Propositional Attitude Operators in Homotopy Type Theory
Extended Abstract

Colin Zwanziger

Department of Philosophy
Carnegie Mellon University, Pittsburgh, PA, USA
zwanzig@cmu.edu

1 Précis

Since it interprets propositions by sets of possible worlds, the intensional logic of Montague (*locus classicus* 1973) does not distinguish propositions which are true in the same possible worlds. Because of this, the system does not satisfactorily interpret propositional attitude verbs, a fact which has motivated the development of 'hyperintensional' logics (see Fox and Lappin 2008 for a survey).

Below, I isolate a hyperintensional system, Comonadic Homotopy Type Theory (**CHoTT**), which naturally incorporates the intensional logic of Montague with the usual notions of homotopy type theory (see UFP 2013). This system is a fragment of Shulman (2017). From homotopy type theory, we inherit two notions of equality, $\equiv$ and $=$, which we think of as expressing intensional and extensional equalities, respectively. From Montague, we inherit a syntax for intensional operators, which for us will mean operators which respect intensional but not (necessarily) extensional equality. These are used to interpret propositional attitude operators. When interpreting natural language, the intensional equality is chosen to be sufficiently 'granular' that the usual issues are avoided.

2 Comonadic Homotopy Type Theory

As intimated, our system of study, **CHoTT**, combines a version of Montague's intensional logic with homotopy type theory. Since **CHoTT** is a homotopy type theory, it includes two notions of equality: definitional equality, written $\equiv$, and thought of as intensional equality, and typal equality, written $=$, thought of as extensional equality. Of course, $\equiv$ is stronger than $=$ in a suitable sense. By integrating Montague, we permit 'intensional' operators which do not respect $=$.

In condensed terms, **CHoTT** is the fragment of Shulman (*op. cit.*) consisting of the usual notions of homotopy type theory, together with the comonadic type operator $\flat$, which we think of as an intension operator performing the role of Montague's $\langle s, - \rangle$.

This system is now delineated.

Following in the tradition of Pfenning and Davies (2001) (and including Shulman *op. cit.*), **CHoTT** has two variable judgements,

$$u :: A$$

and

$$x : A$$

We will say (at variance with prior terminology) that "u is an intensional variable of type A," when $u :: A$ and that "x is an extensional variable of type A," when $x : A$. A function in an intensional variable will not be required to respect extensional equality with respect to that variable ('in that argument').

The hypothetical judgements of **CHoTT** have the form

$$\Delta \mid \Gamma \vdash t : B$$

and

$$\Delta \mid \Gamma \vdash t \equiv u : B$$

where Δ represents a list of intensional typed variables, and Γ a list of extensional typed variables. The Γ can depend on the Δ, but not vice versa.

Due to the two variable judgements, there is a duplication of the rules for context extension and variables, with variants for extensional and intensional variables. These are given in the figure below.

$$\frac{}{\cdot \mid \cdot \ \mathsf{ctx}}\ \text{ctx-Emp.}$$

$$\frac{\Delta \mid \Gamma \vdash B : U}{\Delta \mid \Gamma, x : B\ \mathsf{ctx}}\ \text{ctx-Ext.}^e \qquad \frac{\Delta \mid \Gamma, x : A, \Gamma'\ \mathsf{ctx}}{\Delta \mid \Gamma, x : A, \Gamma' \vdash x : A}\ \text{Var.}^e$$

$$\frac{\Delta \mid \cdot \vdash B : U}{\Delta, u :: B \mid \ \cdot\ \mathsf{ctx}}\ \text{ctx-Ext.}^i \qquad \frac{\Delta, u :: A, \Delta' \mid \Gamma\ \mathsf{ctx}}{\Delta, u :: A, \Delta' \mid \Gamma \vdash u : A}\ \text{Var.}^i$$

Fig. 1. The Extensional ($-^e$) and Intensional ($-^i$) Context Rules

We will import the usual homotopy type theoretical notions (as found in UFP *op. cit.*), including $\prod$- and $\sum$-types (corresponding to the quantifiers $\forall$ and $\exists$), universe polymorphism, $=$-types, higher inductive types (HITs), and univalence. However, the typing rules are assumed to affect *extensional variables only*. For instance, the formation rule for $\prod$ is

$$\frac{\Delta \mid \Gamma \vdash A : U \qquad \Delta \mid \Gamma, x : A \vdash B : U}{\Delta \mid \Gamma \vdash \prod_{x:A} B : U}$$

where $x : A$ is required to be extensional.

Of course, the $=$-types provide our notion of extensional equality. It is the crucial restriction of the $=$-rules to extensional variables that ensures that only functions in extensional variables provably respect $=$.

Finally, we have a comonad $\flat$ corresponding to Montague's $\langle s, - \rangle$. Functions in the intensional variable $u :: A$ will correspond to functions in the extensional variable $x : \flat A$.

We have the following rules:

$$\frac{\Delta \mid \cdot \vdash B : U}{\Delta \mid \Gamma \vdash \flat B : U}\ \flat\text{-Form.} \qquad \frac{\Delta \mid \cdot \vdash t : B}{\Delta \mid \Gamma \vdash t^\flat : \flat B}\ \flat\text{-Intro. (Montague's intension operator } {}^\frown(-))$$

$$\frac{\Delta \mid \Gamma, x : \flat A \vdash B : U \qquad \Delta \mid \Gamma \vdash s : \flat A \qquad \Delta, u :: A \mid \Gamma \vdash t : B[u^\flat/x]}{\Delta \mid \Gamma \vdash (\mathsf{let}\ u^\flat := s\ \mathsf{in}\ t) : B[s/x]}\ \flat\text{-Elim.}$$

$$\frac{\Delta \mid \Gamma, x : \flat A \vdash B : U \qquad \Delta \mid \cdot \vdash s : A \qquad \Delta, u :: A \mid \Gamma \vdash t : B[u^\flat/x]}{\Delta \mid \Gamma \vdash \mathsf{let}\ u^\flat := s^\flat\ \mathsf{in}\ t \equiv t[s/u] : B[s^\flat/x]}\ \flat\text{-}\beta\text{-Conversion}$$

Fig. 2. The Rules for $\flat$

Note that the formation and introduction (intension) rules only apply when no extensional variables are present ("in an intensional context").

3 Interpreting Propositional Attitude Operators

We will now see how to interpret propositional attitude operators via the example of a belief operator.

If "Jane believes that The Morning Star is a planet," and, "The Morning Star is the Evening Star," are both true, it does not follow that "Jane believes that The Evening Star is a planet." **CHoTT** gives a natural interpretation to these sentence where this inference indeed fails.

Like in Montague, we assume a type E of entities. Technically, this is a HIT which has among its constructors $j, m, e : E$ interpreting "Jane," "The Morning Star," and "The Evening Star" and $p : m =_E e$ witnessing the truth of $m =_E e$, which is, of course, the interpretation of "The Morning Star is the Evening Star." We further assume a term $B : E \to \flat U \to U$ interpreting "believes" and a predicate $P : E \to U$ interpreting "is a planet."

This allows us to interpret the sentence, "Jane believes that The Morning Star is a planet," compositionally as

$$B(j, P(m)^\flat)$$

and the sentence, "Jane believes that The Evening Star is a planet," compositionally as

$$B(j, P(e)^\flat)$$

Why does the second not follow from the first? In order to derive the inference, it must be that $B(j, P(x)^\flat)$ is a function in an extensional variable, so that it respects the equality $p : m =_E e$. However, $B(j, P(x)^\flat)$ is ill-typed in this case, since the intension operator only applies in intensional contexts.

Note finally that we have implicitly assumed *de dicto* readings in this section.

4 Future Work

Several issues remain to be addressed.

The model theory of **CHoTT** is deferred to later work, and will involve the homotopy-theoretic models used for homotopy type theory.

In its basic use of distinct intensional and extensional equalities, the present work bears a relation to other work on (hyper)intensional semantics, including the system of Fox and Lappin (2008). The exact relation to Fox and Lappin is of interest. **CHoTT** does enjoy several advantages; for instance, it allows multiple terms of type $a = b$, which can be thought of as distinct pieces of evidence that a has the same extension as b.

Finally, no interpretation has been suggested for *de re* propositional attitude sentences. To do so satisfactorily would likely involve a more elaborate type theory.

Acknowledgments

Steve Awodey and Jonas Frey provided valuable advice on this project, in the course of our parallel joint work on comonadic type theory. Any errors are, of course, my own.

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