** *is true, then D is approximable. In particular U_∞ is approximable.*

Conjecture 2. *Consider the two partial DEFIM generated by the equations:*

$$\alpha = \omega{\to}\alpha \qquad \beta = \omega{\to}\alpha \qquad \gamma = (\gamma \wedge \delta){\to}\beta \qquad \delta = \omega{\to}\alpha{\to}\alpha \qquad (1)$$

$$\alpha = \omega{\to}\alpha \qquad \beta = (\beta{\to}\alpha){\to}\alpha \qquad \gamma = (\gamma \wedge \delta){\to}\beta \qquad \delta = \omega{\to}\alpha{\to}\alpha \qquad (2)$$

The first one is conjectured both sensible and approximable while the last example is also conjectured sensible but can be shown non-approximable.

References

[1] Henk Barendregt, Wil Dekkers, and Richard Statman. *Lambda calculus with types*. Cambridge University Press, 2013.

[2] H.P. Barendregt. *The Lambda Calculus, Its Syntax and Semantics*. Studies in Logic and the Foundations of Mathematics, 1984.

[3] Chantal Berline. From computation to foundations via functions and application: The λ-calculus and its webbed models. *Theoretical Computer Science*, 249:81–161, 2000.

[4] Flavien Breuvart. On the characterization of models of $\mathcal{H}^*$. In *Joint Meeting of Computer Science Logic (CSL) and Logic in Computer Science (LICS)*, page 24. ACM, July 2014.

[5] Flavien Breuvart. *dissecting denotational semantics: from the well-established $\mathcal{H}^*$ to the more recent quantitative coeffects*. PhD thesis, Université Paris Diderot, 2015.

[6] Flavien Breuvart. On the characterization of models of $\mathcal{H}^*$: the semantical aspect. *Lecture and Methodes in Computer Sciences*, 2016. Invited.

[7] Flavien Breuvart. On the characterization of models of H* : The operational aspect. *CoRR*, abs/1801.05150, 2018. URL: `http://arxiv.org/abs/1801.05150`, `arXiv:1801.05150`.

[8] Flavien Breuvart. Refining Properties of Filter Models: Sensibility, Approximability and Reducibility. *CoRR*, abs/1801.05153, 2018. URL: `http://arxiv.org/abs/1801.05153`, `arXiv:1801.05153`.

[9] Flavien Breuvart, Giulio Manzonetto, Polonsky Andrew, and Domenico Ruoppolo. New Results on Morris's Observational Theory: the benefits of separating the inseparable. In *FSCD*, 2016.

[10] Antonio Bucciarelli, Alberto Carraro, Thomas Ehrhard, and Giulio Manzonetto. Full abstraction for resource calculus with tests. In Marc Bezem, editor, *Computer Science Logic*, volume 12, pages 97–111, 2011.

[11] Alberto Carraro and Antonino Salibra. Reflexive scott domains are not complete for the extensional lambda calculus. In *Logic In Computer Science, 2009. LICS'09. 24th Annual IEEE Symposium on*, pages 91–100. IEEE, 2009.

[12] M. Coppo, M. Dezani-Ciancaglini, F. Honsell, and G. Longo. Extended Type Structures and Filter Lambda Models. In *Logic Colloquium 82*, pages 241–262, 1984.

[13] Mario Coppo, Mariangiola Dezani-Ciancaglini, and Maddalena Zacchi. Type theories, normal forms, and D_∞ lambda-models. *Information and Computation*, 72(2):85–116, 1987.

[14] Lorenzo Tortora de Falco. *Réseaux, cohérence et expériences obsessionnelles*. PhD thesis, University Paris 7, 2000.

[15] Mariangiola Dezani-Ciancaglini, Ugo de'Liguoro, and Adolfo Piperno. A filter model for concurrent lambda-calculus. *SIAM Journal on Computing*, 27(5):1376–1419, 1998.

[16] Thomas Ehrhard. The Scott model of linear logic is the extensional collapse of its relational model. *Theoretical Computer Science*, 424:20–45, 2012.

[17] Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

[18] Rainer Kerth. *Isomorphisme et équivalence équationnelle entre modèles de λ-calcul*. PhD thesis, Université Paris VII, 1995.

[19] Rainer Kerth. Isomorphism and equational equivalence of continuous λ-models. *Studia Logica*, 61(3):403–415, 1998.

[20] Giulio Manzonetto. A general class of models of $\mathcal{H}^\star$. In *Mathematical Foundations of Computer Science*, volume 5734 of *Lecture Notes in Computer Science*, pages 574–586. Springer, 2009.

[21] David MR Park. The Y-combinator in Scott's lambda-calculus models. Technical Report 13, Dep. of Computer Science, Univ. of Warwick, 1976.

[22] Dana Scott. *Continuous lattices*. Springer, 1972.

[23] William W Tait. Intensional interpretations of functionals of finite type I. *The Journal of Symbolic Logic*, 32(02):198–212, 1967.

[24] Philip Wadler. Theorems for free! In Joseph E. Stoy, editor, *Proceedings of the fourth international conference on Functional programming languages and computer architecture, FPCA 1989, London, UK, September 11-13, 1989*, pages 347–359. ACM, 1989. URL: `http://doi.acm.org/10.1145/99370.99404`, `doi:10.1145/99370.99404`.

[25] Christopher P. Wadsworth. The relation between computational and denotational properties for Scotts D_∞-models of the lambda-calculus. *SIAM J. Comput.*, 5(3):488521, 1976.

[26] X.Gouy. *Etude des théories équationnelles et des propriétés algébriques des modèles stables du λ-calcul*. PhD thesis, Université de Paris 7, 1995.