On quasi-ordinal diagram systems

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The purposes of this note are the following two; we first generalize Okada-Takeuti's well-quasi-ordinal diagram theory by allowing any well-partial ordering for the inner node labels of the trees. Second, we discuss possible use of such strong ordinal notation systems for the purpose of a typical traditional termination proofs method for term rewriting systems, especially for second-order (pattern-matching-based) rewriting systems.

1 Introduction

A typical termination proof method for the traditional first-order term rewriting system is to show the termination of a term rewrite system R by verifying that for each rule $l(\vec{x}) \to r(\vec{x})$ of R, $f(l(\vec{x}) > f(r(\vec{x}))$ holds, where f is a strictly order-preserving mapping and the ordering $<$ is a well founded ordering with the substitution property and the monotonicity property. Here, the substitution property and the monotonicity property mean

(i) for any substitution (for the list of variables) σ, if $\alpha < \beta$ holds then $\alpha\sigma < \beta\sigma$ holds, and

(ii) for any context u, if $\alpha < \beta$ holds then $u[\alpha] < u[\beta]$ holds,

respectively. The properties (i) and (ii) guarantee the termination of the whole R because any application of (first-order) rewrite rule $l(\vec{x}) \to r(\vec{x})$ has a form $u[l\sigma] \to u[r\sigma]$ for some context u and some substitution σ. In theory and practice, the identity mapping for f above is known useful. In this note, we restrict our attention to the identity f for our basic argument.

The method has been widely used for termination proofs as well as a tool for Knuth-Bendix completion. The method itself would be attractive not only for the traditional first-order rewriting but also for higher-order or graphic-pattern-matching-based rewriting. One could expect that strong and general ordering structures in proof theory would be useful for this termination proof method of higher-order pattern-matching-based rewrite systems.

However, the use of strong orderings $<$ such as Takeuti's ordinal diagram systems, which are non-simplification ordering, cannot satisfy the two basic properties above. Because of this difficulty, instead of the traditional termination proof method, various different techniques have been utilized; for example, in an early studies of higher-order rewriting Jouannaud-Okada introduced a generalized form of Tait-Girard's saturated sets method and reducibility candidates method ([7, 2]).

Hence at a first look, it seems very hard to adapt much stronger well-founded ordering structures to the traditional first-order termination method. It is a natural question how we could adapt them to the termination proof method for higher-order rewriting systems.

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In this note, we first generalize the well-quasi-ordinal diagram theory ([9, 10]) by allowing any well-partial ordering for the inner node labels of the trees, being inspired from the Dershowitz-Tzameret tree embedding theorem with gap-condition ([5]). After defining our quasi-ordinal diagram system in Subsection 2.1, we give two proofs of (weakly-)well-quasi-orderedness of our generalized quasi-ordinal systems. In Subsection 2.2, we give a short proof as a corollary of the Dershowitz-Tzameret's version of tree embedding theorem, which presumes existence of an uncountable set. In Subsection 2.3, we give a more constructive proof using principles of a generalized version of transfinitely iterated inductive definitions. In Section 3, we give some examples of restricted substitution and monotonicity properties which are still satisfied with the ordinal diagram systems and their quasi-ordering variants introduced in Section 2. We take some versions of Buchholz game as examples of pattern-matching-based second-order rewrite systems and show how one could use the termination proof method, mentioned at the beginning of this Introduction, with our strong orderings.

2 Well-quasi-ordering and weak-well-quasi-ordering proofs for generalized systems of quasi-ordinal diagrams

We, in Section 2.1, generalize the systems $\mathsf{Q}(I,A)$ of quasi-ordinal diagrams of Okada-Takeuti [9] (Takeuti [10]) by allowing inner node labels from a given arbitrary well-partial-ordering I (instead of a well-ordering I). These generalized systems are denoted by $\mathsf{GQ}(I,A)$. Next, in Section 2.2, we give a short proof of well-quasi-orderedness of a certain restricted tree-domain, the *path comparable* tree-domain of $\mathsf{GQ}(I,A)$ as a direct corollary of the Dershowitz-Tzameret's version [5] of tree embedding theorem with gap-condition. In Section 2.3, we give a proof of weakly-well-quasi-orderedness of the full domain of $\mathsf{GQ}(I,A)$ by the use of iterated inductive definitions below the greatest element of I. Note that a quasi-ordering $(I,\leq)$ is *weakly well-quasi-ordered* if for any infinite $\leq$-decreasing sequence $i_0 \geq i_1 \geq \dots$ from I, there are n and m such that $n < m$ and $i_n \leq i_m$ holds.

The first proof shows the usefulness of the Dershowitz-Tzameret's tree embedding theorem, while it involves rather big set theoretical operations; for example, their proof needs the set theoretical assumption of existence of an uncountable set. On the other hand, the second proof needs less assumption (related only the inductive definition systems).

2.1 Generalized systems of quasi-ordinal diagrams

The definition of $\mathsf{GQ}(I,A)$ is as follows.

Definition 2.1. Let $(I,\leq_I)$ be a well-partial-ordering and $(A,\leq_A)$ be a weak-well-quasi-ordering. The generalized system $\mathsf{GQ}(I,A)$ of quasi-ordinal diagrams with respect to I and A is defined as follows.

1. If $a \in A$ then a is a connected gqod.
2. If $i \in I$ and α is a gqod then (i,α) is a connected gqod.
3. If $\alpha_1,\dots,\alpha_n$ are connected gqod's then $\alpha_1\#\dots\#\alpha_n$ is an unconnected gqod.

According to the generalization of a well-ordering I to a well-partial-ordering I, we need a slight modified form of definition of "section" as follows.

Definition 2.2. For every $\alpha,\beta \in \mathsf{GQ}(I,A)$ and every $i \in I$, the relation $\alpha \subset_i \beta$ is defined as follows.

1. If $\beta \in A$ holds then $\alpha \subset_i \beta$ never holds.

2. If $\beta \equiv (j, \beta')$ holds then
   (a) when $i = j$ holds, $\alpha \subset_i \beta$ if and only if $\alpha = \beta'$ or $\alpha \subset_i \beta'$,
   (b) when $i < j$ holds, $\alpha \subset_i \beta$ if and only if $\alpha \subset_i \beta'$,
   (c) when $i \not\leq j$ holds, $\alpha \subset_i \beta$ never holds.
3. If $\beta \equiv \beta_1 \# \ldots \# \beta_m$ $(m > 1)$ holds, then $\alpha \subset_i \beta$ if and only if for some β_l $(1 \leq l \leq m)$, $\alpha \subset_i \beta_l$.

We say α is an *i-section* of β when $\alpha \subset_i \beta$ holds. In addition, we say i is an *index* of α if there is a gqod β such that β is an i-section of α. Set $\tilde{I} := I \cup \{\infty\}$. For given gqod's $\alpha_0, \ldots, \alpha_n$ and a given $i \in \tilde{I}$, we define

$$\mathsf{s}(i, \alpha_0, \ldots, \alpha_n) := \begin{cases} \{j \mid i < j,\ j \text{ is an index of } \alpha_0 \text{ or } \ldots \text{ or } \alpha_n\}, & \text{if } i \in I, \\ \emptyset, & \text{if } i = \infty. \end{cases}$$

We denote the cardinality of $\mathsf{s}(i, \alpha_0, \ldots, \alpha_n)$ by $\#\mathsf{s}(i, \alpha_0, \ldots, \alpha_n)$ and the set of all minimal elements of $\mathsf{s}(i, \alpha_0, \ldots, \alpha_n)$ by $\mathsf{s}^{min}(i, \alpha_0, \ldots, \alpha_n)$. For every gqod's $\alpha_0, \ldots, \alpha_n$, the total number of all occurrences of $()$ and $\#$ in $\alpha_0, \ldots, \alpha_n$ is denoted by $l(\alpha_0, \ldots, \alpha_n)$.

The i-nested definition of orderings on $\mathsf{GQ}(I,A)$ needs the following modification in the case where $\alpha \leq_i^{\mathsf{q}} \beta$ with α, β connected and $i \in I$, while the other part is essentially the same as [9, 10].

Definition 2.3. For every $i \in \tilde{I}$, the relation $\leq_i^{\mathsf{q}}$ on $\mathsf{GQ}(I,A)$ is defined by induction on $\omega \cdot (l(\alpha, \beta)) + \#\mathsf{s}(i, \alpha, \beta)$.

1. If $\alpha, \beta \in A$ holds then for every $i \in \tilde{I}$, $\alpha \leq_i^{\mathsf{q}} \beta$ if and only if $\alpha \leq_A \beta$.
2. If $\alpha \in A$ and $\beta \notin A$ hold then for every $i \in \tilde{I}$, $\alpha \leq_i^{\mathsf{q}} \beta$ and $\beta \not\leq_i^{\mathsf{q}} \alpha$ hold.
3. If $\alpha \equiv \alpha_1 \# \ldots \# \alpha_n$ and $\beta \equiv \beta_1 \# \ldots \# \beta_m$ $(n + m > 2)$ hold, then for every $i \in \tilde{I}$, $\alpha \leq_i^{\mathsf{q}} \beta$ if and only if one of the following conditions holds:
   (a) there is a β_l $(1 \leq l \leq m)$ such that for every k $(1 \leq k \leq n)$, $\alpha_k \leq_i^{\mathsf{q}} \beta_l$ and $\beta_l \not\leq_i^{\mathsf{q}} \alpha_k$ hold,
   (b) there is a β_l $(1 \leq l \leq m)$ such that $\alpha_1 \leq_i^{\mathsf{q}} \beta_l$, and if $n \geq 2$ then
   $$\alpha_2 \# \ldots \# \alpha_n \leq_i^{\mathsf{q}} \beta_1 \# \ldots \# \beta_{l-1} \# \beta_{l+1} \# \ldots \# \beta_m$$
   holds.
4. If $\alpha \equiv (j, \alpha_0)$, $\beta \equiv (j', \beta_0)$ and $i = \infty$ hold, then $\alpha \leq_\infty^{\mathsf{q}} \beta$ if and only if either $j < j'$ holds or both $j = j'$ and $\alpha_0 \leq_j^{\mathsf{q}} \beta_0$ hold.
5. If $\alpha \equiv (j, \alpha_0)$, $\beta \equiv (j', \beta_0)$ and $i \in I$ then $\alpha \leq_i^{\mathsf{q}} \beta$ if and only if either
   (a) there is a $\beta' \subset_i \beta$ such that $\alpha \leq_i^{\mathsf{q}} \beta'$, or
   (b) for every $\alpha' \subset_i \alpha$, $\alpha' \leq_i^{\mathsf{q}} \beta$, $\beta \not\leq_i^{\mathsf{q}} \alpha'$ and
      - $\alpha \leq_j^{\mathsf{q}} \beta$ for any element j of $\mathsf{s}^{min}(i, \alpha, \beta)$, if $\mathsf{s}(i, \alpha, \beta) \neq \emptyset$,
      - otherwise, $\alpha \leq_\infty^{\mathsf{q}} \beta$.

Lemma 2.4. For every $i \in \tilde{I}$, $(\mathsf{GQ}(I,A), \leq_i^{\mathsf{q}})$ is a quasi-ordering.

Proof. One can prove the lemma by using the following sublemmas:

1. For every $i \in \tilde{I}$, the relation $\leq_i^{\mathsf{q}}$ satisfies the transitivity, that is, for every $\alpha, \beta, \gamma \in \mathsf{GQ}(I,A)$ and every $i \in \tilde{I}$, if $\alpha \leq_i^{\mathsf{q}} \beta$ and $\beta \leq_i^{\mathsf{q}} \gamma$ hold, then $\alpha \leq_i^{\mathsf{q}} \gamma$ holds.

2. For every $\alpha \in \mathsf{GQ}(I,A)$ and every $i \in \tilde{I}$,
   (a) the relation $\leq^{\mathrm{q}}_i$ satisfies the reflexivity, that is, $\alpha \leq^{\mathrm{q}}_i \alpha$ holds, and
   (b) if $i \in I$, then every $\beta \in \mathsf{GQ}(I,A)$ with $\alpha \subset_i \beta$, $\alpha \leq^{\mathrm{q}}_i \beta$ and $\beta \not\leq^{\mathrm{q}}_i \alpha$ hold.

□

Remark 2.5. To verify the transitivity of the relation $\leq^{\mathrm{q}}_{\infty}$, we need the anti-symmetricity of I in $\mathsf{GQ}(I,A)$. This is why we define I of $\mathsf{GQ}(I,A)$ as not a well-quasi-ordering but a well-partial ordering. In addition, note that the above proof for the quasi-orderedness of $\mathsf{GQ}(I,A)$ neither depends on the well-quasi-orderedness of I nor the weakly-well-quasi-orderedness of A.

2.2 Well-quasi-ordering proof for the subsystem $\mathsf{QPC}(I,A)$ of $\mathsf{GQ}(I,A)$ via Dershowitz and Tzameret's tree embedding theorem

In this section, we give a short proof of well-quasi-orderedness of the path comparable tree-domain of $\mathsf{GQ}(I,A)$ as a direct corollary of Dershowitz-Tzameret's tree embedding theorem. A labeled tree in $\mathsf{GQ}(I,A)$ is called a *path comparable tree* if for any path of the tree and any two inner (non-leaf) nodes a and b in the path, a and b are comparable in the partial ordering I (cf. [5]). Let $\mathsf{QPC}(I,A)$ be the set of all forests from path comparable trees in $\mathsf{GQ}(I,A)$.

Note that $\mathsf{QPC}(I,A)$ is located between Okada-Takeuti's quasi-ordinal diagram system $\mathsf{Q}(I,A)$ and our $\mathsf{GQ}(I,A)$: $\mathsf{QPC}(I,A)$ is a generalization of Okada-Takeuti's quasi-ordinal diagram system $\mathsf{Q}(I,A)$ as I is generalized from a well-ordering to a well-partial ordering. On the other hand, our $\mathsf{GQ}(I,A)$ is a generalization of $\mathsf{QPC}(I,A)$.

We say $\alpha \lll \beta$ holds if for every $i \in \tilde{I}$, $\alpha \leq^{\mathrm{q}}_i \beta$ holds. It is obvious that the pair $(\mathsf{GQ}(I,A), \lll)$ is a quasi-ordering. Let $\hookrightarrow_g$ be Dershowitz-Tzameret's forest-embedding with the gap-condition on $\mathsf{QPC}(I,A)$, which is obtained from their tree-embedding (cf. [5]) in a natural way. We have the following proposition.

Proposition 2.6. For every $\alpha, \beta \in \mathsf{QPC}(I,A)$, if $\alpha \hookrightarrow_g \beta$ holds, then $\alpha \lll \beta$ holds.

Proof. Use the following properties of $\mathsf{GQ}(I,A)$: For every $\alpha, \beta \in \mathsf{GQ}(I,A)$ and every $i, j \in I$ with $i \leq j$,

1. $\alpha <^{\mathrm{q}}_i (j, \alpha)$ holds.
2. If β is connected and $(i, \alpha) \lll \beta$ holds, then $(i, \alpha) \lll (j, \beta)$ holds.
3. If β is of the form $\beta_1 \# \cdots \# \beta_n$ $(n > 1)$ and for some m $(1 \leq m \leq n)$, $(i, \alpha) \lll \beta_m$ holds, then $(i, \alpha) \lll (j, \beta)$ holds.

□

Then, the well-quasi-orderedness of $(\mathsf{QPC}(I,A), \lll)$ is an immediate corollary of the tree-embedding theorem of [5].

Corollary 2.7. The quasi-ordering $(\mathsf{QPC}(I,A), \lll)$ is well-quasi-ordered.

Remark 2.8. The Dershowitz-Tzameret's tree embedding theorem also shows the well-orderedness of the lexicographic ordering $(\mathsf{OT}(I,A), <_{\mathbf{B}})$ defined in [8], because for each $\mathsf{OT}(I,A)$ one can take a system $\mathsf{QPC}(I',A)$ that includes $\mathsf{OT}(I,A)$ as a subset and $<_{\mathbf{B}}$ coincides with $<^{\mathrm{q}}_{\infty}$ on $\mathsf{OT}(I,A)$. Note that the lexicographic ordering $<_{\mathbf{B}}$ on the larger domains $\mathsf{O}(I,A)$ or $\mathsf{QPC}(I,A)$ is not well-founded.

2.3 Weak-well-quasi-ordering proof for the full system $\mathsf{GQ}(I,A)$

In this section, we give a direct proof for the weakly-well-quasi-orderedness of the full systems $\mathsf{GQ}(I,A)$ by using iterated inductive definitions below the greatest element of I. Okada-Takeuti gave the minimal-bad sequence argument for the direct proof of the weakly-well-quasi-orderedness of the subsystems $\mathsf{Q}(I,A)$ of $\mathsf{GQ}(I,A)$ (cf. [9, 10]). On the other hand, the usual well-ordering proof of ordinal diagrams are typically proved explicitly or implicitly in a system of inductive definitions. Below we expose the latter method by following [1].

However, we cannot use the Gentzen-style method in [1] that lifts up the well-foundedness of α to the one of $\alpha+(i,\beta)$, because the orderings on $\mathsf{GQ}(I,A)$ are not linearly ordered but quasi-ordered. To complete our proof, another method is exploited. This is the only substantial difference from the usual proofs.

Let I be an arbitrary well-partial-ordering with a minimal element 0 and the greatest element ξ. In addition, let A be an arbitrary weak-well-quasi-ordering. We suppose a fixed binary relation Acc on $I \times \mathsf{GQ}(I,A)$ with the following principles (CL) and (LE)

(CL) $\forall i<\xi(\mathsf{Prog}[\mathsf{Fan}_i,<_i^{\mathsf{q}},\mathsf{Acc}_i])$, (LE) $\forall P \subseteq \mathsf{GQ}(I,A)\forall i<\xi(\mathsf{Prog}[\mathsf{Fan}_i,<_i^{\mathsf{q}},P] \to \mathsf{Acc}_i \subseteq P)$,

where for every $i \in I$ and $\alpha,\beta \in \mathsf{GQ}(I,A)$,

- $\mathsf{Acc}_i(\alpha) :\Longleftrightarrow \mathsf{Acc}(i,\alpha)$, $\mathsf{Fan}_i(\alpha) :\Longleftrightarrow \forall j<i\forall\beta \subset_j \alpha\mathsf{Acc}_j(\beta)$,
- $\mathsf{Prog}[X,R,Y] :\Longleftrightarrow \forall\alpha(X(\alpha) \wedge \forall\beta(\beta R\alpha \wedge X(\beta) \to Y(\beta)) \to Y(\alpha))$.

We also assume the following principle of transfinite induction on I:

$$\text{(TI)} \quad \forall P \subseteq \mathsf{GQ}(I,A)\forall i<\xi(\forall j<iP(j) \to P(i)) \to \forall i<\xi(P(i)).$$

Then, we have the following proposition in a similar way to [1] with an exception mentioned above.

Proposition 2.9. For every $\alpha \in \mathsf{GQ}(I,A)$, $\mathsf{Acc}_0(\alpha)$ holds.

Corollary 2.10. The quasi-ordering $(\mathsf{GQ}(I,A),\leq_0^{\mathsf{q}})$ is weakly-well-quasi-ordered.

Proof. Define

$\mathsf{Wwqo}_i(\alpha) :\Longleftrightarrow$ For any function f from $\mathbb{N}$ to $\mathsf{GQ}(I,A)$,
$[f(0) \leq_i^{\mathsf{q}} \alpha \wedge \forall n(\mathsf{Fan}_i(f(n)) \wedge f(n+1) \leq_i^{\mathsf{q}} f(n)) \to \exists n\exists m(n<m \wedge f(m) \geq_i^{\mathsf{q}} f(n))]$.

Then, use Proposition 2.9 and the fact that $\forall i<\xi(\mathsf{Acc}_i \subseteq \mathsf{Wwqo}_i)$.

□

3 Application of the weakly-well-quasi-orderedness of $\mathsf{GQ}(I,A)$ to the termination proof method for higher-order rewriting; Buchholz Game

In this section, we discuss the possibility of application of the weakly-well-quasi-orderedness of $\mathsf{GQ}(I,A)$ to higher-order pattern-matching-based rewriting systems. We formulate the restricted versions of Substitution Property and Monotonicity Property of $\mathsf{GQ}(I,A)$ to give a traditional termination proof method for such rewrite systems. As an example, Buchholz hydra game-style rewrite system is considered (cf. [3, 6]).

First, we formulate Buchholz hydra game-style rewrite system. Let V be an infinite set of variables $x_1,x_2,\ldots$. In this subsection, we consider the system $\mathsf{GQ}(I,A)$ of gqod's such that

- $A = V \cup \{\rho\}$ and for any $a_1, a_2 \in A$, $a_1 \leq_A a_2$ holds if and only if $a_1 = a_2$ holds,
- I is a well-partial ordering with the least element ρ.

For any $n \in \mathbb{N}$, let $\alpha \cdot (n+1)$ be $\underbrace{\alpha\#\cdots\#\alpha}_{n+1 \text{ times}}$. We denote the set of all contexts by $\mathscr{C}$. For a given $u[*] \in \mathscr{C}$, we define $u[\alpha]$ as the gqod obtained by substituting α for $*$ in $u[*]$. In addition, we inductively define $\mathscr{C}_i$ as follows: $* \in \mathscr{C}_i$, and if $u[*] \in \mathscr{C}_i$ and $\alpha, \beta \in \mathsf{GQ}(I, A)$, then $\alpha\#u[*], u[*]\#\alpha \in \mathscr{C}_i$. Finally, if $j > i$ and $u[*] \in \mathscr{C}_i$, then $(j, u[*]) \in \mathscr{C}_i$.

The following is a generalization of Buchholz game rules that involve pattern-matching-based reductions.

Definition 3.1 (**Buchholz game-style rewrite rules**)**.** Buchholz game-style rewrite rules are as follows: Let a be an arbitrary element of A, i' be an arbitrary successor element of i, λ be an arbitrary limit element, $u[*]$ be an arbitrary element of $\mathscr{C}$ and $u_i[*]$ be an arbitrary element of $\mathscr{C}_i$.

(1) If $n+m > 0$, then
$(\rho, u[(i, \alpha_1\#\cdots\#\alpha_n\#(\rho, a)\#\beta_1\#\cdots\#\beta_m)]) \rhd (\rho, u[(i, \alpha_1\#\cdots\#\alpha_n\#\beta_1\#\cdots\#\beta_m\#\rho \cdot 3) \cdot (k+1)])$,
otherwise $(\rho, u[(i, (\rho, a))]) \rhd (\rho, u[(i, a\#\rho \cdot 2) \cdot (k+1)])$.

(1)' If $n+m > 0$, then
$(\rho, u[\alpha_1\#\cdots\#\alpha_n\#(\rho, a)\#\beta_1\#\cdots\#\beta_m]) \rhd (\rho, u[\alpha_1\#\cdots\#\alpha_n\#\beta_1\#\cdots\#\beta_m\#\rho \cdot 3])$,
otherwise $(\rho, u[(\rho, a)]) \rhd (\rho, u[a\#\rho \cdot 2])$.

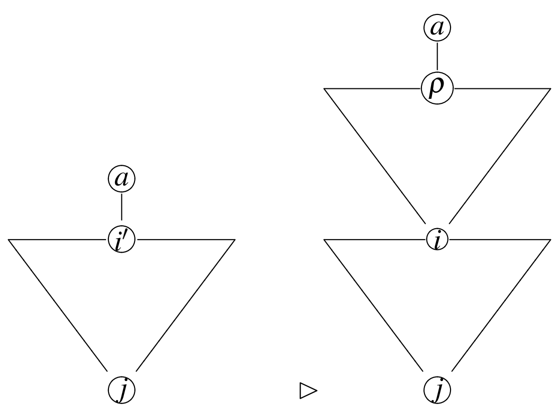
(2) $(\rho, u[(j, u_i[(i', a)])]) \rhd (\rho, u[(j, u_i[(i, u_i[(\rho, a)])])])$ with an arbitrary $u_i \in \mathscr{C}_i$ and an arbitrary $j \not> i$.

(3) $(\rho, u[(\lambda, a)]) \rhd (\rho, u[(i', a\#\rho)])$ with an arbitrary $i < \lambda$.

Remark 3.2. To apply the restricted Monotonicity Property lemma below (Lemma 3.9), we restricted ourselves to hydras with the root ρ in the rules above. Because of this restriction, we cannot rewrite hydras without the label ρ. By omitting this restriction, one can rewrite hydras without ρ to *numeral trees*, which are normal forms in our game. The notion of numeral trees is defined below.

In addition, the reason we attached one, two or three ρ's to the right-hand side in some of the rules above is that we record how many times we at most applied the rules.

In graph notation, a part of the rule schema (2) is expressed as follows.



Example 3.3 (**A concrete example of the rule schema (2)**)**.** Let us suggest the correspondence of the rule schema (2) to higher-order rewriting by means of a toy example. In $\mathsf{GQ}(\omega+1, V \cup \{0\})$, the following is an instance of the rule schema (2), where $j = 2$, $u = *$, $i' = 3$:

$$(0,(2,X(3,0))) \rhd (0,(2,X(2,X(0,0)))),$$

where X is a higher-order variable and corresponds to the context $u_i \in \mathscr{C}_i$.

To give a traditional termination proof method for pattern-matching-based rewrite systems such as Buchholz game, we first formulate the restricted Substitution Property of $\mathsf{GQ}(I,A)$. The following is a counterexample for the full Substitution Property of $\mathsf{GQ}(I,A)$.

Example 3.4 (A counterexample for the full Substitution Property of $\mathsf{GQ}(I,A)$). Set $\alpha :\equiv x$ and $\beta :\equiv (j,x)$, and take a substitution σ assigning (j',γ) to x with $j < j'$. Then, $\alpha \equiv x \leq^{\mathsf{q}}_{\rho} (j,x) \equiv \beta$ holds, but $\alpha\sigma = (j',\gamma) \leq^{\mathsf{q}}_{\rho} (j,(j',\gamma)) = \beta\sigma$ does not hold whenever there is an $i \in I$ such that $\rho < i$ but i and j are incomparable.

A *numeral tree* is a connected gqod that consists of ρ's only. A *numeral substitution* is a substitution assigning a numeral tree to each variable. Note that a numeral tree can be seen as a generalized numeral. For numerical substitutions, we have Substitution Property.

Lemma 3.5 (The numeral Substitution Property lemma). For any numeral substitution σ, any $i \in I$, any α and β in $\mathsf{GQ}(I,A)$, if $\alpha \leq^{\mathsf{q}}_{i} \beta$ holds, then $\alpha\sigma \leq^{\mathsf{q}}_{i} \beta\sigma$ holds.

Proof. Induction on $\omega \cdot l(\alpha,\beta) + \#\mathsf{s}(i,\alpha,\beta)$. We consider only the case where $i \neq \infty$ and α,β are of the forms (j,α') and (j',β'), respectively. First, Suppose that there is a γ such that γ is an i-section of β and $\alpha \leq^{\mathsf{q}}_{i} \gamma$ holds. By IH, we have $\alpha\sigma \leq^{\mathsf{q}}_{i} \gamma\sigma$. It is obvious that $\gamma\sigma \subset_i \beta\sigma$ holds, so it follows that $\alpha\sigma \leq^{\mathsf{q}}_{i} \beta\sigma$ holds. Next, Suppose that for any $\delta \subset_i \alpha$, $\delta <^{\mathsf{q}}_{i} \beta$ holds and that for any minimal element l of $\mathsf{s}(i,\alpha,\beta)$, $\alpha \leq^{\mathsf{q}}_{l} \beta$ holds. In this case, we immediately have $\alpha \leq^{\mathsf{q}}_{i} \beta$ by IH. □

One can observe the following fact, which is crucial for application of the weak-well-quasi-orderedness of $\mathsf{GQ}(I,A)$ to the termination proof of Buchholz game-style rewrite system.

Fact 3.6. For any two elements l and r of $\mathsf{GQ}(I,A)$, if $l \rhd r$ holds, then $l >^{\mathsf{q}}_{\rho} r$ holds.

Proposition 3.7. For any numeral substitution σ, any $i \in I$, any l and r in $\mathsf{GQ}(I,A)$, if $l \rhd r$ holds, then $l\sigma >^{\mathsf{q}}_{\rho} r\sigma$ holds.

Proof. By the numeral Substitution Property lemma (Lemma 3.5) and Fact 3.6. □

Next, we formulate the restricted Monotonicity Property of $\mathsf{GQ}(I,A)$. The following is a counterexample for the full Monotonicity Property of $\mathsf{GQ}(I,A)$.

Example 3.8 (A counterexample for the full Monotonicity Property of $\mathsf{GQ}(I,A)$). Let i be an element of i with $\rho < i$. A counterexample for Monotonicity Property of $<_{\rho}$ in $\mathsf{GQ}(I,A)$ is as follows. Set $u[x] :\equiv (i,x)$. We have $(i,\rho) <_{\rho} (\rho,(i,\rho))$, but $u[(i,\rho)] = (i,(i,\rho)) \not<_{\rho} (i,(\rho,(i,\rho))) = u[(\rho,(i,\rho))]$ holds because we have $u[(\rho,(i,\rho))] <_{\rho} u[(i,\rho)]$.

For connected gqod's with the root ρ, we have Monotonicity Property.

Lemma 3.9 (The restricted Monotonicity Property lemma). For any two elements α and β of $\mathsf{GQ}(I,A)$, if $(\rho,\alpha) \leq^{\mathsf{q}}_{\rho} (\rho,\beta)$ holds, then for any context u, $u[(\rho,\alpha)] \leq^{\mathsf{q}}_{\rho} u[(\rho,\beta)]$ holds.

By the proposition below, one can prove the termination of Buchholz game-style rewrite relation $\rightarrow$ restricted to numeral substitutions.

Proposition 3.10. For any two elements l and r of $\mathsf{GQ}(I,A)$, if $l \rhd r$ holds, then for any context u and any numeral substitution σ, $u[l\sigma] >^{\mathsf{q}}_{\rho} u[r\sigma]$ holds.

Proof. By Fact 3.6, the numeral Substitution Property lemma (Lemma 3.5) and the restricted Monotonicity Property lemma (Lemma 3.9). $\square$

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