The equivalence between opetopic sets and many-to-one polygraphs

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Abstract

From the polynomial approach to the definition of opetopes of Kock et al., we derive a category $\mathbb{O}$ of opetopes, and show that its $\mathcal{S}$et-valued presheaves, or opetopic sets, are equivalent to many-to-one polygraphs.

As an immediate corollary, we establish that opetopic sets are equivalent to multitopic sets, introduced and studied by Harnick et al.

§1. Introduction

Opetopes were originally introduced by Baez and Dolan in [5] as a algebraic structure to describe compositions and coherence laws in weak higher dimensional categories. They differ from other shapes (such as globular or simplicial) by their (higher) tree structure, giving them the informal designation of "many-to-one". Pasting opetopes give rise to opetopes of higher dimension (it is in fact how they are defined!), thus the analogy between opetopes and cells in a free higher category starts to emerge. On the other hand, polygraphs (also called computads) are higher dimensional directed graphs used to generate free higher categories by specifying generators and the way they may be pasted together (by means of source and targets).

In this paper, we relate opetopes and polygraphs in a formal way. Namely, we define a category $\mathbb{O}$ whose objects are opetopes, in such a way that the category of its $\mathcal{S}$et-valued presheaves, or opetopic sets, is equivalent to the category of many-to-one polygraphs.

The notion of multitope [4, 3] is related to that of opetope, and has been developed based on similar motivations. However the approaches used are very different: *ope*topes are based on *ope*rads [9], while *multi*topes are based on *multi*categories. It is known that multitopic sets are equivalent to many-to-one polygraphs [4, 3, 2], and thus together with our present contribution, we obtain an equivalence between multitopic sets and opetopic sets.

We begin by recalling elements of the theory of polygraphs and polynomial trees in Section 2. We then give the definition of polynomial opetopes from [8] in Section 3. Lastly, we outline the proof of the equivalence in Section 4, by introducing the "opetal" functor $\mathrm{O}[-] : \mathbb{O} \longrightarrow \mathcal{P}\mathrm{ol}^{\nabla}$ from opetopes to many-to-one polygraphs, and the auxiliary notion of shape of a generator in a many-to-one polygraph.

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§2. Preliminaries

§2.1. Polygraphs

A *polygraph* (also called a *computad*) P consists of a small ω-category P^* and sets $P_n \subseteq P^*_n$ for $n \in \mathbb{N}$, such that P_0 is the set of objects of P^*, and such that the underlying $(n+1)$-category $P^*|_{n+1}$ is freely generated by P_{n+1} over its underlying n-category $P^*|_n$, for all $n \in \mathbb{N}$. A morphism of polygraphs is an ω-functor mapping generators to generators. Let $\mathcal{P}\mathrm{ol}$ be the category of polygraphs and morphisms between them.

A polygraph P is said to be *many-to-one* if the target of a generator is also a generator, i.e. if each target map $t : P_{n+1} \longrightarrow P^*_n$ corestricts as $t : P_{n+1} \longrightarrow P_n$. Let $\mathcal{P}\mathrm{ol}^\nabla$ be the corresponding full subcategory.

For more comprehensive surveys about polygraphs, we refer to [10] or [3].

§2.2. Polynomial trees

We review some elements of the theory of polynomial functors and polynomial trees. For more complete references, see [7] or [6].

Definitions. A *polynomial endofunctor*[1] is a $\mathcal{S}\mathrm{et}$-diagram of the form

$$\begin{tikzcd} & E \arrow[r, "p"] \arrow[dl, "s"'] & B \arrow[dr, "t"] & \\ I & & & I. \end{tikzcd} \tag{2.1}$$

Elements of B are called *nodes*, elements of the fiber $E(b) = p^{-1}(b)$ are the *inputs* of b, and elements of I are *edges*. Let $\mathcal{P}\mathrm{olyEnd}$ be the category of polynomial endofunctors and morphisms of polynomial functors.

A *polynomial tree* (or just *tree*) T is a polynomial endofunctor as above where all sets are finite, where $I \neq \emptyset$ (by convention), where all nodes can be reached from a certain edge called the *root* by a unique sequence of adjacent edges, and where no two distinct nodes have the same associated sequence. For $b \in B$, and $e_0, \ldots, e_k$ this sequence of edges, where e_1 is the root edge, where there is $x_i \in E$ such that $e_{i-1} = tp(x_i)$ and $e_i = s(x_i)$, and $t(b) = e_k$, denote by $\&b = [e_0 \cdots e_k]$ the address of b. We also call b the $[e_0 \cdots e_k]$-source of T and write $b = s_{[e_0 \cdots e_k]} T$. The path from the root node (that whose target is the root edge) to itself is empty, so we write the address of the root node as $[\varepsilon]$. Let $T^\bullet$ be the set of node addresses of T. A leaf is an edge $e \in I$ that is not the target of any node, i.e. there is no $b \in B$ such that $t(b) = e$. Clearly, edges can be assigned addresses too, and we let $T^|$ be the set of leaf addresses of T.

F-trees. Given $F \in \mathcal{P}\mathrm{olyEnd}$ as in equation (2.1), an F-tree T is a polynomial tree $\langle T \rangle$ together with a morphism of polynomial functors $T : \langle T \rangle \longrightarrow F$. We denote by $\operatorname{tr} F$ a chosen skeleton of the category of F-trees. Nodes of T are thought of as "decorated" in B via T, and likewise for edges.

Some trees deserve particular attention. Let I be the *trivial tree*, containing only one edge and no node (hence its set E is empty). Let Y_n be a *corolla*, containing only one node b with n inputs. Then, with F as above, and $i \in I$, let I_i be the trivial F-tree whose only edge is decorated in i. We have $\langle \mathsf{I}_i \rangle = \mathsf{I}$. Likewise, for $b \in B$, let Y_b be the corolla whose only node is decorated in b, so that $\langle \mathsf{Y}_b \rangle = \mathsf{Y}_n$, where $n = \#E(b)$.

Grafting. For $F \in \mathcal{P}\mathrm{olyEnd}$ as in equation (2.1), $S, T \in \operatorname{tr} F$, and $[l] \in S^|$ such that the $[l]$-leaf of S and the root edge of T are both decorated by the same $i \in I$. We may form the grafting $S \circ_{[l]} T$, defined by the

[1] The denomination "functor" is comes from the fact that such a diagram induces a functor $\mathcal{S}\mathrm{et}/I \xrightarrow{s^*} \mathcal{S}\mathrm{et}/E \xrightarrow{\forall_p} \mathcal{S}\mathrm{et}/B \xrightarrow{\exists_t} \mathcal{S}\mathrm{et}/I$ by composition of the pullback along s, dependent product along p, and dependent sum along t, respectively.

following pushout:

$$\begin{tikzcd}
\mathsf{I}_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & S \arrow[d] \\
T \arrow[r] & S \circ_{[l]} T.
\end{tikzcd}$$

Every P-tree is either of the form I_i, or obtained by iterated grafting of corollas.

§3. Opetopes

§3.1. The Baez–Dolan construction

Polynomial monads. A polynomial monad is a strong cartesian monad whose underlying endofunctor is polynomial, and let $\mathcal{P}\mathrm{olyMnd}$ be the category of polynomial monads and morphisms of polynomial functors that are also morphisms of monads. Any polynomial endofunctor F as in equation (2.1) admits a free polynomial monad $F^\star$, whose underlying polynomial endofunctor is given by

$$\begin{tikzcd}
& \mathrm{tr}^{|} F \arrow[r, "p"] \arrow[dl, "s"'] & \mathrm{tr}\, F \arrow[dr, "t"] & \\
I & & & I
\end{tikzcd} \tag{3.1}$$

where $\mathrm{tr}^{|} F$ are F-trees with a marked leaf, s maps an F-tree with marked leaf to the label of that leaf, p forgets the marking, and t maps a tree to the label of its root. Remark that for $T \in \mathrm{tr}\, F$ we have $p^{-1}T = T^{|}$. The adjunction $(-)^\star : \mathcal{P}\mathrm{olyEnd} \xrightarrow{\dashv} \mathcal{P}\mathrm{olyMnd} : U$ is monadic, and we abuse notation by letting $(-)^\star$ be the associated monad on $\mathcal{P}\mathrm{olyEnd}$.

Let M be a polynomial monad as in equation (2.1), or equivalently, a $(-)^\star$-algebra. Write its structure map $M^\star \longrightarrow M$ as:

$$\begin{tikzcd}
& \mathrm{tr}^{|} M \arrow[r] \arrow[dl] \arrow[d, "\wp"] \arrow[dr, phantom, "\lrcorner"] & \mathrm{tr}\, M \arrow[d, "t"] \arrow[dr] & \\
I \arrow[d, equal] & E \arrow[r] \arrow[dl] & B \arrow[dr] & I \arrow[d, equal] \\
I & & & I.
\end{tikzcd} \tag{3.2}$$

For $T \in \mathrm{tr}\, M$, the node $t\,T \in B$ is called the *target* of T, while the map $\wp_T : T^{|} \xrightarrow{\cong} E(t\,T)$ is called the *readdressing map of* T. If we think of the element of B as corollas, with leaves (or input edges) indexed in the relevant fiber in E, then M-trees are indeed trees obtained by coherent graftings of those corollas. The target map t then contracts a tree to a corolla, and since the middle square is cartesian, the number of leaves is preserved. The map $\wp$ establishes a coherent correspondence between the leaf addresses of a tree, and the node addresses of its target. The importance of that correspondence is argued in the proof of theorem 3.3.

The $(-)^+$ construction. For $M \in \mathcal{P}\mathrm{olyMnd}$ as in equation (2.1), define M^+, the *Baez–Dolan construction on* M, as the following polynomial endofunctor:

$$\begin{tikzcd}
& \mathrm{tr}^{\bullet} M \arrow[r, "p"] \arrow[dl, "s"'] & \mathrm{tr}\, M \arrow[dr, "t"] & \\
B & & & B,
\end{tikzcd}$$

where $\mathrm{tr}^\bullet M$ is the set of M-trees with a marked node, where s maps such an M-tree decoration of that node, p forgets the marking, and t is part of the structure map of M as in equation (3.2). If $T \in \mathrm{tr}\, M$, remark that $T^\bullet = p^{-1}T$, and thus if $[p] \in T^\bullet$ (i.e. p is a path in T), then $s[p] = s_{[p]} T$.

Theorem 3.3 ([8]). *The polynomial functor M^+ has a canonical structure of a polynomial monad.*

Proof (sketch). The partial law $\mu^+ : \mathrm{tr}^\bullet M \times_B \mathrm{tr}\, M \longrightarrow \mathrm{tr}\, M$ is given by substitution as we now explain. Take $U \in \mathrm{tr}^\bullet M$, $T \in \mathrm{tr}\, M$ such that $s\, U = b = t\, T$, i.e. $(U, T) \in \mathrm{tr}^\bullet M \times_B \mathrm{tr}\, M$. We may think of U as a context corresponding to the selected node: $U = U[Y_b]$. The readdressing map $\wp_T$ of T gives a bijection between $Y_b^|$ and $T^|$, and thus specifies "rewiring instructions" to replace Y_b by T in U: $\mu^+(U, T) = U[T]$. ✓

§3.2. Polynomial approach

We make use of the polynomial functor approach to the definition of opetopes as presented in [8]: let $\mathfrak{Z}^0$ be the identity polynomial monad, $\mathfrak{Z}^n = (\mathfrak{Z}^{n-1})^+$, and expand $\mathfrak{Z}^n$ as

$$\begin{tikzcd} & \mathbb{O}_{n+1}^\bullet \arrow[r, "p"] \arrow[dl, "s"'] & \mathbb{O}_{n+1} \arrow[dr, "t"] & \\ \mathbb{O}_n & & & \mathbb{O}_n . \end{tikzcd}$$

An n-opetope is then an element of $\mathbb{O}_n$, or equivalently a $\mathfrak{Z}^{n-2}$-tree, if $n \geq 2$. In the latter case, an n-opetope is then a tree whose nodes are (labeled in) $(n-1)$-opetopes, and edges are (labeled in) $(n-2)$-opetopes. For $\omega \in \mathbb{O}_n$ with $n \geq 2$, an element of $\mathbb{O}_n^\bullet(\omega)$ is a morphism of $\mathfrak{Z}^{n-2}$-trees of the form $Y_\psi \longrightarrow \omega$, where $\psi \in \mathbb{O}_{n-1}$.

Let $\omega \in \mathbb{O}_n$ with $n \geq 2$, $[p] \in \omega^\bullet$, and $\psi = s_{[p]}\, \omega \in \mathbb{O}_{n-1}$. Then by construction, there is a bijection between the input edges of the node at address $[p]$ in ω and $\psi^\bullet$. If $[q] \in \psi^\bullet$, we call $[[q]]$ the associated input edge, so that the address of that specific edge in ω is $[p[q]]$. Moreover, the $(n-2)$-opetope decorating that edge is by construction $s_{[q]}\, \psi = s_{[q]}\, s_{[p]}\, \omega$.

An opetope $\omega \in \mathbb{O}_n$ with $n \geq 2$ is called *degenerate* if it is of the form $\omega = I_\phi$ for some $\phi \in \mathbb{O}_{n-2}$. We call an edge *inner* if it is neither the root nor a leaf. Inner edges of ω are exactly those whose address is of the form $[p[q]]$, with $[p] \in \omega^\bullet$, $[q] \in (s_{[p]}\, \omega)^\bullet$, and $[p[q]] \in \omega^\bullet$.

§3.3. The category of opetopes

Akin to the work of Cheng [1], we define a category of opetopes by means of generators and relations. The difference with the aforementioned is our use of polynomial opetopes (also equivalent to Leinster's definition [9, 8]), while Cheng uses an approach by multicategorical slicing, yielding symmetric opetopes.

Theorem 3.4 (Opetopic identities). *Let $\omega \in \mathbb{O}_n$ with $n \geq 2$.*

(i) (Inner edge) For $[p[q]] \in \omega^\bullet$ we have $t\, s_{[p[q]]}\, \omega = s_{[q]}\, s_{[p]}\, \omega$.

(ii) (Globularity 1) If ω is non degenerate, we have $t\, s_{[\varepsilon]}\, \omega = t\, t\, \omega$.

(iii) (Globularity 2) If ω is non degenerate, and $[p[q]] \in \omega^|$, we have $s_{[q]}\, s_{[p]}\, \omega = s_{\wp_\omega [p[q]]}\, t\, \omega$.

(iv) (Degeneracy) If ω is degenerate, then $t\, \omega$ is not (so that $[\varepsilon] \in (t\, \omega)^\bullet$), and we have $s_{[\varepsilon]}\, t\, \omega = t\, t\, \omega$.

With those identities in mind, we define the category $\mathbb{O}$ of opetopes by generators and relations as follows.

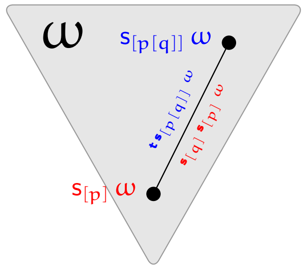
(i) Objects: We set $\mathrm{ob}\, \mathbb{O} = \coprod_{n \in \mathbb{N}} \mathbb{O}_n$.

(ii) Generators: Let $\omega \in \mathbb{O}_n$ with $n \geq 1$. We introduce a generator, called target embedding: $t\, \omega \xrightarrow{t} \omega$. If $[p] \in \omega^\bullet$, then we introduce a generator, called source embedding: $s_{[p]}\, \omega \xrightarrow{s_{[p]}} \omega$. A face embedding is either a source or target embedding.

(iii) Relations: We impose 4 relations described by commutative squares, that are well defined thanks to theorem 3.4. To help with intuition, we give on the right informal diagrams that depicts an opetope as a tree (reverse triangle with some features explicitly drawn). Let $\omega \in \mathbb{O}_n$ with $n \geq 2$

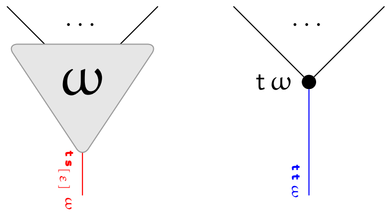
(a) **[Inner]** for $[p[q]] \in \omega^\bullet$ (forcing ω to be non degenerate),

$$\begin{tikzcd} s_{[q]}\, s_{[p]}\, \omega \arrow[r, "s_{[q]}"] \arrow[d, "t"'] & s_{[p]}\, \omega \arrow[d, "s_{[p]}"] \\ s_{[p[q]]}\, \omega \arrow[r, "s_{[p[q]]}"] & \omega \end{tikzcd}$$



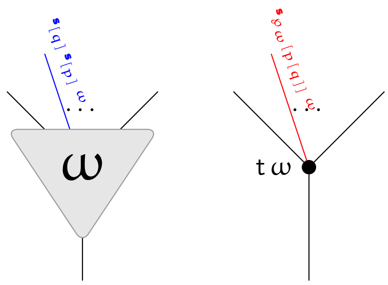
(b) **[Glob1]** if ω is non degenerate,

$$\begin{tikzcd} t\,t\,\omega \arrow[r, "t"] \arrow[d, "t"'] & t\,\omega \arrow[d, "t"] \\ s_{[\varepsilon]}\, \omega \arrow[r, "s_{[\varepsilon]}"] & \omega. \end{tikzcd}$$



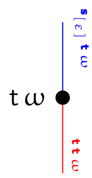
(c) **[Glob2]** if ω is non degenerate, and for $[p[q]] \in \omega^|$,

$$\begin{tikzcd} s_{\wp_\omega[p[q]]}\, t\,\omega \arrow[r, "s_{\wp_\omega[p[q]]}"] \arrow[d, "s_{[q]}"'] & t\,\omega \arrow[d, "t"] \\ s_{[p]}\, \omega \arrow[r, "s_{[p]}"] & \omega. \end{tikzcd}$$



(d) **[Degen]** if ω is degenerate,

$$\begin{tikzcd} t\,t\,\omega \arrow[r, "t"] \arrow[d, "s_{[\varepsilon]}"'] & t\,\omega \arrow[d, "t"] \\ t\,\omega \arrow[r, "t"] & \omega. \end{tikzcd}$$



§4. Outline of the equivalence

We now aim to prove that the category of opetopic sets, i.e. $\mathcal{S}et$-presheaves over the category $\mathbb{O}$ defined previously, is equivalent to the category of many-to-one polygraphs $\mathcal{P}ol^\nabla$. We achieve this by first constructing the *opetal*[2] functor $O[-] : \mathbb{O} \longrightarrow \mathcal{P}ol^\nabla$ that "realizes" an opetope as a polygraph, in that it freely implements all its tree structure by means of adequately chosen generators in each dimension. Secondly, writing $\hat{\mathbb{O}} = \mathcal{S}et^{\mathbb{O}^{op}}$ (as per French tradition), we consider the "polygraphic realization" $|-| : \hat{\mathbb{O}} \longrightarrow \mathcal{P}ol^\nabla$, which is the left Kan extension of $O[-]$ along the Yoneda embedding. This realization has a right adjoint, the "opetopic nerve" $N : \mathcal{P}ol^\nabla \longrightarrow \hat{\mathbb{O}}$, and we prove this adjunction to be an adjoint equivalence. This is done using the shape function, which to any generator x of a many-to-one polygraph P associates an opetope $x^\natural$ along with a canonical projection $\tilde{x} : O[x^\natural] \longrightarrow P$.

[2]The name intends to follow the unofficial "-al" convention e.g. *cubical*, *dentroidal*, *oriental*, *simplicial*, etc.

§4.1. The opetal functor

An opetope $\omega \in \mathbb{O}_n$, with $n \geq 1$, has one target $\mathsf{t}\,\omega$, and sources $\mathsf{s}_{[p]}\,\omega$ laid out in a tree. If the sources $\mathsf{s}_{[p]}\,\omega$ happened to be generators in some polygraph, then that tree would describe a way to compose them. With this in mind, we define a many-to-one polygraph $O[\omega]$, whose generators are essentially iterated faces (i.e. sources or targets) of ω. Moreover, $O[\omega]$ will be "maximally unfolded" (or "free") in that two (iterated) faces that are the same opetope, but located at different addresses, will correspond to distinct generators. The opetal functor $O[-]$ is defined inductively, together with its boundary $\partial O[-]$.

Initial cases. For $\blacklozenge$ the unique 0-opetope, let $\partial O[\blacklozenge]$ be the polygraph with no generators in any dimension, and $O[\blacklozenge]$ be the polygraph with a unique generator in dimension 0, which we denote by $\blacklozenge$.

For $\blacksquare$ the unique 1-opetope, let $\partial O[\blacksquare] = O[\blacklozenge] + O[\blacklozenge]$, and let $O[\blacksquare]$ be the obvious cellular extension $\left(\partial O[\blacksquare] \xleftarrow{\mathsf{s},\mathsf{t}} \{\blacksquare\}\right)$, where s and t map $\blacksquare$ to distinct 0-generators. There are obvious functors $O[\mathsf{s}_{[\varepsilon]}], O[\mathsf{t}] : O[\blacklozenge] \longrightarrow O[\blacksquare]$, mapping $\blacklozenge$ to $\mathsf{s}\,\blacksquare$ and $\mathsf{t}\,\blacksquare$ respectively.

Inductive step. Let $n \geq 2$ and assume by induction that $\partial O[-]$ and $O[-]$ are defined on $\mathbb{O}_{<n}$, the full subcategory of $\mathbb{O}$ spanned by opetopes of dimension strictly less than n. Let $\omega \in \mathbb{O}_n$ and start by defining

$$\partial O[\omega] = \operatorname*{colim}_{\mathbb{O}_{<n}/\omega} O[-].$$

This extends as a functor $\partial O[-] : \mathbb{O}_{\leq n} \longrightarrow \mathcal{P}\mathrm{ol}^{\triangledown}$, mapping a k-opetope to a $(k-1)$-polygraph, for $k \leq n$.

In $\partial O[\omega]$, each source $\mathsf{s}_{[p]}\,\omega$ is an $(n-1)$-generator, while ω itself is a tree whose nodes are its sources. Thus, ω can be thought of as a composition tree of $(n-1)$-generators, and the result of that composition is a cell $\mathsf{s}\,\omega \in \partial O[\omega]^{\triangledown}_{n-1}$. On the other hand, the target of ω is a generator: $\mathsf{t}\,\omega \in \partial O[\omega]_{n-1}$. One can show that $\mathsf{s}\,\omega$ and $\mathsf{t}\,\omega$ are parallel, and thus there is a well defined extension

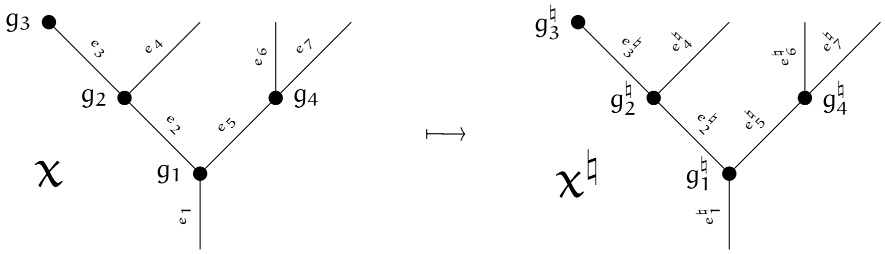
$$O[\omega] = \left(\partial O[\omega] \xleftarrow{\mathsf{s},\mathsf{t}} \{\omega\}\right),$$

where s and t map ω to $\mathsf{s}\,\omega$ and $\mathsf{t}\,\omega$ respectively.

The opetopic "realization–nerve" adjunction. We have a functor $O[-] : \mathbb{O} \longrightarrow \mathcal{P}\mathrm{ol}^{\triangledown}$, and since $\mathcal{P}\mathrm{ol}^{\triangledown}$ is cocomplete, we consider the "opetopic realization" $|-| : \hat{\mathbb{O}} \longrightarrow \mathcal{P}\mathrm{ol}^{\triangledown}$, which is the left Kan extension of $O[-]$ along the Yoneda embedding $\mathbb{O} \hookrightarrow \hat{\mathbb{O}}$. It has a right adjoint $N : \mathcal{P}\mathrm{ol}^{\triangledown} \longrightarrow \hat{\mathbb{O}}$, the "opetopic nerve", given by $NP = \mathcal{P}\mathrm{ol}^{\triangledown}(O[-], P)$, for $P \in \mathcal{P}\mathrm{ol}^{\triangledown}$.

§4.2. The shape function

Take $P \in \mathcal{P}\mathrm{ol}^{\triangledown}$. We now define functions $(-)^{\natural} : P_n \longrightarrow \mathbb{O}_n$ by induction. The cases $n = 0, 1$ are trivial, since there is a unique 0-opetope and a unique 1-opetope. Assume $n \geq 2$, and take $x \in P_n$. Then the composition tree of $\mathsf{s}\,x$ is a coherent tree whose nodes are $(n-1)$-generators, and edges are $(n-2)$-generators. Replacing those $(n-1)$ and $(n-2)$-generators by their respective shape, we obtain a coherent tree whose nodes are $(n-1)$-opetopes, and edges are $(n-2)$-opetopes, in other words, we obtained an n-opetope, which we shall denote by $x^{\natural}$.



The fact that $x^{\natural}$ corresponds to the intuitive notion of "shape" of x is argued by the following result.

Proposition 4.1. *Let $P \in \mathcal{P}\mathrm{ol}^{\nabla}$ and $x \in P_n$, for any $n \in \mathbb{N}$.*

(i) There is a unique morphism of polygraphs $\tilde{x} : O[x^\natural] \longrightarrow P$ mapping $x^\natural \in O[x^\natural]_n$ to x.

(ii) If $\omega \in \mathbb{O}_n$ and $f : O[\omega] \longrightarrow P$ maps ω to x, then $\omega = x^\natural$ and $f = \tilde{x}$.

Corollary 4.2. *The function $\widetilde{(-)}$, that maps a generator x of P to $\tilde{x} : O[x^\natural] \longrightarrow P$, is a bijection $P_n \longrightarrow \coprod_{\omega \in \mathbb{O}_n} NP_\omega$.*

Theorem 4.3. *The adjunction $|-| : \hat{\mathbb{O}} \overset{\dashv}{\longleftrightarrow} \mathcal{P}\mathrm{ol}^{\nabla} : N$ is an adjoint equivalence of categories.*

Proof (sketch). The unit η is proved to be an isomorphism by inspection. For the counit ε, consider the following diagrams.

$$\begin{tikzcd} N \arrow[rr, "\mathrm{id}_N"] \arrow[dr, "\eta N"'] & & N \\ & N|N-|, \arrow[ur, "N\varepsilon"'] & \end{tikzcd} \qquad \begin{tikzcd} |NP| \arrow[r, "\varepsilon"] \arrow[d, "\widetilde{(-)}"'] & P \\ N|NP| \arrow[r, "N\varepsilon"] & NP. \arrow[u, "\widetilde{(-)}^{-1}"'] \end{tikzcd}$$

The first is a triangle identity, and shows that $N\varepsilon$ is a natural isomorphism. It is routine verification to prove that the second commutes, for $P \in \mathcal{P}\mathrm{ol}^{\nabla}$, and it shows that ε itself is an isomorphism. ✓

Many-to-one polygraphs have been the subject of other work [4, 3], and proved to be equivalent to the notion of *multitopic sets*. This, together with our present contribution, proves the following:

Corollary 4.4. *The category $\hat{\mathbb{O}}$ of opetopic sets is equivalent to the category $\mathcal{M}\mathrm{ltSet}$ of multitopic sets.*

§5. Conclusion

We proved the equivalence between opetopic sets (where "opetope" is understood in the sense of Leinster [9, 8]) and many-to-one polygraphs. Along the way, we introduced formal tools and notations to ease the manipulation of opetopes, and demonstrated that the shape they represent are indeed very present in higher category theory.

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