Internal Universes in Models of Homotopy Type Theory

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Abstract

We begin by recalling the essentially global character of universes in various models of homotopy type theory, which prevents a straightforward axiomatization of their properties using the internal language of the presheaf toposes from which these model are constructed. We get around this problem by extending the internal language with a modal operator for expressing properties of global elements. In this setting we show how to construct a universe that classifies the Cohen-Coquand-Huber-Mörtberg (CCHM) notion of fibration from their cubical sets model, starting from the assumption that the interval is tiny—a property that the interval in cubical sets does indeed have. This leads to an elementary axiomatization of that and related models of homotopy type theory within what we call *crisp type theory*.

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1 Introduction

Voevodsky's univalence axiom in Homotopy Type Theory (HoTT) [39] is motivated by the fact that constructions on structured types should be invariant under isomorphism. From a programming point of view, such constructions can be seen as type-generic programs. For example, if G and H are isomorphic groups, then for any construction C on groups, an instance $C(G)$ can be *transported* to $C(H)$ by lifting this isomorphism using a type-generic program corresponding to C. As things stand, there is no single definition of the semantics of such generic programs; instead there are several variations on the theme of giving a computational interpretation to the new primitives of HoTT (univalence and higher inductive types) via different constructive models [9, 13, 6, 5], the pros and cons of which are still being explored.

As we show in this paper, that exploration benefits from being carried out in a type-theoretic language. This is different from developing the consequences of HoTT itself using a type-theoretic language, such as intensional Martin-Löf type theory with axioms for univalence and higher inductive types, as used in [39]. There *all* types have higher-dimensional structure, or "are fibrant" as one says, via the structure of the iterated identity types associated with them. Contrastingly, when using type theory to describe models of HoTT, being fibrant is an explicit structure external to a type; and that structure can itself be classified by a type, so that users of the type theory can *prove* that a type is fibrant by inhabiting a certain other type. As an example, consider the cubical sets model of type theory introduced by Cohen, Coquand, Huber and Mörtberg (CCHM) [13]. This model uses a presheaf topos on a particular category of cubes that we denote by $\square$, generated by an interval object $\mathbb{I}$, maps out of which represent paths. The corresponding presheaf topos $\widehat{\square}$ has an associated category with families (CwF) [15] structure that gives a model of Extensional Martin-Löf Type Theory [27] in a standard way [19]. While not all types in this presheaf topos have a fibration structure in the CCHM sense, working within constructive set theory, CCHM show how to make a new CwF of fibrant types out of this presheaf CwF, one which is a model of Intensional Martin-Löf Type Theory with univalent universes and (some) higher inductive types [39]. Their model construction is rather subtle and complicated. Coquand noticed that the CCHM version of Kan fibration could be more simply described in terms of partial elements in the *internal language* of the topos. Some of us took up and expanded upon that suggestion in [30] and [10, Section 4]. Using Extensional Martin-Löf Type Theory with an impredicative universe of propositions (one candidate for the internal language of toposes), those works identify some relatively simple axioms for an interval and a collection of Kan-filling shapes (*cofibrant* propositions) that are sufficient to define a CwF of CCHM fibrations and prove most of its properties as a model of univalent foundations, for example, that Π, Σ, path and other types are fibrant. These internal language constructions can be used as an intermediate point in constructing a concrete model in cubical sets: the type theory of HoTT [39] can be translated into the internal language of the topos, which has a semantics in the topos itself in a standard way. The advantages of this indirection are two-fold. First, the definition and properties of the notion of fibration (both the CCHM notion [13] and other related ones [5, 34]) are simpler when expressed in the internal language; and secondly, so long as the axioms are not too constraining, it opens up the possibility of finding new models of HoTT. Indeed, since our axioms do not rely on the infinitary aspects of Grothendieck toposes (such as having infinite colimits), it is possible to consider models of them in elementary toposes, such as Hyland's effective topos [16, 38].

From another point of view, the internal language of the presheaf topos can itself be

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viewed as a two-level type theory [4, 40] with fibrant and non-fibrant types, where being fibrant is classified by a type, and the constructions are a library of fibrancy instances for all of the usual types of type theory. Directed type theory [34] has a very similar story: it adds a directed interval type and a logic of partial elements to homotopy type theory, and using them defines some new notions of higher-dimensional structure, including co- and contravariant fibrations.

However, the existing work describing models using an internal language [30, 10, 5] does not encompass *universes* of fibrant types. The lack of universes is a glaring omission for making models of HoTT, due to both their importance and the difficulty of defining them correctly. Moreover, it is an impediment to using internal language presentations of cubical type theory as a two-level type theory. For example, most constructions on higher inductive types, like calculating their homotopy groups, require a fibrant universe of fibrant types; and adding universes to directed type theory would have analogous applications. Finally, packaging the fibrant types together into a universe restores much of the convenience of working in a language where all types are fibrant: instead of passing around separate fibrancy proofs, one knows that a type is fibrant by virtue of the universe to which it belongs.

In this paper, we address this issue by studying universes of fibrant types expressed in internal languages for models of cubical type theories. CCHM [13] define a universe concretely using a version of the Hofmann-Streicher universe construction in presheaf toposes [20]. This gives a classifier for their notion of fibration—the universe is equipped with a CCHM fibration that gives rise to every fibration (with small fibres) by re-indexing along a function into the universe. In this way one gets a model of a Tarski-style universe closed under whatever type-forming operations are supported by CCHM fibrations. Thus, there is an appropriate semantic target for a universe of fibrant types, but neither [30], nor [10] gave a version of such a universe expressed in the internal language. This is for a good reason: [32, Remark 7.5] points out that there can be no *internal* universe of types equipped with a CCHM fibration that weakly classifies fibrations. We recall in detail why this is the case in Section 3, but the essence is that naïve axioms for a weak classifier for fibrations imply that a family of types, each member of which is fibrant, has to form a fibrant family; but this is not true for many notions of fibration, such as the CCHM one.

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To fix this issue, in Section 4 we enrich the internal language to a *modal* type theory with two context zones [33, 14, 36], inspired in particular by the fact that cubical sets are a model of Shulman's spatial type theory. In a judgement $\Delta \mid \Gamma \vdash a : A$ of this modal type theory, the context Γ represents the usual local elements of types in the topos, while the new context Δ represents global ones. The dual context structure is that of an S4 necessity modality in modal logic, because a global element determines a local one, but global elements cannot refer to local elements. We use Shulman's term "crisp" for variables from Δ, and call the type theory *crisp type theory*, because we do not in fact use any of the modal type operators of his spatial type theory, but just Π-types whose domains are crisp. Using these crisp Π-types, we give axioms that specify a universe that classifies global fibrations—the modal structure forbids the internal substitutions that led to inconsistency.

One approach to validating these universe axioms would be to check them directly in a cubical set model; but we can in fact do more work using crisp type theory as the internal language and reduce the universe axioms to a structure that is simpler to check in models. Specifically, in Theorem 5.2, we construct such a universe from the assumption that the interval $\mathbb{I}$ is *tiny*, which by definition means that its exponential functor $\mathbb{I} \to \_$ has a right adjoint (a global one, not an internal one—this is another example where crisp type theory is needed to express this distinction). The ubiquity of right adjoints to exponential functors was

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first pointed out by Lawvere [23] in the context of synthetic differential geometry. Awodey pointed out their occurrence in interval-based models of type theory in his work on various cube categories [7]. As far as we know, it was Sattler who first suggested their relevance to constructing universes in such models (see [35, Remark 8.3]). It is indeed the case that the interval object in the topos of cubical sets is tiny. Some ingenuity is needed to use the right adjoint to $\mathbb{I} \to \_$ to construct a universe with a fibration that gives rise to every other one up to equality, rather than just up to isomorphism; we employ a technique of Voevodsky [41] to do so.

Finally, we describe briefly some applications in Section 6. First, our universe construction based on a tiny interval is the missing piece that allows a completely internal development of a model of univalent foundations based upon the CCHM notion of fibration, albeit internal to crisp type theory rather than ordinary type theory. Secondly, we describe a preliminary result showing that our axioms for universes are suitable for building type theories with hierarchies of universes, each with a different notion of fibration.

The constructions and proofs in this paper have been formalized in Agda-flat [2], an appealingly simple extension of Agda [3] that implements crisp type theory; see `https://doi.org/10.17863/CAM.22369`. Agda-flat was provided to us by Vezzosi as a by-product of his work on modal type theory and parametricity [29].

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2 Internal description of fibrations

We begin by recalling from [32, 10] the internal description of fibrations in presheaf models, using CCHM fibrations [13, Definition 13] as an example. Rather than using Extensional Martin-Löf Type Theory with an impredicative universe of propositions as in [32, 10], here we use an intensional and predicative version, therefore keeping within a type theory with decidable judgements.[4] Our type theory of choice is the one implemented by Agda [3], whose assistance we have found invaluable for developing and checking the definitions. Adopting Agda-style syntax, dependent function types are written $(x : A) \to B\,x$, or $\{x : A\} \to B\,x$ if the argument to the function is implicit; non-dependent function types are written $(\_ : A) \to B$, or just $A \to B$. There is a non-cumulative hierarchy of Russell-style [25] universe types $\mathsf{Set} = \mathsf{Set}_0 : \mathsf{Set}_1 : \mathsf{Set}_2 : \mathsf{Set}_3 \ldots$ Among Agda's inductive types we need identity types $\_ \equiv \_ : \{A : \mathsf{Set}_n\} \to A \to A \to \mathsf{Set}_n$, which form the inductively defined family of types with a single constructor $\mathsf{refl} : \{A : \mathsf{Set}_n\}\{x : A\} \to x \equiv x$; and we need the empty inductive type $\bot : \mathsf{Set}$, which has no constructors. Among Agda's record types (inductive types with a single constructor for which η-expansion holds definitionally) we need the unit type $\top : \mathsf{Set}$ with constructor $\mathsf{tt} : \top$; and dependent products (Σ-types), that we write as $\Sigma\, x : A\,,B\,x$ and which are dependent record types with constructor $(\_\,,\_) : (x : A)(\_ : B\,x) \to \Sigma\, x : A\,, B\,x$ and fields (projections) $\mathsf{fst} : (\Sigma\, x : A\,, B\,x) \to A$ and $\mathsf{snd} : (z : \Sigma\, x : A\,, B\,x) \to B(\mathsf{fst}\, z)$.

This type theory can be interpreted in (the category with families of) any presheaf topos, such as the one defined below, so long as we assume that the ambient set theory has a countable hierarchy of Grothendieck universes; in particular, one could use a constructive ambient set theory such as IZF [1] with universes. We will use the fact that the interpretation of the type theory in presheaf toposes satisfies *function extensionality* and *uniqueness of*

[4] Albeit at the expense of some calculations with universe levels; Coq's universe polymorphism would probably deal with this aspect automatically.

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identity proofs:

$$\mathsf{funext} : \{A : \mathsf{Set}_n\}\{B : A \to \mathsf{Set}_m\}\{f\ g : (x : A) \to B\,x\}((x : A) \to f\,x \equiv g\,x) \to f \equiv g \quad (1)$$

$$\mathsf{uip} : \{A : \mathsf{Set}_n\}\{x\ y : A\}\{p\ q : x \equiv y\} \to p \equiv q \quad (2)$$

▶ **Definition 2.1** (**Presheaf topos of de Morgan cubical sets**)**.** Let $\square$ denote the small category with finite products which is the Lawvere theory of De Morgan algebra (see [8, Chap. XI] and [37, Section 2]). Concretely, $\square^{\mathrm{op}}$ consists of the free De Morgan algebras on n generators, for each $n \in \mathbb{N}$, and the homomorphisms between them. Thus $\square$ contains an object I that generates the others by taking finite products, namely the free De Morgan algebra on one generator. This object is the generic De Morgan algebra and in particular it has two distinct global elements, corresponding to the constants for the greatest and least elements. The *topos of cubical sets* [13], which we denote by $\widehat{\square}$, is the category of $\mathcal{S}et$-valued functors on $\square^{\mathrm{op}}$ and natural transformations between them. The Yoneda embedding, written $\mathrm{y} : \square \hookrightarrow \widehat{\square}$, sends $\mathrm{I} \in \square$ with its two distinct global elements to a representable presheaf $\mathbb{I} = \mathrm{yI}$ with two distinct global elements. This interval $\mathbb{I}$ is used to model path types: a path in A from a_0 to a_1 is any morphism $\mathbb{I} \to A$ that when composed with the distinct global elements gives a_0 and a_1.

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The toposes used in other cubical models [9, 6, 5] vary the choice of algebra from the De Morgan case used above; see [11]. To describe all these cubical models using type theory as an internal language, we postulate the existence of an *interval* type $\mathbb{I}$ with two distinct elements, which we write as `O` and `I`:

$$\mathbb{I} : \mathsf{Set} \qquad \mathtt{O} : \mathbb{I} \qquad \mathtt{I} : \mathbb{I} \qquad \mathtt{O}{\not\equiv}\mathtt{I} : (\mathtt{O} \equiv \mathtt{I}) \to \bot \quad (3)$$

Apart from an interval, the other data needed to define a cubical sets model of homotopy type theory is a notion of *cofibration*, which specifies the shapes of filling problems that can be solved in a dependent type. For this, CCHM [13] use a particular subobject of $\Omega \in \widehat{\square}$ (the subobject classifier in the topos $\widehat{\square}$), called the *face lattice*; but other choices are possible [32]. Here, we avoid the use of the impredicative universe of propositions Ω and just assume the existence of a collection of "cofibrant" types in the first universe Set, including at least the empty type $\bot$ (in Section 6, we will introduce more cofibrations, needed to model various type constructs):

$$\mathsf{cof} : \mathsf{Set} \to \mathsf{Set} \qquad \mathsf{cof}_\bot : \mathsf{cof}\ \bot \quad (4)$$

We call $\varphi : \mathsf{Set}$ *cofibrant* if $\mathsf{cof}\,\varphi$ holds, that is, if we can supply a term of that type. To define the fibrations as a type in the internal language we use two pieces of notation. First, the *path functor* associated with the interval $\mathbb{I}$ is

$$\wp : \mathsf{Set}_n \to \mathsf{Set}_n \qquad \wp\text{`} : \{A : \mathsf{Set}_n\}\{B : \mathsf{Set}_m\}(f : A \to B) \to \wp\,A \to \wp\,B \quad (5)$$
$$\wp\,A = \mathbb{I} \to A \qquad \wp\text{`}\,f\,p\,i = f(p\,i)$$

Secondly, we define the following *extension* relation

$$\_ \nearrow \_ : \{\varphi : \mathsf{Set}\}\{A : \mathsf{Set}_n\}(t : \varphi \to A)(x : A) \to \mathsf{Set}_n \qquad t \nearrow x = (u : \varphi) \to t\,u \equiv x \quad (6)$$

Thus $t \nearrow x$ is the type of proofs that the partial element $t : \varphi \to A$ extends to the (total) element $x : A$. We will use this when t denotes a partial element of A of cofibrant extent, that is when we have a proof of $\mathsf{cof}\,\varphi$.

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▶ **Definition 2.2 (fibrations).** The type $\mathsf{isFib}\,A$ of *fibration structures* for a family of types $A : \Gamma \to \mathsf{Set}_n$ over some type $\Gamma : \mathsf{Set}_m$ consists of functions taking any path $p : \wp\,\Gamma$ in the base type to a *composition structure* in $\mathsf{C}(A \circ p)$:

$$\mathsf{isFib} : (\Gamma : \mathsf{Set}_m)(A : \Gamma \to \mathsf{Set}_n) \to \mathsf{Set}_{1 \sqcup m \sqcup n} \qquad \mathsf{isFib}\,\Gamma\,A = (p : \wp\,\Gamma) \to \mathsf{C}(A \circ p) \tag{7}$$

Here C is some given function $\wp\,\mathsf{Set}_n \to \mathsf{Set}_{1 \sqcup n}$ (polymorphic in the universe level n) which parameterizes the notion of fibration. Then for each type Γ, the type $\mathsf{Fib}_n\,\Gamma$ of *fibrations* over it with fibers in Set_n consists of families equipped with a fibration structure

$$\mathsf{Fib}_n : (\Gamma : \mathsf{Set}_m) \to \mathsf{Set}_{m \sqcup (n+1)} \qquad \mathsf{Fib}_n\,\Gamma = \Sigma\,A : (\Gamma \to \mathsf{Set}_n)\,,\, \mathsf{isFib}\,\Gamma\,A \tag{8}$$

and there are *re-indexing* functions, given by composition of dependent functions $(\_ \circ \_)$

$$\begin{aligned} &\_[\_] : \{\Gamma : \mathsf{Set}_k\}\{\Gamma' : \mathsf{Set}_m\}(\Phi : \mathsf{Fib}_n\,\Gamma)(\gamma : \Gamma' \to \Gamma) \to \mathsf{Fib}_n\,\Gamma' \\ &(A\,,\alpha)[\gamma] = (A \circ f\,,\alpha \circ \wp`\,f) \end{aligned} \tag{9}$$

A *CCHM fibration* is the above notion of fibration for the composition structure $\mathsf{CCHM} : \wp\,\mathsf{Set}_n \to \mathsf{Set}_{1 \sqcup n}$ from [13]:

$$\begin{aligned} \mathsf{CCHM}\,P = (\varphi : \mathsf{Set})(\_ : \mathsf{cof}\,\varphi)(p : (i : \mathbb{I}) \to \varphi \to P\,i) \to (\Sigma\,a_0 : P\,\mathtt{0}\,, p\,\mathtt{0} \nearrow a_0) \to \\ (\Sigma\,a_1 : P\,\mathtt{1}\,, p\,\mathtt{1} \nearrow a_1) \end{aligned} \tag{10}$$

Thus the type $\mathsf{CCHM}\,P$ of CCHM composition structures for a path of types $P : \wp\,\mathsf{Set}_n$ consists of functions taking any dependently-typed path of partial elements $p : (i : \mathbb{I}) \to \varphi \to P\,i$ of cofibrant extent to a function mapping extensions of the path at one end $p\,\mathtt{0} \nearrow a_0$, to extensions of it at the other end $p\,\mathtt{1} \nearrow a_1$. When the cofibration is $\bot$, this $\mathsf{isFib}\,\Gamma\,A$ expands to the statement that for all paths $p : \mathbb{I} \to \Gamma$, $A(p\,\mathtt{0}) \to A(p\,\mathtt{1})$, so that this internal language type says that A is equipped with a transport function along paths in Γ. The use of cofibrant partial elements generalizes transport with a notion of path composition, which is used to show that path types are fibrant.

Other notions of fibration follow the above definitions but vary the definition of $\mathsf{C} : \wp\,\mathsf{Set}_n \to \mathsf{Set}_{1 \sqcup n}$; for example, generalized diagonal Kan composition [5]. Co/contravariant fibrations in directed type theory [34] also have the form of isFib for some C, but with $\wp$ being directed paths. Definition 2.2 illustrates the advantages of internal-language presentations; in particular, uniformity [13] is automatic.

If Γ denotes an object of the cubical sets topos $\widehat{\square}$, then $\mathsf{Fib}_0\,\Gamma$ denotes an object whose global sections correspond to the elements of the set $\mathrm{FTy}(\Gamma)$ of families over Γ equipped with a composition structure as defined in [13, Definition 13]. Our goal now is to first recall that there can be no universe that weakly classifies these CCHM fibrations in an internal sense, and then move to a modal type theory where such a universe can be expressed.

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3 The "no-go" theorem for internal universes

In this section we recall from [32, Remark 7.5] why there can be no universe that weakly classifies CCHM fibrations in an internal sense. Such a weak classifier would be given by the following data

$$\begin{aligned} &\mathsf{U} : \mathsf{Set}_2 && \mathsf{code} : \{\Gamma : \mathsf{Set}\}(\Phi : \mathsf{Fib}_0\,\Gamma) \to \Gamma \to \mathsf{U} \\ &\mathsf{El} : \mathsf{Fib}_0\,\mathsf{U} && \mathsf{Elcode} : \{\Gamma : \mathsf{Set}\}(\Phi : \mathsf{Fib}_0\,\Gamma) \to \mathsf{El}[\mathsf{code}\,\Phi] \equiv \Phi \end{aligned} \tag{11}$$

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where for simplicity we restrict attention to fibrations whose fibers are in the lowest universe, $\mathsf{Set} = \mathsf{Set}_0$. Here U is the universe[5] and El is a CCHM fibration over it which is a weak classifier in the sense that any fibration $\Phi : \mathsf{Fib}_0\,\Gamma$ can be obtained from it (up to equality) by re-indexing along some function $\mathsf{code}\,\Phi : \Gamma \to \mathsf{U}$. (The word "weak" refers to the fact that we do not require there to be a *unique* function $\gamma : \Gamma \to \mathsf{U}$ with $\mathsf{El}[\gamma] \equiv \Phi$.)

We will show that the data in (11) implies that the interval must be trivial ($\mathtt{0} \equiv \mathtt{I}$), contradicting the assumption in (3). This is because (11) allows one to deduce that if a family of types $A : \Gamma \to \mathsf{Set}$ has the property that each $A\,x$ has a fibration structure when regarded as a family over the unit type $\top$, then there is a fibration structure for the whole family A; and yet there are families where this cannot be the case. For example, consider the family $P : \mathbb{I} \to \mathsf{Set}$ with $P\,i = (\mathtt{0} \equiv i)$. For each $i : \mathbb{I}$, the type $P\,i$ has a fibration structure $\pi\,i : \mathsf{isFib}\,\top\,(\lambda\,\_ \to P\,i)$, because of uniqueness of identity proofs (2). But the family as a whole satisfies $\mathsf{isFib}\,\mathbb{I}\,P \to \bot$, because if we had a fibration structure $\alpha : \mathsf{isFib}\,\mathbb{I}\,P$, then we could apply it to

$$
\begin{array}{lllll}
\mathsf{id} : \wp\,\mathbb{I} & \varphi : \mathsf{Set} & u : \mathsf{cof}\,\varphi & p : (i : \mathbb{I}) \to \varphi \to P\,i & z : \Sigma\,a_0 : P\,\mathtt{0}\,,\, p\,\mathtt{0} \nearrow a_0 \\
\mathsf{id}\,i = i & \varphi = \bot & u = \mathsf{cof}_\bot & p\,i = \lambda\,\_ \to \bot\mathsf{elim} & z = (\mathsf{refl}\,,\,\lambda\,\_ \to \mathsf{uip})
\end{array}
$$

(where $\bot\mathsf{elim} : \{A : \mathsf{Set}\} \to \bot \to A$ is the elimination function for the empty type) to get $\alpha\,\mathsf{id}\,\varphi\,u\,p\,z : (\Sigma\,a_1 : P\,\mathtt{I}\,,\, p\,\mathtt{I} \nearrow a_1)$ and hence $\mathtt{0}{\not\equiv}\mathtt{I}\,(\mathsf{fst}\,(\alpha\,\mathsf{id}\,\varphi\,u\,p\,z)) : \bot$. From this we deduce the following "no-go" [6] theorem for internal universes of CCHM fibrations.

▶ **Theorem 3.1.** [32, Remark 7.5] *The existence of types and functions as in* (11) *for CCHM fibrations is contradictory. More precisely, if* $\mathsf{IntUniv} : \mathsf{Set}_3$ *is the dependent record type with fields* U, El, code *and* Elcode *as in* (11), *then there is a term of type* $\mathsf{IntUniv} \to \bot$.

Proof.[7] Suppose we have an element of $\mathsf{IntUniv}$ and hence functions as in (11). Then taking P to be $\lambda i \to (\mathtt{0} \equiv i)$ and using the family $\pi\,i$ of fibration structures on each type $P\,i$ mentioned above, we get:

$$
\Phi : \mathsf{Fib}_0\,\mathbb{I} \qquad \Phi = \mathsf{El}[(\lambda\,i \to \mathsf{code}\,((\lambda\,\_ \to P\,i)\,,\,\pi\,i)\,\mathsf{tt})] \tag{12}
$$

Using Elcode and function extensionality (1), it follows that there is a proof $u : \mathsf{fst}\,\Phi \equiv P$, namely $u = \mathsf{funext}\,(\lambda\,i \to \mathsf{cong}\,(\lambda\,x \to \mathsf{fst}\,x\,\mathsf{tt})\,(\mathsf{Elcode}\,((\lambda\,\_ \to P\,i)\,,\,\pi\,i)))$, where cong is the usual congruence property of equality. From that and $\mathsf{snd}\,\Phi$ we get an element of $\mathsf{isFib}\,\mathbb{I}\,P$. But we saw above how to transform such an element into a proof of $\bot$. So altogether we have a proof of $\mathsf{IntUniv} \to \bot$. ◀

▶ **Remark 3.2.** This counterexample generalizes to other notions of fibration: it is not usually the case that any type family $A : \Gamma \to \mathsf{Set}$ for which $A\,x$ is fibrant over $\top$ for all $x : \Gamma$, is fibrant over Γ. The above proof should be compared with the proof that there is no "fibrant replacement" type-former in *Homotopy Type System* (HTS); see `https://ncatlab.org/homotopytypetheory/show/Homotopy+Type+System#fibrant_replacement`. Theorem 5.1 below provides a further example of a global construct that does not internalize.

[5] Our predicative treatment of cofibrant types makes it necessary to place U in Set_2 rather than Set_1.
[6] We are stealing Shulman's terminology [36, section 4.1].
[7] See the file `theorem-3-1.agda` at `https://doi.org/10.17863/CAM.22369` for an Agda version of this proof.

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4 Crisp type theory

The proof of Theorem 3.1 depends upon the fact that in the internal language the code function can be applied to elements with free variables. In this case it is the variable $i : \mathbb{I}$ in $\mathsf{code}\,((\lambda\,\_ \to P\,i)\,,\pi\,i)\,\mathsf{tt}$; by abstracting over it we get a function $\mathbb{I} \to \mathsf{U}$ and re-indexing El along this function gives the offending fibration (12). Nevertheless, the cubical sets presheaf topos does contain a (univalent) universe which is a CCHM fibration classifier, but only in an *external* sense. Thus there is an object U in $\widehat{\square}$ and a global section $\mathsf{El} : 1 \to \mathsf{Fib}_0\,\mathsf{U}$ with the property that for any object Γ and morphism $\Phi : 1 \to \mathsf{Fib}_0\,\Gamma$, there is a morphism $\mathsf{code}\,\Phi : \Gamma \to \mathsf{U}$ so that Φ is equal to the composition $\mathsf{Fib}_0\,(\mathsf{code}\,\Phi) \circ \mathsf{El} : 1 \to \mathsf{Fib}_0\,\Gamma$; see [13, Definition 18] for a concrete description of U. The internalization of this property replaces the use of global elements $1 \to \Gamma$ of an object by local elements, that is, morphisms $X \to \Gamma$ where X ranges over a suitable collection of generating objects (for example, the representable objects in a presheaf topos); and we have seen that such an internalized version cannot exist.

Nevertheless, we would like to explain the construction of universes like $\mathsf{U} \in \widehat{\square}$ using some kind of type-theoretic language that builds on Section 2. So we seek a way of manipulating global elements of an object Γ, within the internal language. One cannot do so simply by quantifying over elements of the type $\top \to \Gamma$, because of the isomorphism $\Gamma \cong (\top \to \Gamma)$. Instead, we pass to a modal type theory that can speak about global elements, which we call *crisp type theory*. Its judgements, such as $\Delta \mid \Gamma \vdash a : A$, have two context zones—where Δ represents global elements and Γ the usual, local ones. The context structure is that used for an S4 necessitation modality [33, 14, 36], because a global element from Δ can be used locally, but global elements cannot depend on local variables from Γ. Following [36], we say that the left-hand context Δ contains *crisp* hypotheses about the types of variables, written $x :: A$.

The interpretation of crisp type theory in cubical sets makes use of the comonad $\flat : \widehat{\square} \to \widehat{\square}$ that sends a presheaf A to the constant presheaf on the set of global sections of A; thus $\flat A(X) \cong A(1)$ for all $X \in \square$ (where $1 \in \square$ is terminal). Then a judgement $\Delta \mid \Gamma \vdash a : A$ describes the situation where Δ is a presheaf, Γ is a family of presheaves over $\flat\Delta$, A is a family over $\Sigma(\flat\Delta)\Gamma$ and a is an element of that family. The rules of crisp type theory are designed to be sound for this interpretation. Compared with ordinary type theory, the key constraint is that *types in the crisp context and terms substituted for crisp variables depend only on crisp variables.* The crisp variable and (admissible) substitution rules are:

$$\frac{}{\Delta, x :: A, \Delta' \mid \Gamma \vdash x : A} \qquad \frac{\Delta \mid \diamond \vdash a : A \qquad \Delta, x :: A, \Delta' \mid \Gamma \vdash b : B}{\Delta, \Delta'[a/x] \mid \Gamma[a/x] \vdash b[a/x] : B[a/x]} \tag{13}$$

The semantics of the variable rule, which says that global elements can be used locally, uses the counit $\varepsilon\,A : \flat A \to A$ of the comonad $\flat$ mentioned above. In the substitution rule, $\diamond$ stands for the empty list, so a and A may only depend upon the crisp variables from Δ. The other rules of crisp type theory (those for Π types, Σ types, etc.) carry the crisp context along. For our application we do not need a type-former for $\flat$, but instead make use of crisp Π types (see, e.g. [14, 28]), that is, Π types whose domain is crisp

$$\frac{\Delta \mid \diamond \vdash A : \mathsf{Set}_m \qquad \Delta, x :: A \mid \Gamma \vdash B : \mathsf{Set}_n}{\Delta \mid \Gamma \vdash (x :: A) \to B : \mathsf{Set}_{m \sqcup n}} \qquad \frac{\Delta, x :: A \mid \Gamma \vdash b : B}{\Delta \mid \Gamma \vdash \lambda x :: A.b : (x :: A) \to B} \qquad \frac{\Delta \mid \Gamma \vdash f : (x :: A) \to B \qquad \Delta \mid \diamond \vdash a : A}{\Delta \mid \Gamma \vdash f\,a : B[a/x]} \tag{14}$$

with $\beta\eta$ judgemental equalities. In these rules, because the argument variable x is crisp, its

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type A, and the term a to which the function f is applied, must also be crisp. We also use *crisp induction* for identity types [36]—identity elimination with a family $y :: A, p :: x \equiv y \vdash C(y, p)$ whose parameters are crisp variables, which is given by a term of type

$$\{A :: \mathsf{Set}_n\}\{x :: A\}(C : (y :: A)(p :: x \equiv y) \to \mathsf{Set}_n)(z : C\, x\, \mathsf{refl})(y :: A)(p :: x \equiv y) \to C\, y\, p \tag{15}$$

together with a β judgemental equality.

▶ **Remark 4.1 (Presheaf models of crisp type theory).** Crisp type theory is motivated by the specific presheaf topos $\widehat{\Box}$ from Definition 2.1. However, it seems that very little is required of a category $\mathbf{C}$ for the presheaf topos $\widehat{\mathbf{C}}$ to soundly interpret it using the comonad $\flat = p^* \circ p_*$, where p_* takes the global sections of a presheaf and its left adjoint p^* sends sets to constant presheaves. This $\flat$ preserves finite limits (because it is the composition of functors with left adjoints—p^* is isomorphic to the functor given by precomposition with $\mathbf{C} \to 1$ and hence has a left adjoint given by left Kan extension along $\mathbf{C} \to 1$). Although the details remain to be worked out, it appears that to model crisp type theory with crisp Π types and crisp identification induction (and moreover a $\flat$ modality with crisp $\flat$ induction, which we do not use here), the only additional condition needed is that this comonad is idempotent (meaning that the comultiplication $\delta : \flat \to \flat \circ \flat$ is an isomorphism). This idempotence holds iff $\widehat{\mathbf{C}}$ is a connected topos, which is the case iff $\mathbf{C}$ is a connected category—for example, when $\mathbf{C}$ has a terminal object. If it does have a terminal object, then $\widehat{\mathbf{C}}$ is a local topos [21, Sect. C3.6] and $\flat$ has a right adjoint; in which case, conjecturally [36, Remark 7.5], one gets a model of the whole of Shulman's spatial type theory, of which crisp type theory is a part. In fact $\Box$ does not just have a terminal object, it has all finite products (as does any Lawvere theory) and from this it follows that $\widehat{\Box}$ is not just local, but also cohesive [24].

▶ **Remark 4.2 (Agda-flat).** Vezzosi has created a fork of Agda, called *Agda-flat* [2], which allows us to explore crisp type theory. It adds the ability to use crisp variables[8] $x :: A$ in places where ordinary variables $x : A$ may occur in Agda, and checks the modal restrictions in the above rules. For example, Agda-flat quite correctly rejects the following attempted application of a crisp-Π function to an ordinary argument

$$\mathsf{wrong} : (A :: \mathsf{Set}_n)(B : \mathsf{Set}_m)(f : (\_ :: A) \to B)(x : A) \to B \qquad \mathsf{wrong}\, A\, B\, f\, x = f(x)$$

while the variant with $x :: A$ succeeds. This is a simple example of keeping to the modal discipline that crisp type theory imposes; for more complicated cases, such as occur in the proof of Theorem 5.2 below, we have found Agda-flat indispensable for avoiding errors. However, Agda-flat implements a superset of crisp type theory and more work is needed to understand their precise relationship. For example, Agda's ability to define inductive types leads to new types in Agda-flat, such as the $\flat$ modality itself; and its pattern-matching facilities allow one to prove properties of $\flat$ that go beyond crisp type theory. Agda allows one to switch off pattern-matching in a module; to be safe we do that as far as possible in our development. Installation instructions for Agda-flat can be found at `https://doi.org/10.17863/CAM.22369`.

5 Universes from tiny intervals

In crisp type theory, to avoid the inconsistency in the "no-go" Theorem 3.1, we can weaken the definition of a universe in (11) by taking code and Elcode to be crisp functions of

[8] The Agda-flat concrete syntax for "$x :: A$" is "x :{$\flat$} A".

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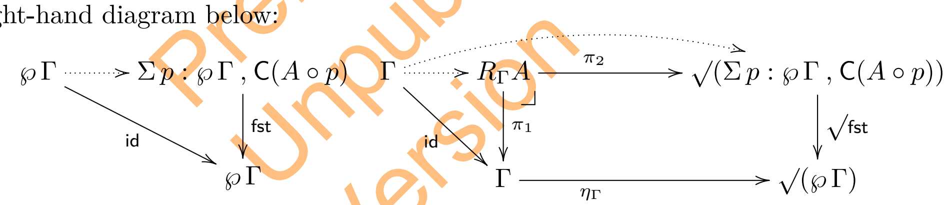
fibrations Φ (and implicitly, of the base type Γ of the fibration). For if code has type $\{\Gamma :: \mathsf{Set}\}(\Phi :: \mathsf{Fib}_0\,\Gamma)(x : \Gamma) \to \mathsf{U}$, then the proof of a contradiction is blocked when in (12) we try to apply code to $\Phi = ((\lambda\,\_ \to P\,i)\,,\pi\,i)$, which depends upon the local variable $i : \mathbb{I}$. Indeed we show in this section that given an extra assumption about the interval type $\mathbb{I}$ that holds for cubical sets, it is possible to define a universe with such crisp coding functions which moreover are unique, so that one gets a classifying fibration, rather than just a weakly classifying one.

Recall from Definition 2.1 that in the cubical sets model, the type $\mathbb{I}$ denotes the representable presheaf $\mathrm{yI} \in \widehat{\square}$ on the object $\mathrm{I} \in \square$. Since $\square$ has finite products, there is a functor $\_ \times \mathrm{I} : \square \to \square$. Pre-composition with this functor induces an endofunctor on presheaves $(\_ \times \mathrm{I})^* : \widehat{\square} \to \widehat{\square}$ which has left and right adjoints, given by left and right Kan extension [26, Chap. X] along $\_ \times \mathrm{I}$. Hence by the Yoneda Lemma, for any $F \in \widehat{\square}$ and $X \in \square$

$$(\mathbb{I} \to F)\,X \cong \widehat{\square}(\mathrm{y}X, \mathbb{I} \to F) \cong \widehat{\square}(\mathrm{y}X \times \mathrm{yI}, F) \cong \widehat{\square}(\mathrm{y}(X \times \mathrm{I}), F) = ((\_ \times \mathrm{I})^* F)\,X$$

naturally in both X and F. It follows that the exponential functor $\wp = \mathbb{I} \to \_ : \widehat{\square} \to \widehat{\square}$ is naturally isomorphic to $(\_ \times \mathrm{I})^*$ and hence not only has a left adjoint (corresponding to product with $\mathbb{I}$) but also a right adjoint. The significance of objects in a category with finite products that are not only exponentiable (product with them has a right adjoint), but also whose exponential functor has a right adjoint was first pointed out by Lawvere in the context of synthetic differential geometry [23]. He called such objects "atomic", but we will follow later usage [42] and call them *tiny*.[9] Thus the interval in cubical sets is tiny and we have a right adjoint to the path functor $\wp$ that we denote by $\surd : \widehat{\square} \to \widehat{\square}$. So for each $B \in \widehat{\square}$, the functor $\widehat{\square}(\wp\,\_, B) : \widehat{\square} \to \mathit{Set}$ is representable by $\surd B$, that is, there are bijections $\widehat{\square}(\wp\,A, B) \cong \widehat{\square}(A, \surd B)$, natural in A.

Given Γ and $A : \Gamma \to \mathsf{Set}$ in $\widehat{\square}$, from Definition 2.2 we have that fibration structures $1 \to \mathsf{isFib}\,\Gamma\,A$ correspond to sections of $\mathsf{fst} : (\Sigma\,p : \wp\,\Gamma\,, \mathsf{C}(A \circ p)) \to \wp\,\Gamma$ and hence, transposing across the adjunction $\wp \dashv \surd$, to morphisms making the outer square commute in the right-hand diagram below:



We therefore have that fibration structures for A correspond to sections of the pullback $\pi_1 : R_\Gamma A \to \Gamma$ of $\surd\mathsf{fst}$ along the unit $\eta_\Gamma : \Gamma \to \surd(\wp\,\Gamma)$ of the adjunction at Γ (which is the adjoint transpose of $\mathsf{id} : \wp\,\Gamma \to \wp\,\Gamma$). This characterization of fibration structure does not depend on the particular definition of C, so should apply to many notions of fibration. We will show how it leads to the construction of a universe $\mathsf{U} = R_{\mathsf{Set}}\mathsf{id}$ and family $\pi_1 : R_{\mathsf{Set}}\mathsf{id} \to \mathsf{Set}$ which is a classifier for fibrations. However, there are two problems that have to be solved in order to carry out the construction within type theory:

- First, for Elcode in (11) to be an equality (rather than just an isomorphism), one needs the choice of $R_\Gamma A$ to be strictly functorial with respect to re-indexing along Γ (and hence to be a dependent right adjoint in the sense of [12]).

[9] Warning: the adjective "tiny" is sometimes used to describe an object X of a $\mathcal{V}$-enriched cocomplete category $\mathcal{C}$ for which the hom $\mathcal{V}$-functor $\mathcal{C}(X, \_) : \mathcal{C} \to \mathcal{V}$ preserves colimits; see [35] for example. We prefer Kelly's term *small-projective object* for this property. In the special case that $\mathcal{V} = \mathcal{C}$ and $\mathcal{C}$ is cartesian closed and has sufficient properties for there to be an adjoint functor theorem, then a small-projective object is in particular a tiny one in the sense we use here.

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$\surd : (A :: \mathsf{Set}_n) \to \mathsf{Set}_n$
$\mathsf{R} : \{A :: \mathsf{Set}_n\}\{B :: \mathsf{Set}_m\}(f :: \wp\, A \to B) \to A \to \surd B$
$\mathsf{L} : \{A :: \mathsf{Set}_n\}\{B :: \mathsf{Set}_m\}(g :: A \to \surd B) \to \wp\, A \to B$
$\mathsf{LR} : \{A :: \mathsf{Set}_n\}\{B :: \mathsf{Set}_m\}\{f :: \wp\, A \to B\} \to \mathsf{L}(\mathsf{R}\, f) \equiv f$
$\mathsf{RL} : \{A :: \mathsf{Set}_n\}\{B :: \mathsf{Set}_m\}\{g :: A \to \surd B\} \to \mathsf{R}(\mathsf{L}\, g) \equiv g$
$\mathsf{R}\wp : \{A :: \mathsf{Set}_n\}\{B :: \mathsf{Set}_m\}\{C :: \mathsf{Set}_k\}(g :: A \to B)(f :: \wp\, B \to C) \to \mathsf{R}(f \circ \wp`g) \equiv \mathsf{R} f \circ g$

Figure 1 Axioms for tinyness of the interval in crisp type theory

- Secondly, one cannot use ordinary type theory as the internal language to formulate the construction, because the right adjoint to $\wp$ does not internalize, as the following theorem shows.

▶ **Theorem 5.1.** *There is no* internal *right adjoint to the path functor* $\wp : \widehat{\square} \to \widehat{\square}$ *for cubical sets. In other words, there is no family of natural isomorphisms* $(\wp\,\_ \to B) \cong (\_ \to \surd B) : \widehat{\square} \to \widehat{\square}$ *(for* $B \in \widehat{\square}$*).*

Proof. It is an elementary fact about adjoint functors that such a family of natural isomorphisms is also natural in B. Note that $\wp\top \cong \top$. So if we had such a family, then we would also have isomorphisms $B \cong (\top \to B) \cong (\wp\top \to B) \cong (\top \to \surd B) \cong \surd B$ which are natural in B. Therefore $\surd$ would be isomorphic to the identity functor and hence so would be its left adjoint $\wp$. Hence $\mathbb{I} \to \_$ and $\top \to \_$ would be isomorphic functors $\widehat{\square} \to \widehat{\square}$, which implies (by the internal Yoneda Lemma) that $\mathbb{I}$ is isomorphic to the terminal object $\top$, contradicting the fact that $\mathbb{I}$ has two distinct global elements. ◀

We will solve the first of the two problems mentioned above in the same way that Voevodsky [41] solves a similar strictness problem (see also [12, Section 6]): apply $\surd$ once and for all to the displayed universe and then re-index, rather than *vice versa* (as done above). The second problem is solved by using the crisp type theory of the previous section to make the right adjoint $\surd$ suitably global. The axioms we use are given in Fig. 1. The function R gives the operation for transposing (global) morphisms across the adjunction $\wp \dashv \surd$, with inverse L (the bijection being given by RL and LR); and $\mathsf{R}\wp$ is the naturality of this operation. The other properties of an adjunction follow from these, in particular its functorial action $\surd` : \{A :: \mathsf{Set}_n\}\{B :: \mathsf{Set}_m\}(f :: A \to B) \to \surd A \to \surd B$. Note that Fig. 1 assumes that the right adjoint to $\mathbb{I} \to (\_)$ preserves universe levels. The soundness of this for $\widehat{\square}$ relies on the fact that this adjoint is given by right Kan extension [26, Chap. X] along $\_ \times \mathrm{I} : \square \to \square$ and hence sends a presheaf valued in the nth Grothendieck universe to another such.

▶ **Theorem 5.2 (Universe construction[10]).** *For fibrations as in Definition 2.2 with any definition of composition structure* C *(e.g. the CCHM one in* (10)*), assuming axioms* (1)–(4) *and a tiny (Fig. 1) interval, there is a universe* U *equipped with a fibration* El *which is*

[10] We just construct a universe for fibrations with fibers in Set_0; similar universes $\mathsf{U}_n : \mathsf{Set}_{2\sqcup n}$ can be constructed for fibrations with fibers in Set_n, for each n; see `theorem-5-2.agda` at `https://doi.org/10.17863/CAM.22369`.

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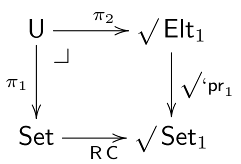
classifying *in the sense that we have*

$$\begin{aligned}&\mathsf{U} : \mathsf{Set}_2\\ &\mathsf{El} : \mathsf{Fib}_0\,\mathsf{U}\\ &\mathsf{code} : \{\Gamma :: \mathsf{Set}\}(\Phi :: \mathsf{Fib}_0\,\Gamma) \to \Gamma \to \mathsf{U}\\ &\mathsf{Elcode} : \{\Gamma :: \mathsf{Set}\}(\Phi :: \mathsf{Fib}_0\,\Gamma) \to \mathsf{El}[\mathsf{code}\,\Phi] \equiv \Phi\\ &\mathsf{codeEl} : \{\Gamma :: \mathsf{Set}\}(\gamma :: \Gamma \to \mathsf{U}) \to \mathsf{code}(\mathsf{El}[\gamma]) \equiv \gamma\end{aligned} \tag{16}$$

Proof. Consider the display function associated with the first universe:

$$\begin{aligned}&\mathsf{Elt}_1 : \mathsf{Set}_2 && \mathsf{pr}_1 : \mathsf{Elt}_1 \to \mathsf{Set}_1\\ &\mathsf{Elt}_1 = \Sigma\, A : \mathsf{Set}_1\,,\, A && \mathsf{pr}_1(A, x) = A\end{aligned} \tag{17}$$

We have $\mathsf{C} : \wp\,\mathsf{Set}_0 \to \mathsf{Set}_1$ and hence using the transpose operation from Fig. 1, $\mathsf{R}\,\mathsf{C} : \mathsf{Set}_0 \to \surd\,\mathsf{Set}_1$. We define $\mathsf{U} : \mathsf{Set}_2$ by taking a pullback:

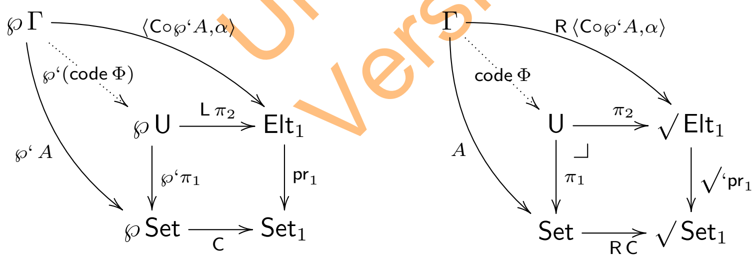


$$\begin{aligned}&\mathsf{U} = \Sigma\, A : \mathsf{Set}\,,\, (\Sigma\, B : \surd\,\mathsf{Elt}_1\,,\, (\surd\text{‘}\mathsf{pr}_1\, B \equiv \mathsf{R}\,\mathsf{C}\, A))\\ &\pi_1(A\,,\,(\_\,,\,\_)) = A\\ &\pi_2(\_\,,\,(B\,,\,\_)) = B\end{aligned} \tag{18}$$

Transposing this square across the adjunction $\wp \dashv \surd$ gives $\mathsf{pr}_1 \circ \mathsf{L}\,\pi_2 = \mathsf{C} \circ \wp\text{‘}\pi_1 : \wp\,\mathsf{U} \to \mathsf{Set}_1$. Considering the first and second components of $\mathsf{L}\,\pi_2$, we have $\mathsf{L}\,\pi_2 \equiv \langle \mathsf{C} \circ \wp\text{‘}\pi_1\,,\, \upsilon\rangle$ for some $\upsilon : (p : \wp\,\mathsf{U}) \to \mathsf{C}(\wp\text{‘}\pi_1\, p)$; hence υ is an element of $\mathsf{isFib}\, U\, \pi_1$ and so we can define

$$\mathsf{El} : \mathsf{Fib}_0\,\mathsf{U} \qquad \mathsf{El} = (\pi_1, \upsilon) \tag{19}$$

So it just remains to construct the functions in (16). Given $\Gamma :: \mathsf{Set}$ and $\Phi = (A, \alpha) :: \mathsf{Fib}_0\,\Gamma$, we have $\alpha :: \mathsf{isFib}\,\Gamma\, A = (p : \wp\,\Gamma) \to \mathsf{C}(A \circ p)$. So the outer square in the left-hand diagram below commutes:



(20)

Transposing across the adjunction $\wp \dashv \surd$, this means that the outer square in the right-hand diagram also commutes and therefore induces a function $\mathsf{code}\,\Phi : \Gamma \to \mathsf{U}$ to the pullback. So there are proofs of $\pi_1 \circ \mathsf{code}\,\Phi \equiv A$ and $\pi_2 \circ \mathsf{code}\,\Phi \equiv \mathsf{R}\,\langle \mathsf{C} \circ \wp\text{‘}A\,,\,\alpha\rangle$. Transposing the latter back across the adjunction gives a proof of $\mathsf{L}\,\pi_2 \circ \wp\text{‘}(\mathsf{code}\,\Phi) \equiv \langle \mathsf{C} \circ \wp\text{‘}A\,,\,\alpha\rangle$; and since $\mathsf{L}\,\pi_2 \equiv \langle \mathsf{C} \circ \wp\text{‘}\pi_1\,,\,\upsilon\rangle$, this in turn gives a proof of $\upsilon \circ \wp\text{‘}(\mathsf{code}\,\Phi) \equiv \alpha$. Combining this with the proof of $\pi_1 \circ \mathsf{code}\,\Phi \equiv A$, we get the desired element $\mathsf{Elcode}\,\Phi$ of $\mathsf{El}[\mathsf{code}\,\Phi] = (\pi_1 \circ \mathsf{code}\,\Phi\,,\, \upsilon \circ \mathsf{code}\,\Phi) \equiv (A\,,\,\alpha) = \Phi$. Finally, taking $\Gamma = \mathsf{U}$ and $\Phi = \mathsf{El}$ in (20), the uniqueness property of the pullback implies that $\mathsf{code}\,\mathsf{El} \equiv \mathsf{id}$; and similarly, for any $\gamma :: \Delta \to \Gamma$ we have that $(\mathsf{code}\,\Phi) \circ \gamma \equiv \mathsf{code}(\Phi[\gamma])$. Together these properties give us the desired element codeEl of $\mathsf{code}(\mathsf{El}[\gamma]) \equiv (\mathsf{code}\,\mathsf{El}) \circ \gamma \equiv \mathsf{id} \circ \gamma = \gamma$. ◀

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$$
\begin{array}{lll}
\_\sqcap\_ : \mathbb{I}\to\mathbb{I}\to\mathbb{I} & 0\sqcap : (i:\mathbb{I})\to 0\sqcap i\equiv 0 & \sqcap 0 : (i:\mathbb{I})\to i\sqcap 0\equiv 0\\
 & \mathtt{I}\sqcap : (i:\mathbb{I})\to \mathtt{I}\sqcap i\equiv i & \sqcap\mathtt{I} : (i:\mathbb{I})\to i\sqcap \mathtt{I}\equiv i\\
\mathsf{rev} : \mathbb{I}\to\mathbb{I} & \mathsf{revrev} : (i:\mathbb{I})\to \mathsf{rev}\,(\mathsf{rev}\,i)\equiv i & \mathsf{rev0} : \mathsf{rev}\,0\equiv \mathtt{I}
\end{array}
$$

$$
\begin{array}{l}
\mathsf{isPropcof} : (\varphi:\mathsf{Set})(u\ v:\mathsf{cof}\,\varphi)\to u\equiv v\\
\mathsf{cofisProp} : (\varphi:\mathsf{Set})(\_:\mathsf{cof}\,\varphi)(x\ y:\varphi)\to x\equiv y\\
\mathsf{cofExt} : (\varphi\ \psi:\mathsf{Set})(\_:\mathsf{cof}\,\varphi)(\_:\mathsf{cof}\,\psi)(\_:\varphi\to\psi)(\_:\psi\to\varphi)\to\varphi\equiv\psi\\
\mathsf{cofO} : (i:\mathbb{I})\to\mathsf{cof}\,(0\equiv i)\\
\mathsf{cofI} : (i:\mathbb{I})\to\mathsf{cof}\,(\mathtt{I}\equiv i)\\
\mathsf{cofOr} : (\varphi\ \psi:\mathsf{Set})(\_:\mathsf{cof}\,\varphi)(\_:\mathsf{cof}\,\psi)\to\mathsf{cof}\,(\varphi\vee\psi)\\
\mathsf{cofAnd} : (\varphi\ \psi:\mathsf{Set})(\_:\mathsf{cof}\,\varphi)(\_:\varphi\to\mathsf{cof}\,\psi)\to\mathsf{cof}\,(\varphi\times\psi)\\
\mathsf{cof}\forall\mathbb{I} : (\varphi:\mathbb{I}\to\mathsf{Set})(\_:(i:\mathbb{I})\to\mathsf{cof}\,(\varphi\,i))\to\mathsf{cof}\,((i:\mathbb{I})\to\varphi\,i)\\
\mathsf{strax} : (\varphi{:}\mathsf{Set})(\_{:}\mathsf{cof}\,\varphi)(A:\mathsf{Set}_n)(t:\varphi\to(\Sigma\,B:\mathsf{Set}_n\ ,\ A\cong B))\to\\
\qquad\qquad\qquad\qquad\qquad\qquad \Sigma\,T:(\Sigma\,B:\mathsf{Set}_n\ ,\ A\cong B)\ ,\ t\nearrow T
\end{array}
$$

Figure 2 Further axioms needed for the CCHM model

▶ **Remark 5.3.** The above theorem can be generalized by replacing the particular universe $\mathsf{id} : \mathsf{Set} \to \mathsf{Set}$ by an arbitrary one $E_0 : U_0 \to \mathsf{Set}$. So long as the composition structure C lands in U_0, one can use the above method to construct a universe of fibrant types from among the U_0 types.[11] The application of this generalization we have in mind is to directed type theory; for example one can first construct the universe of fibrant types in the CCHM sense and then make a universe of covariant discrete fibrations in the Riehl-Shulman [34] sense from the fibrant types (repeating the construction with a different interval object).

▶ **Remark 5.4.** The results in this section only make use of the fact that the functor $\surd : \widehat{\square} \to \widehat{\square}$ is right adjoint to the exponential $\mathbb{I} \to (\_)$ and we saw at the beginning of this section why such a right adjoint exists. It is possible to give an explicit description of presheaves of the form $\surd\Gamma$, but so far we have not found such a description to be useful.

6 Applications

Models. Theorem 5.2 is the missing piece that allows a completely internal development of a model of univalent foundations based upon the CCHM notion of fibration, albeit internal to crisp type theory rather than ordinary type theory. One can define a CwF in crisp type theory whose objects are crisp types $\Gamma :: \mathsf{Set}_2$, whose morphisms are crisp functions $\gamma :: \Gamma' \to \Gamma$, whose families are crisp CCHM fibrations $\Phi = (A, \alpha) :: \mathsf{Fib}_0\,\Gamma$ and whose elements are crisp dependent functions $f :: (x : \Gamma) \to A\,x$. To see that this gives a model of univalent foundations one needs to prove:

(a) The CwF is a model of intensional type theory with Π-types and inductive types (Σ-types, identity types, booleans, W-types, ...).
(b) The type $\mathsf{U} :: \mathsf{Set}_2$ constructed in Theorem 5.2 is fibrant (as a family over the unit type).
(c) The classifying fibration $\Phi :: \mathsf{Fib}_0\,\mathsf{U}$ satisfies the univalence axiom in this CwF.

Although we have yet to complete the formal development in Agda-flat, these should be provable from axioms (1)–(4) and Fig. 1, together with some further assumptions about

[11] See `theorem-5-2-relative.agda` at `https://doi.org/10.17863/CAM.22369`.

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the interval object and cofibrant types listed in Fig. 2. Part (a) was carried out in prior work, albeit in the setting with an impredicative universe of propositions [32]. In the predicative version considered here, we replace the impredicative universe of propositions with axioms asserting that being cofibrant is a mere proposition (isPropcof), that cofibrant types are mere propositions (cofisProp) and satisfy propositional extensionality (cofExt). These axioms are satisfied by $\widehat{\square}$ provided we interpret $\mathsf{cof} : \mathsf{Set} \to \mathsf{Set}$ as $\mathsf{cof}\, A = \exists \varphi : \Omega\, , \varphi \in \mathsf{Cof} \wedge A \equiv \{\_ : \top \mid \varphi\}$, using the subobject $\mathsf{Cof} \rightarrowtail \Omega$ corresponding to the face lattice in [13] (see [32, Definition 8.6]). Axioms cofO, cofI, cofOr, cofAnd, cof$\forall\mathbb{I}$ and strax correspond to the axioms ax_5–ax_9 from [32]; in strax, $\cong$ is the usual internal statement of isomorphism. cofAnd is the dominance axiom that guarantees that cofibrations compose. Note that axiom cofOr uses an operation sending mere propositions φ and ψ to the mere proposition $\varphi \vee \psi$ that is the propositional truncation of their disjoint union; the existence of this operation either has to be postulated, or one can add axioms for *quotient types* [18, Section 3.2.6.1] to crisp type theory, (of which propositional truncation is an instance), in which case function extensionality (1) is no longer needed as an axiom, since it is provable using quotient types [39, Section 6.3]. Since in this paper we have taken a CCHM fibration to just give a composition operation for cofibrant partial paths from `0` to `I` and not vice versa, in Fig. 2 we have postulated a path-reversal operation rev; this and the other axioms for $\mathbb{I}$ in that figure suffice to give a "connection algebra" structure on $\mathbb{I}$ [32, axioms ax_3 and ax_4].

Part (b) can be proved using a version of the *glueing* operation from [13], which is definable within crisp type theory as in [32, Section 6] and [10, Section 4.3.2]. The strictness axiom strax in Fig. 2 is needed to define this; and the assumption that cofibrant types are closed under $\mathbb{I}$-indexed $\forall$ (cof$\forall\mathbb{I}$) is used to define the appropriate fibration structure for glueing.

Part (c) can be proved as in [31, Section 6] using a characterization of univalence somewhat simpler than the original definition of Voevodsky [39, Section 2.10]. The axiom strax gets used to turn isomorphisms into paths; and the axiom cof$\forall\mathbb{I}$ is used to "realign" fibration structures that agree on their underlying types (see [31, Lemma 6.2]).

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▶ **Remark 6.1 (The interval is connected).** Fig. 2 does not include an axiom asserting that the interval is connected, because that is implied by its tinyness (Fig. 1). Connectedness was postulated as ax_1 in [32] and used to prove that CCHM fibrations are closed under inductive type formers (and in particular that the natural number object is fibrant). The proof [32, Thm 8.2] that the interval in cubical sets is connected essentially uses the fact that $\widehat{\square}$ is a cohesive topos (Remark 4.1). However it also follows directly from the tinyness property: connectedness holds iff $(\mathbb{I} \to \mathbb{B}) \cong \mathbb{B}$, where $\mathbb{B} = \top + \top$ is the type of Booleans. Since we postulate that $\mathbb{I} \to \_$ has a right adjoint, it preserves this coproduct and hence $(\mathbb{I} \to \mathbb{B}) \cong (\mathbb{I} \to \top) + (\mathbb{I} \to \top) \cong \top + \top = \mathbb{B}$.

▶ **Remark 6.2 (Alternative models).** We have focussed on axioms satisfied by $\widehat{\square}$ and the CCHM notion of fibration in that presheaf topos. However, the universe construction in Theorem 5.2 also applies to the cartesian cubical set models [5], and we expect it is possible to give proofs in crisp type theory of its fibrancy and univalence as well.

In this paper we only consider "cartesian" path-based models of type theory, in which a path is an arbitrary function out of an interval object, or in other words, the path functor is given by an exponential. The models in [22] and [9] are not cartesian in that sense—the path functors they use are right adjoint to certain functorial cylinders [17] not given by cartesian

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product.[12] However, those path functors do have right adjoints (given by right Kan extension to suitable "shift" functors on the domain category of the presheaf toposes involved) and universes in these models can be constructed using the method of Theorem 5.2. (Our Agda proof of that theorem does not depend upon the path functor being an actual exponential.) A proof in crisp type theory that those universes are fibrant and univalent may require a modification of our axiomatic treatment of cofibrancy; we leave this for future work.

Universe hierarchies. Given that there are many notions of fibration that one may be interested in, it is natural to ask how relationships between them induce relationships between universes of fibrant types. As motivating examples of this, we might want a cubical type theory with a universe of fibrations with *regularity*, an extra strictness corresponding to the computation rule for identity types in intensional type theory; or a three-level directed type theory with non-fibrant, fibrant, and co/contravariant universes. Towards building such hierarchies, in the companion code[13] we have shown in crisp type theory that universes are functorial in the notion of fibration they encapsulate: when one notion of fibrancy implies another, the first universe includes the second.

▶ **Proposition 6.3.** *Let* $\mathsf{C}^1, \mathsf{C}^2 : \wp\,\mathsf{Set}_n \to \mathsf{Set}_{1\sqcup n}$ *be two notions of composition,* isFib^1 *and* isFib^2 *the corresponding fibration structures, and* U^1 *and* U^2 *the corresponding classifying universes. A morphism of fibration structures is a function* $f_{\Gamma,A} : \mathsf{isFib}^1\,\Gamma\,A \to \mathsf{isFib}^2\,\Gamma\,A$ *for all* Γ *and* A*, such that* f *is stable under reindexing (given* $h : \Delta \to \Gamma$*, and* $\phi : \mathsf{isFib}^1\,\Gamma\,A$*,* $f_{\Gamma,A}(\phi) \circ (\wp\text{`}h) \equiv f_{\Delta,A\circ f}(\phi[h])$*). Then a morphism of fibrations* f *induces a function* $\mathsf{U}^1 \to \mathsf{U}^2$*, and this preserves identity and composition.* ◀

7 Conclusion

Since the appearance of the CCHM [13] constructive model of univalence, there has been a lot of work aimed at analysing what makes this model tick, with a view to simplifying and generalizing it. Some of that work, for example by Gambino and Sattler [17, 35], uses category theory directly, and in particular techniques associated with the notion of Quillen model structure. Here we have continued to pursue the approach that uses a form of type theory as an internal language in which to describe the constructions associated with this model of univalent foundations [32, 10]. For those familiar with the language of type theory, we believe this provides an appealingly simple and accessible description of the notion of fibration and its properties in the CCHM model and in related models. We recalled why there can be no internal description of the univalent universe itself if one uses ordinary type theory as the internal language. Instead we extended ordinary type theory with a suitable modality and then gave a universe construction that hinges upon the tinyness property enjoyed by the interval in cubical sets. We call this language *crisp type theory* and our work inside it has been carried out and checked using an experimental version of Agda provided by Vezzosi [2].

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[12] Furthermore, obvious candidates for an interval object are not necessarily tiny in those models—for example, for the 1-simplex $\Delta[1]$ the exponential $\Delta[1] \to (\_)$ in the topos $\hat{\Delta}$ of simplicial sets does not have a right adjoint; thanks to a referee for pointing this out.

[13] see `proposition-6-2.agda` at `https://doi.org/10.17863/CAM.22369`

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