Merge-bicategories: towards semi-strictification of higher categories

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This is a partial abstract of my article *Weak units, divisible cells, and coherence via universality for bicategories* [8], focussing on its last two sections.

In [6, Chapter 3], I outlined a new approach to higher categories founded on a notion of *regular polygraph*, where the shapes of n-dimensional cells are restricted to ones whose k-boundaries have geometric realisations homeomorphic to k-balls, for all $k < n$. These are described using ideas from poset topology [19] and will be the subject of an upcoming paper [7]. The advantages of this model of higher categories are that

1. unlike most algebraic approaches, for example those based on globular operads [15, Part III], it has a clear notion of geometric realisation;

2. similarly to cubical approaches (limited to strict ω-categories [1]), but unlike, for example, the opetopic or multitopic approaches [2, 12], there is a natural monoidal biclosed structure, giving access to higher morphisms;

3. similarly to the opetopic or multitopic approaches, but unlike cubical approaches, a coherent algebra of composition can be induced by the existence of cells satisfying certain universal properties ("coherence via universality" [11]).

The first aim of this approach is to answer certain open questions regarding the semi-strictifiability of higher categories and groupoids, synthesising the combinatorial-topological aspects of more explicit coherence results, like those based on string diagrams [5, 13], with the essential features of Hermida's abstract coherence proof for monoidal categories [10]. In particular, I am interested in proving a variant of C. Simpson's conjecture [18, Conjecture 6.5.1], that there exists an algebraic notion of higher groupoids which

1. have a composition satisfying strict associativity and interchange axioms, but only weak unitality, and

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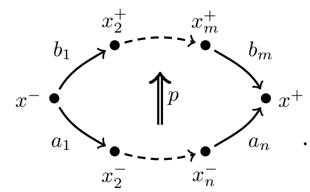
2. admit a geometric realisation functor that realises all homotopy types, in such a way that the "algebraic" homotopy groups of a higher groupoid coincide with the homotopy groups of its realisation.

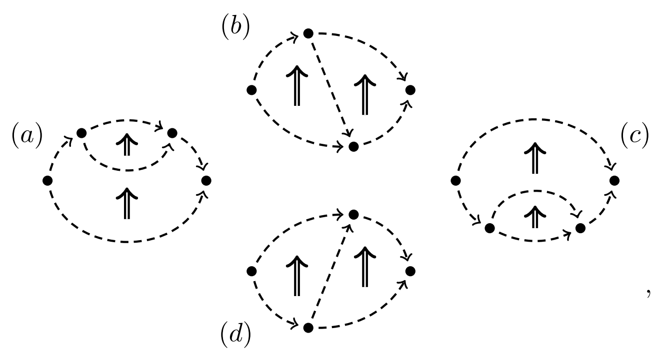
A proof of C. Simpson's conjecture based on regular polygraphs is parallelly being pursued by S. Henry [9].

A first benchmark of any definition of higher category is the equivalence of its 2-dimensional instance with the standard algebraic definition of bicategory. For this, I endow regular 2-polygraphs with an algebraic composition, where composable diagrams of 2-cells are the "regular" ones that can function as boundaries of regular 3-cells. Picture a 2-cell in a regular 2-polygraph with n input and m output 1-cells, $n, m > 0$, as



A dashed arrow stand for a (possibly empty, compatibly with the regularity constraint) sequence of 1-cells. Such 2-cells will be also denoted by $p : (a_1, \ldots, a_n) \to (b_1, \ldots, b_m)$, and 1-cells by $a : x \to y$.

Composable pairs of 2-cells are those in any of the following setups:



(1)

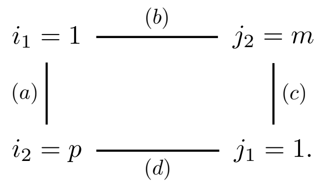
the shared boundary consisting of one or more 1-cells. I call this kind of composition a *merger*.

Let $X_2^{(n,m)}$ be the set of 2-cells of a regular 2-polygraph with n inputs and m outputs; given $p : (a_1, \ldots, a_n) \to (b_1, \ldots, b_m)$, for $1 \leq i_1 \leq i_2 \leq n$ and $1 \leq j_1 \leq j_2 \leq m$, let $\partial^-_{[i_1,i_2]}(p) = (a_{i_1}, \ldots, a_{i_2})$, and $\partial^+_{[j_1,j_2]}(p) = (b_{j_1}, \ldots, b_{j_2})$.

Definition 1. A *merge-bicategory* is a regular 2-polygraph X together with "merge" functions

$$X_2^{(n,m)} \;{}_{\partial^+_{[j_1,j_2]}}\!\times_{\partial^-_{[i_1,i_2]}} X_2^{(p,q)} \xrightarrow{\mathrm{mrg}^{[i_1,i_2]}_{[j_1,j_2]}} X_2^{(n+p-\ell, m+q-\ell)},$$

whenever $1 \leq j_1 \leq j_2 \leq m$ and $1 \leq i_1 \leq i_2 \leq p$, such that $\ell := j_2 - (j_1 - 1) = i_2 - (i_1 - 1)$, satisfy the two conditions on any side of the following square:



The interactions of merge functions and boundaries are the ones evident from diagram (1); the $\mathrm{mrg}^{[i_1,i_2]}_{[j_1,j_2]}$ satisfy associativity and interchange equations that guarantee the uniqueness of the merger of three or more 2-cells, whenever they can be merged in different orders.

Given two merge-bicategories X, Y, a *morphism* $f : X \to Y$ is a morphism of the underlying regular 2-polygraphs that commutes with the $\mathrm{mrg}^{[i_1,i_2]}_{[j_1,j_2]}$ functions. Merge-bicategories and their morphisms form a category **MrgBiCat**.

Notice that, in particular, there is no horizontal composition, nor whiskering with 1-cells as part of the structure of a merge-bicategory.

The fundamental notion connecting merge-bicategories to bicategories is *divisibility* of cells, as in the ability to factor other cells through them ("divide" the cell *from* a composite) at a certain location in their boundary; "a cell which is divisible both at its input boundary and at its output boundary" is an elementary notion of equivalence.

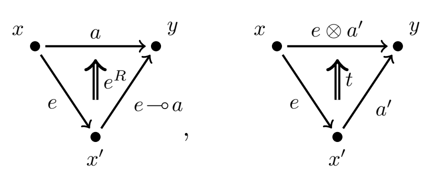
For 2-cells, this encompasses both "universal" or "representing" cells as in [10, 4], and "(absolute) Kan extensions" or "internal homs" [17]. Divisibility for 2-cells is formulated with respect to the algebraic composition, and for 1-cells with respect to an internal composition, witnessed by divisible 2-cells.

Definition 2. A 2-cell $t \in X_2^{(n,m)}$ is *divisible at* $\partial^+_{[j_1,j_2]}$ if, for all 2-cells s and well-formed equations $\mathrm{mrg}^{[i_1,i_2]}_{[j_1,j_2]}(t, x) = s$, there exists a unique 2-cell r such that $\mathrm{mrg}^{[i_1,i_2]}_{[j_1,j_2]}(t, r) = s$.

Dually, t is *divisible at* $\partial^-_{[i_1,i_2]}$ if, for all 2-cells s and well-formed equations $\mathrm{mrg}^{[i_1,i_2]}_{[j_1,j_2]}(x, t) = s$, there exists a unique 2-cell r such that $\mathrm{mrg}^{[i_1,i_2]}_{[j_1,j_2]}(r, t) = s$.

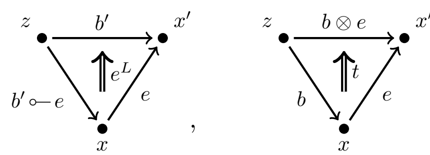
A 2-cell $t \in X_2^{(n,m)}$ is *divisible* if it is divisible at $\partial^-_{[1,n]}$ and at $\partial^+_{[1,m]}$.

Definition 3. A 1-cell $e : x \to x'$ in a merge-bicategory X is *tensor left divisible* if, for each $a : x \to y$ and each $a' : x' \to y$, there exist divisible 2-cells



(2)

that are also divisible at $\partial^-_{[2,2]}$. Dually, e is *tensor right divisible* if, for each $b : z \to x$ and each $b' : z \to x'$, there exist divisible 2-cells



(3)

that are also divisible at $\partial^-_{[1,1]}$. The 1-cell e is *tensor divisible* if it is both tensor left and tensor right divisible.

A 1-cell $e : x \to x'$ is *par divisible* if it is tensor divisible in X^{co}, which is X with the direction of 2-cells reversed. Finally, e is *divisible* if it is both tensor and par divisible.

Definition 4. A merge-bicategory X is *representable* if

1. for all 0-cells x in X, there exist a 0-cell $\overline{x}$ and a divisible 1-cell $e : x \to \overline{x}$, or a divisible 1-cell $e' : \overline{x} \to x$;
2. for all 1-cells a in X, there exist a 1-cell $\overline{a}$ and a divisible 2-cell $p : (a) \to (\overline{a})$, or a divisible 2-cell $p' : (\overline{a}) \to (a)$;
3. for all pairs of 1-cells $a : x \to y$, $b : y \to z$ in X, there exist a 1-cell $a \otimes b$ and a divisible 2-cell $t : (a, b) \to (a \otimes b)$, or a divisible 2-cell $t' : (a \otimes b) \to (a, b)$.

A morphism $f : X \to Y$ of representable merge-bicategories is *strong* if it maps divisible cells of X to divisible cells of Y. Representable merge-bicategories and strong morphisms form a category $\mathbf{MrgBiCat}_\otimes$.

In the definition of representability, $\overline{x}$ can in fact be chosen to be equal to x, and $\overline{a}$ to a, but it is one of the main conceptual points of this approach that this is not necessary: it suffices to postulate that "equivalences" (which are not endomorphisms) exist, and units will follow.

Theorem 5. [8, Theorem 5.17] *The category* $\mathbf{MrgBiCat}_\otimes$ *is equivalent to the category* **BiCat** *of bicategories and (pseudo)functors.*

The construction of a weakly associative horizontal composition from "binary" divisible 2-cells is imported straightforwardly from [10] (or, in higher dimensions, from the opetopic/multitopic approach). However, in the opetopic or multitopic approach, horizontal units (generally, n-units) are constructed from "nullary" universal 2-cells ($(n+1)$-cells), while here they come from "unary" divisible 1-cells (n-cells). The key observation is that, given a divisible 1-cell $e : x \to \overline{x}$, both the 1-cells $e \multimap e : x' \to x'$ and $e \circ\!\!- e : x \to x$, as in diagrams (2) and (3), have the properties of a *Saavedra unit*, an alternative definition of bicategorical unit, due to J. Kock [14].

The category **MrgBiCat** has monoidal biclosed structure [8, Proposition 5.24], a truncated version of the monoidal biclosed structure on regular polygraphs [6, Definition 3.33]. Hence, given merge-bicategories X, Y, we can form a merge-bicategory $[X, Y]$ of morphisms, (op)lax transformations, and modifications; moreover, representable merge-bicategories are an exponential ideal in **MrgBiCat**, in the sense that $[X, Y]$ is representable whenever Y is representable.

This allows us to recover all the higher structure of **BiCat** in $\mathbf{MrgBiCat}_{\otimes}$. In particular, (op)lax transformations of functors of bicategories correspond to (op)lax transformations of strong morphisms of representable merge-bicategories which assign divisible 2-cells to divisible 1-cells [8, Theorem 5.32]. Then the definition of equivalence of representable merge-bicategories can also be imported from bicategories.

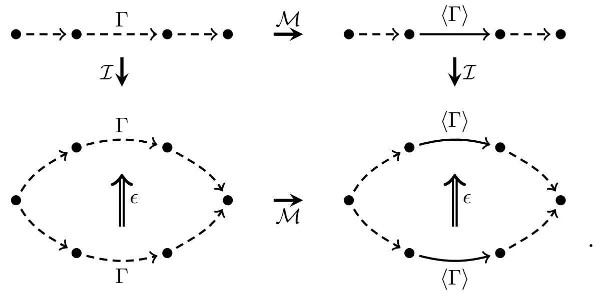
Similarly to the case of representable multicategories and monoidal categories [10], the definition of a bicategory structure from a representable merge-bicategory is not canonically determined, but depends on a choice of divisible cells. I define a monad $(\mathcal{T}, \tilde{\mu}, \tilde{\eta})$ on **MrgBiCat** with the property that any $\mathcal{T}$-algebra $\alpha : \mathcal{T}X \to X$ on a merge-bicategory X, such that α is a strong morphism, determines the structure of a *strictly associative* bicategory with weak units.

The key point is that $\mathcal{T}$ splits as a composite $\mathcal{M}\mathcal{I}$ of two monads, $(\mathcal{I}, \mu, \eta)$, for *inflate*, and $(\mathcal{M}, \nu, \zeta)$, for *merge*, related by a distributive law [3]: roughly, the first encodes the structure relative to units, and the second the structure relative to composition.

The idea — extended to higher dimensions — is that "merge" compositions are those that can be implemented, in the geometric realisation of the shapes, as homeomorphic mappings of two or more n-balls onto a single n-ball; while units are created by collapses of cells with identical shapes in the input and the output boundary ("inflate" as dual to "collapse"). These can be realised directly for cells with a single input and a single output, or by first merging parallelly the inputs and the outputs, then collapsing, and finally "un-merging" homeomorphically, in the general case.

For example, given any pair of 1-cells $a : x \to y$ and $b : y \to z$ of X, the merge-bicategory $\mathcal{I}X$ contains a divisible 2-cell $\epsilon : (a, b) \to (a, b)$, corresponding to the collapse of a 2-cell with two inputs and two outputs; then, $\mathcal{M}\mathcal{I}X$ contains a 1-cell $\langle a, b\rangle$ — a merger of the sequence a, b — and a divisible 2-cell $\tilde{\epsilon}$: $(a, b) \to (\langle a, b\rangle)$, via the merger of a, b in the output of ϵ. The strong morphism $\alpha : \mathcal{M}\mathcal{I}X \to X$ maps this to a divisible 2-cell $(a, b) \to (a \otimes b)$, effectively picking a composite of a, b.

The distributive law $\sigma : \mathcal{I}\mathcal{M} \to \mathcal{M}\mathcal{I}$ encodes the fact that, whenever we merge some 1-cells in a sequence, and then "inflate" it to a unit, we can first inflate the original sequence, then merge on both sides of the unit's boundary, instead:



The decomposition of $\mathcal{T}$ leads to a semi-strictification argument that, while sharing the essential structure of Hermida's proof, has an explicit combinatorial content in common with Mac Lane's original proof [16]. The main steps are the following:

1. any representable merge-bicategory X admits an $\mathcal{I}$-algebra structure $\alpha : \mathcal{I}X \to X$ such that α is a strong morphism;
2. given an $\mathcal{I}$-algebra $\alpha : \mathcal{I}X \to X$ such that α is a strong morphism, there is a canonical $\mathcal{T}$-algebra structure $\beta : \mathcal{T}X \to \mathcal{M}X$ on $\mathcal{M}X$ such that β is a strong morphism;
3. if X is representable, the unit $\zeta_X : X \to \mathcal{M}X$ is an equivalence of representable merge-bicategories.

In fact, the individual steps hold under weaker conditions. Together, they imply constructively the following result.

Theorem 6. *Let X be a representable merge-bicategory. Then there exist an equivalence $f : X \to Y$ of representable merge-bicategories, and a $\mathcal{T}$-algebra $\beta : \mathcal{T}Y \to Y$ such that β is a strong morphism.*

If time permits, I will sketch how this argument may be generalised to the higher-dimensional analogues of $\mathcal{I}$ and $\mathcal{M}$, as outlined above.

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