Termination in linear $(2,2)$-categories with braidings and duals

Benjamin DUPONT, *bdupont@math.univ-lyon1.fr*
Université Claude Bernard Lyon 1, CNRS UMR 5208,
Institut Camille Jordan, 43 boulevard du 11 novembre 1918, F-69622 Villeurbanne cedex, France

1. Introduction

This work is part of a research project aiming at developing rewriting methods to study diagrammatic algebras. These diagrammatic algebras appear in various domains of mathematics and physics, as for instance Temperley-Lieb algebras [16] in statistical mechanics, Brauer algebras [4] in representation theory or Birman-Wenzl algebras [3] and Jones' planar algebras [8] in knot theory. Moreover, in representation theory, a new approach has emerged with the idea of studying categorifications of algebras, that is actions of algebras on higher dimensional categories. In this process, some new diagrammatic algebras with a categorical structure appear, such as KLR algebras [10, 14] or Khovanov's diagram algebras [5], and one of the main issue is to compute linear bases of these algebras. Higher-dimensional rewriting, see [1], provides new methods to compute these bases by using termination and confluence properties.

In this work, we consider diagrammatic $\mathbb{K}$-algebras that can be seen as $\mathbb{K}$-linear $(2,2)$-categories, that is 2-categories $\mathcal{C}$ with sets of 0-cells $\mathcal{C}_0$, 1-cells $\mathcal{C}_1$ and 2-cells $\mathcal{C}_2$ such that for all 1-cells p and q, the set $\mathcal{C}_2(p,q)$ of 2-cells with source p and target q is a $\mathbb{K}$-vector space. According to Alleaume [1], these categories can be presented by 3-dimensional linear rewriting systems called linear $(3,2)$-polygraphs. These polygraphs provide define the generating 1-cells and 2-cells of a presentation of $\mathcal{C}$ by generators and relations. Considering the algebras coming from representation theory, we will in particular be interested in some particular cases of linear $(2,2)$-categories which are endowed with braidings, adjunctions and duals.

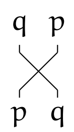
The termination and confluence properties of linear $(3,2)$-polygraphs are essential to construct the required linear bases. In this work, we present new termination proofs for these linear $(2,2)$-categories enriched with braidings and duals using new termination orders which are closely related to monomial orders. A different method already exists to prove termination of 3-polygraphs, [7], but it is hard to use in presence of a great number of relations. Our approach using this new order may be useful in some situations. The main idea is to count the generators on string diagrams and try to find some characteristics which are both stable by adding a context, and strictly greater on sources on 3-cells than on targets. We will at first present the different categorical structures that we will study, and then we will progressively construct the termination heuristics depending on the kind of relations we consider.

2. Some particular cases of linear $(2,2)$-categories

2.1. Categories with braidings. We consider a linear $(2,2)$-category $\mathcal{C}$ with an additional structure given by braidings, that is for any generating 1-cells p and q of $\mathcal{C}$, there is a natural isomorphism

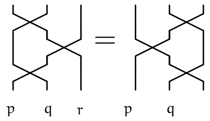
$$\sigma : p \star_0 q \to q \star_0 p$$

where the $\star_0$-composition is diagrammatically depicted by placing two string diagrams next to each other. Such natural morphisms are diagrammatically depicted by crossings:



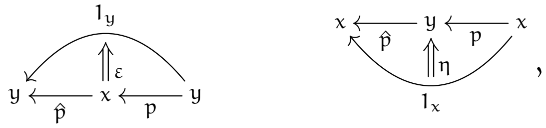
In this work, we use a symmetric notation for braidings, and do not distinguish a braid from its topological inverse because we allow linear $(2,2)$-categories with relations of the form $\sigma_{p,q} \star_1 \sigma_{q,p} = 1_p \star_0 1_q$.

These braidings have to satisfy some naturality axioms, namely the hexagon axiom: $(\mathrm{id}_q \otimes \sigma_{p,r}) \circ \alpha_{q,p,r} \circ (\sigma_{p,q} \otimes \mathrm{id}_r) = \alpha_{q,r,p} \circ (\sigma_{q,r\otimes p} \circ \alpha_{p,q,r})$ where $\alpha_{p,q,r} : (p \otimes q) \otimes r \to p \otimes (q \otimes r)$. This naturality yield to the Yang-Baxter equation, which can be diagrammatically depicted by:

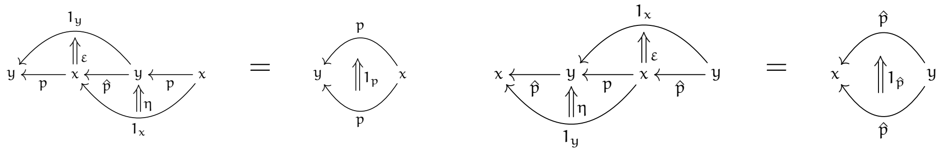


(1)

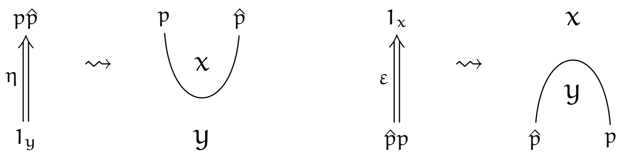
It appears that these braidings structures are often combined with other kinds of operations and in particular, adjunctions on the 1-cells of the linear $(2,2)$-category. These adjunctions are defined as follows: for any 1-cell $p : x \to y$ of $\mathcal{C}$, there is a 1-cell $\hat{p} : y \to x$ and two 2-cells ε and η in $\mathcal{C}$ defined as follows:



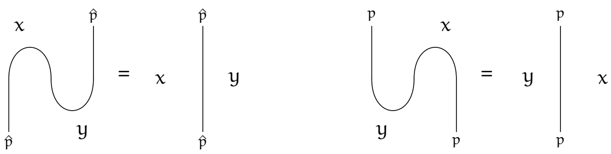
called the counit and unit of the adjunction, such that the equalities



hold. We denote the fact that p is a left adjoint of $\hat{p}$ by $p \dashv \hat{p}$. In a string diagrammatic notation, these units and counits are represented by caps and cups morphisms as follows:



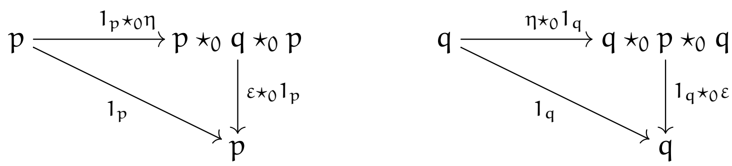
The axioms of an adjunction require that the equalities between composites of 2-morphisms



(2)

are satisfied. When we are in the situation where $\hat{p}$ is also a left-adjoint of p, that is p and $\hat{p}$ are biadjoint, that we denote by $p \dashv \hat{p} \dashv p$, the symmetric zig-zag relations hold similarly.

2.2. Categories with duals. Following [15] for monoidal categories, in a strict linear $(2,2)$-category an exact pairing between two 1-morphisms p and q such that $p \dashv q$ is given by a pair of morphisms $\eta : 1_y \to q \star_0 p$ and $\varepsilon : p \star_0 q \to 1_x$ which are the unit and counit of the adjunction and such that the following two adjuction triangles commute



In such an exact pairing, B is called the *right dual* of A and A is the *left dual* of B.

2.3 Definition. A linear $(2,2)$-category is said *right autonomous* if every 1-morphism p has a right dual, which we then denote by p^*. It is said *left autonomous* if every 1-morphism p has a left dual, which we denote by ${}^*p$. It is *autonomous* if it is both left and right autonomous.

2.4. Mateship under adjunction. We recall following [13] the 2-category theoretic notion of mateship under adjunction introduced by Kelly and Street in [9]. This is a certain correspondence between 2-morphisms in the presence of adjoints.

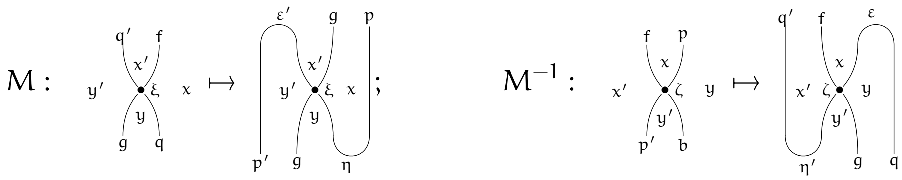
Given adjoints $\eta, \varepsilon \colon p \dashv q \colon x \to y$ and $\eta', \varepsilon' \colon p' \dashv q' \colon x' \to y'$ in the 2-category $\mathcal{C}$, for any 1-cells $f \colon x \to x'$ and $g \colon y \to y'$ in $\mathcal{C}$, there is a bijection M between 2-morphisms $\xi \in \mathcal{C}(g \star_0 q, q' \star_0 f)$ and 2-morphisms $\zeta \in \mathcal{C}(p' \star_0 g, f \star_0 p)$, where ζ is given in terms of ξ by the composite:

$$
\begin{aligned}
M \colon \mathcal{C}(g \star_0 q, q' \star_0 f) &\longrightarrow \mathcal{C}(p' \star_0 g, f \star_0 p) \\
\xi &\mapsto \left(p' \star_0 g \xRightarrow{p' \star_0 g \eta} p' \star_0 g \star_0 q \star_0 p \xRightarrow{p' \star_0 \xi \star_0 p} p' \star_0 q' \star_0 f \star_0 p \xRightarrow{\varepsilon' \star_0 f \star_0 p} f \star_0 p \right) = \zeta ,
\end{aligned}
$$

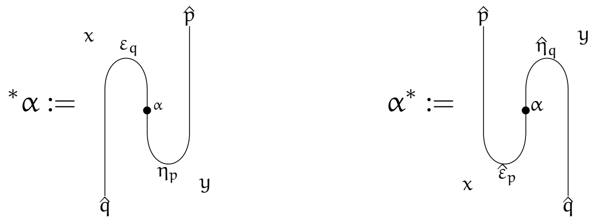
and ξ is given in terms of ζ by the composite:

$$
\begin{aligned}
M^{-1} \colon \mathcal{C}(p' \star_0 g, f \star_0 p) &\longrightarrow \mathcal{C}(g \star_0 q, q' \star_0 f) \\
\zeta &\mapsto \left(g \star_0 q \xRightarrow{\eta' \star_0 g \star_0 q} q' \star_0 p' \star_0 g \star_0 q \xRightarrow{q' \star_0 \zeta \star_0 q} q' \star_0 f \star_0 p \star_0 q \xRightarrow{q' \star_0 f \star_0 \varepsilon} q' \star_0 f \right) = \xi .
\end{aligned}
$$

We then say that ξ and ζ are *mates under adjunction*. Diagrammatically, this notion of mateship under adjunction can be expressed as:



2.5. Cyclic 2-morphisms. Given a pair of 1-morphisms $p, q \colon x \to y$ with chosen biadjoints $(\hat{p}, \eta_p, \hat{\eta}_p, \varepsilon_p, \hat{\varepsilon}_p)$ and $(\hat{q}, \eta_q, \hat{\eta}_q, \varepsilon_q, \hat{\varepsilon}_q)$, then any 2-morphism $\alpha \colon p \Rightarrow q$ has two obvious duals ${}^*\alpha, \alpha^* \colon \hat{q} \Rightarrow \hat{p}$, or mates, one constructed using the left adjoint structure, the other using the right adjoint structure. Diagrammatically the two mates are given by



(3)

We will call α^* the right dual of α because it us obtained from α as its mate using the right adjoints of p and q. Similarly, ${}^*\alpha$ is called the left dual of α because it is obtained from α as its mate using the left adjoints of p and q.

In general there is no reason why ${}^*\alpha$ should be equal to α^*, see [13] for a simple counterexample.

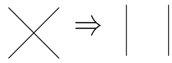
2.6 Definition ([6])**.** Given biadjoints $(p, \hat{p}, \eta_p, \hat{\eta}_p, \varepsilon_p, \hat{\varepsilon}_p)$ and $(q, \hat{q}, \eta_q, \hat{\eta}_q, \varepsilon_q, \hat{\varepsilon}_q)$ and a 2-morphism $\alpha \colon p \Rightarrow q$ define $\alpha^* := \hat{p}\hat{\eta}_q . \hat{\varepsilon}_p \hat{q}$ and ${}^*\alpha := \varepsilon_G \hat{p} . \hat{\eta}_q$ as above. Then a 2-morphism α is called a *cyclic 2-morphism* if the equation ${}^*\alpha = \alpha^*$ is satisfied, or either of the equivalent conditions ${}^{**}\alpha = \alpha$ or $\alpha^{**} = \alpha$ are satisfied.

3. Termination of 3-dimensional linear rewriting systems

We present a method to prove termination of a given linear $(3, 2)$-polygraph presenting a linear $(2, 2)$-category with a particular additionnal structure coming from braidings or duals. This method is based on the construction of a termination order similar to a monomial order, that is compatible with contexts and well-founded but that is not required to be total. Actually, as shown by Alleaume [1] computing such a monomial order on a linear $(2, 2)$-category is not always possible.

3.1. The general idea: counting the generators. In [7], Guiraud and Malbos described a method to prove termination of a 3-dimensional string rewriting system by constructing a derivation defined on generating 2-cells of the category presented and ensuring that for 3-cell α, this derivation is strictly greater on $s(\alpha)$ than $t(\alpha)$. However, this method is not efficient when the number of rules or generating 2-cells is too important, since computing the values of the derivation may become complicated. We want to find another method which could be used for many kinds of such linear $(2,2)$-categories.

We fix a presentation of our linear $(2,2)$-category by choosing an orientation of the 3-cells and we try to find some characteristics of the diagrammatic sources of 3-cells for which they could be compared to the targets. For instance, with a rule of the form



we notice that the number of crossings is strictly greater on the source of the 3-cell (since it is 1) than on the target (since it is 0). Besides, this characteristic is "monomial" in a sense, because it is stable by context. Namely, if we put our source into a given context, that is if we plug with compositions $\star_0$ and $\star_1$ of the category new diagrams to this crossing, the number of crossings of the resulting diagram is strictly greater that on the same diagram with the crossing replaced by the identity.

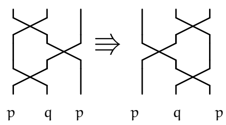
3.2. Decreasing order operators. Given a linear $(3,2)$-polygraph Σ, one defines a *decreasing order operator* for Σ as a family of functions $\Phi_{p,q} : \Sigma_2(p,q) \to \mathbb{N}^{m(p,q)} \times \mathbb{Z}$ indexed by pairs of 1-cells p and q satisfying the following conditions:

- For each diagrams D_1 and D_2 in $\Sigma_2(p,q)$ such that there exists a 3-cell $\alpha : D_1 \Rrightarrow D_2$ in Σ, the function $\Phi_{p,q}$ satisfy $\Phi_{p,q}(D_1) > \Phi_{p,q}(D_2)$ where $>$ is the lexicographic order on $\mathbb{N}^{m(p,q)} \times \mathbb{Z}$. We denote this by $D_1 >_{\text{lex}} D_2$.
- The $\Phi_{p,q}$ are stable by context in the following sense: for any diagrams D_1 and D_2 and any context C of Σ, if $D_1 >_{\text{lex}} D_2$, then $C[D_1] >_{\text{lex}} C[D_2]$.

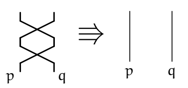
Note that the lexicographic order on $\mathbb{N}^{m(p,q)} \times \mathbb{Z}$ is not well-founded, but in general we use decreasing order operators with no $\mathbb{Z}$-component or with a lower bound on this component as we will explain in 3.9.

3.3. Termination with braidings. We assume that $\mathcal{C}$ is a linear category having for generating 2-cells only the crossings $\sigma_{p,q}$ for each generating 1-cells p and q in $\mathcal{C}_1$. Let us choose Σ a linear $(3,2)$-polygraph presenting $\mathcal{C}$, then the set Σ_3 has to contain all the Yang-Baxter relations with a fixed choice of orientation for each one. We assume that:

- Σ_3 contains only these Yang-Baxter relations (1), and we decide to choose the following orientation for each of these relations:

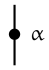
  

  Then, for any 1-cells p and q, the functions $\Phi_{p,q}$ counting the number of occurences of the 2-cell $\sigma_{p,q} \star_0 \text{id}_p$ in a given diagram allows to prove that such a rewrite system is terminating, because the number of such occurences can not increase using the exchange law of the category and is stable by context. This idea is similar to the termination order counting the number of occurences of the generator s in the monoid $B_3^+ = \langle\, s,t \mid sts \Rightarrow tst \,\rangle$.
- Σ_3 contains the Yang-Baxter relations and some other relations making the number of braidings decrease, as double permutations relations in the symmetric group:

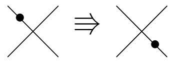
 (4)

Then we can add as first component to the functions $\Phi_{p,q}$ defined above a component counting the number of crossings of any diagram. Similarly, it is clearly stable by exchange law and contexts.

Assume now that there are an additional 2-morphism α in $\mathcal{C}_2(q,q)$ for $q \in \mathcal{C}_1$ diagrammatically depicted by



and that we denote by a simple dot when there is no ambiguity. If α appears in relations with the braidings of type commutation or at least commutation with creation of lower terms for the order already defined. For instance, with a rule of the form $(\alpha \star_0 \mathrm{id}_p) \star_1 \sigma_{p,q} = \sigma_{p,q} \star_1 (\mathrm{id}_p \star_0 \alpha) + (\text{lower terms})$, diagrammatically represented by

 + lower terms (5)

for any $p, q \in \mathcal{C}_1$, the termination of the associated rewriting system can be ensured in the following way: for each 1-cell p in $\mathcal{C}_1$, we denote by $\ell(p)$ the length of p corresponding to the number of generating 1-cells appearing in its decomposition in $\star_0$-composition of these generators. We fix $m(p,q) = \max(\ell(p), \ell(q))$. For a diagram D in $\mathcal{C}_2(p,q)$, we add to our previous order m new components, which are defined as follows: we numerate the strands of the diagram from 1 to $m(p,q)$ (on top or bottom, depending on which 1-cell has the maximal length) and for each $1 \leq k \leq m$, we add the components $(c_k(D))_{1 \leq k \leq m(p,q)}$ defined by:

- 0 if the k-h strand is not a *through strand*, that is a strand linking a point of the top and a point of the bottom boundaries of the diagram; this can not occur in category with only braidings.
- the number of crossings below the upper dot of the k-th strand.

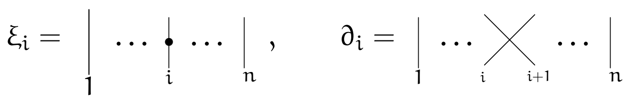
Then we define a decreasing order operator by $\Phi(p,q) = (c(D), \mathrm{ybg}(D), c_1(D), \ldots, c_m(D)$ where $c(D)$ corresponds to the number of crossings, $\mathrm{ybg}(D)$ corresponds to the number of occurences of a $\sigma_{p_1,q_1} \otimes \mathrm{id}_{p_1}$ for generating 1-cells p_1 and q_1. It is clearly stable by context as we add a constant number of crossings under each dot of the diagrams, and is well-founded.

3.4 Nil Hecke algebra. For $n \in \mathbb{N}$, let us consider the *Nil-Hecke algebra* $\mathcal{NH}_n^0$ which is a $\mathbb{K}$-algebra for a given field $\mathbb{K}$ given by:

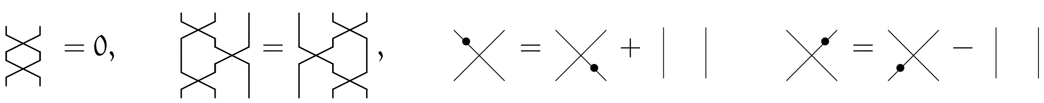
- generators ξ_i for $1 \leq i \leq n$ and ∂_i for $1 \leq i < n$;
- relations:

$$\begin{array}{ll}
\xi_i\xi_j = \xi_j\xi_i, & \\
\partial_i\xi_j = \xi_j\partial_i \quad \text{if } |i-j| > 1, & \partial_i\partial_j = \partial_j\partial_i \quad \text{if } |i-j| > 1, \\
\partial_i^2 = 0, & \partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1}, \\
\xi_i\partial_i - \partial_i\xi_{i+1} = 1, & \partial_i\xi_i - \xi_{i+1}\partial_i = 1.
\end{array}$$

This $\mathbb{K}$-algebra can be seen as a linear $(2,2)$-category with only one 0-cell, the 1-cells are given by permutations of the set $\{1, \ldots, n\}$ and 2-cells between two permutations are given by braiding diagrams linking these two permutations. The generating 2-cells can be pictured as



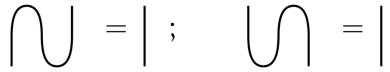
and the local relations are represented by:



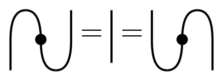
The other algebraic relations correspond to the exchange laws of the 2-category and are implicit in this context. If we define a linear $(3,2)$-polygraph by orienting the relations as suggested above, then we can prove that it is terminating using the following decreasing order operator: for a given diagram D in $\mathcal{NH}_n^0(\sigma,\tau)$,

$$\Phi_{\sigma,\tau}(D) := \Phi = (c(D), \mathrm{ybg}(D), c_1(D), \ldots, c_n(D)).$$

3.5. Categories with adjunctions. We consider a linear $(2,2)$-category $\mathcal{C}$ whose 1-morphisms are equipped with given biadjunctions, which yield isotopy relations of the form


(6)

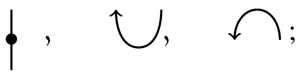
where we omit the labelling on morphisms when not necessary. If we orient these relations from left to right, we have to add new components to "count the degree of isotopy of a diagram", which can be realised by counting its number of caps and cups. It is also stable by context. Now, assuming that there is an additional 2-morphism α which is cyclic with respect to some biadjunction $p \vdash q \vdash p$, we have to impose some new relations of the form:


(7)

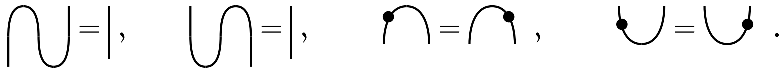
More over, this linear $(3,2)$-polygraph is not confluent, and the first steps of Knuth-Bendix computation forces us to add some relations of the form

allowing to make dots move in each direction on a cup or a cap. These relations also arise naturally in the context of *pivotal categories*, which are a particular case of categories with duals in which $p^{**} \simeq p$ for any 1-cell p, since they are just axiom consequences of the pivotal structure following [13].

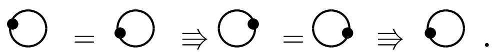
3.6. The $(3,2)$-polygraphs of pearls. In this section, we consider a particular case of a linear $(2,2)$-category with adjunctions and an additionnal cyclic dot 2-morphism, see [7] for details on this rewriting system. Namely, let $\mathcal{C}$ be the linear $(2,2)$-category with:

- only one 0-cell $*$;
- only one 1-cell p;
- generating 2-cells: 

subject to the following relations:

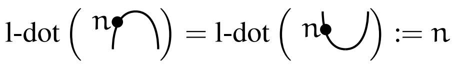


We fix a linear $(3,2)$-polygraph presenting $\mathcal{C}$ by orienting the above relations from left to right. We notice that if we orient the relations in such a way that dots move in different directions on composable cups and caps, we automatically lose termination because we create cycles of the following form:



To prove that such a linear $(3,2)$-polygraph is terminating, we define the following order associated with the functions $\Phi_{p,p}(D) = (I(D), \text{l-dot}(D))$ where:

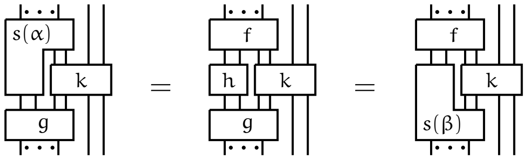
- $I(D)$ corresponds to the isotopy degree of D, that is the number of caps and cups it contains;
- l-dot(D) corresponds to the number of positively left-dotted caps and cups, that is the number of elements  and  with at least one dot appearing in D with the convention that



The lexicographic order defined by this function is stable by context because by composing a diagram using $\star_0$ and $\star_1$-compositions, we add a constant number of cups and caps to the diagram, and so is the number of positively dotted left cups or caps since a dot cannot move from right of a cap/cup to its left even by adding a context.

3.7. Indexed critical branchings. We described a method to study termination of a linear $(3,2)$-polygraph which contains the rules of the linear $(3,2)$-polygraphs of pearls. However, we saw that we had to to orient the dot-moves in the same direction to preserve termination. We will see that this may be an obstruction to prove the confluence of the rewriting system and that we may choose to lose the termination and consider quasi-terminating linear $(3,2)$-polygraphs.

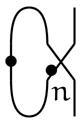
According to [12, 7], in 3-dimensional set-theoretic rewriting there are unusual forms of critical branchings which appear. They are called *indexed critical branchings* and have the following form:

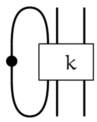


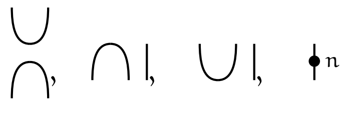
where $f, g, h, k \in \mathcal{C}_2$, and $s(\alpha) \overset{\alpha}{\Rrightarrow} t(\alpha)$ is a 3-cell in the linear $(3,2)$-polygraph Σ.

With isotopy relations given by the biadjunction structure, keeping the dot-move relations with the same orientation may generate a huge number of new indexed critical branchings. If we consider a linear $(2,2)$-category also equipped with braidings and a linear $(3,2)$-polygraph presenting Σ with relations oriented as above, then whenever the following compositions make sense we create for each $n \in \mathbb{N}$ an indexed critical branching in Σ whose form is given in Figure 1 below. This may be source of obstructions to prove the confluence of our system. In fact, for each 2-cell that one can plug in the diagram below, there is an indexed critical branching depicted in Figure 2-below. From [7] these indexed branchings are confluent when the 2-cell k is one of diagrams given in Figure 3 below.

Figure 1 Figure 2 Figure 3



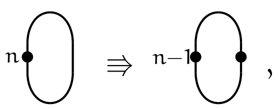




To avoid the computations of critical pairs being too complicated, we sometimes choose to deliberately lose the termination of the rewriting system by allowing new orientations for the dot-move rules:

We will then study the property of *quasi-termination* of the new linear $(3,2)$-polygraph.

3.8. Quasi-termination and quasi-reduced monomials. Following [1], we say that a linear $(3,2)$-polygraph Σ is *quasi-terminating* if for each sequence $(u_n)_{n\in\mathbb{N}}$ of 2-cells such that for each n in $\mathbb{N}$ there is a rewriting step from u_n to u_{n+1}, the sequence $(u_n)_{n\in\mathbb{N}}$ contains an infinite number of occurences of the same 2-cell. A 2-cell ϕ of Σ is called a *quasi-normal form* if for any rewriting step from ϕ to another 2-cell ψ, there exist a rewriting sequence from ψ to ϕ. A quasi-normal form of a 2-cell ϕ is a quasi-normal form $p\tilde{hi}$ such that there exists a rewriting sequence from u to $\tilde{u}$. We say that Σ is *exponentiation free* is for any 2-cell ϕ, there does not exist a 3-cell α in Σ such that $\phi \overset{\alpha}{\Rrightarrow} \lambda\phi + u$ with $\lambda \in \mathbb{K}^*$ and $u \neq 0$. In a quasi-terminating context, there is an equivalent to the critical pair lemma, which states that it remains an interesting property to study since there exists confluence issues in a quasi-terminationg context [2].
These termination heuristics can be extended to prove quasi-termination when the dot-move relations are oriented in different directions. The main idea is to ensure that the linear $(3,2)$-polygraph Σ considered only the bubbles cycles described above and no other rewriting cycles. We define a notion of *quasi-reduced* monomial in Σ which is a monomial on which we can only apply the rules



and we apply the same process of defining a decreasing order operator which does not count the number of left-dotted cups and caps on the set of quasi-reduced monomials to ensure that there is no other obstruction to termination that these cycles.

3.9. A $\mathbb{Z}$-degree component with a lower bound . Some diagrammatic algebras are equipped with a $\mathbb{Z}$-grading, which may be used as a criterion for termination provided that the grading is bounded below. Consider a linear $(2,2)$-category $\mathcal{C}$ is equipped with a $\mathbb{Z}$-grading on 2-cells having caps and cups 2-cells so that we can define *bubbles* in $\mathcal{C}$ (which are $\star_1$-compositions of a cap above a cup in $\mathcal{C}$). As illustrated on the example of $\mathcal{KLR}$ in the appendix, the algebraic context may enforce some bubbles to be 0 whenever their degree decrease too much. In particular, this is the case for a linear $(2,2)$-category in which all bubbles with negative degree are 0.

If we assume that there is a relation implying bubbles in $\mathcal{C}$ for which the degree is strictly decreasing, then there can not exist an infinite rewriting step using this relation since there exists a point from which this degree will be strictly negative, and thus the whole diagram will rewrite to 0. This can be formalized by adding this degree in the last component of a decreasing order operator, with the convention that the degree of 0 is $-\infty$, so that 0 is the minimal element for this order (because each other component of the diagram 0 is 0). We will illustrate this idea on the linear $(3,2)$-polygraphs $\mathcal{KLR}$ in the appendix.

3.10. General heuristics. We generalize the previous heuristics constructed in this section for a linear $(3,2)$-polygraph presenting a linear $(2,2)$-category with braidings and duals. We will illustrate our approach on the linear $(3,2)$-polygraphs $\mathcal{KLR}$ given in the appendix.

3.11 Proposition. *Let $\mathcal{C}$ be a linear $(2,2)$-category endowed with braidings, duals and some additionnal cyclic 2-morphisms which admits a presentation by generators and relations containing further of the following:*

- *Yang-Baxter relations (1);*
- *relations making the number of braidings decrease as symmetric group relations (4);*
- *commutation of some of the cyclic 2-morphisms with the braidings, eventually creating residues with lower crossings (5);*
- *the isotopy relations coming from the adjunctions and the cyclicity of the 2-morphisms (6, 7);*
- *some other relations that make the number of crossings or the number of cups and caps decrease;*
- *in a $\mathbb{Z}$-graded context, some relations making the degree decrease with a lower bound on the degree under which all diagrams are zero as explained in 3.9.*

Then, any linear $(3,2)$-polygraph presenting $\mathcal{C}$ in which the relations are oriented in such a way that the functions $\Phi_{p,q}$ satisfy $\Phi_{p,q}(s(\alpha)) > \Phi_{p,q}(t(\alpha))$ for all 1-cells p,q and 3-cell α is terminating.

The proof of this result is given by our previous heuristics: if we have braidings with relations making the number of crossings decrease, we add a component for the number of crossings and the number of occurences of a $\sigma_{p,q} \star_0 \mathrm{id}_p$ in the diagram. If there is an additionnal 2-morphism which commutes with the braidings, we add the required number of components $c_k(D)$ defined above. If we have the isotopy relations, we add a component counting the number of cups and caps. If we have a dot-move relation of the additionnal 2-morphism, with cups and caps we count the positively left-dotted cups and caps and if we have relations implying decreasing degree conditions we can add a last component counting this degree. The 3-cells are then oriented naturally with respect to this order.

CONCLUSION

In this work, we have presented new heuristics to prove termination of linear $(3,2)$-polygraphs presenting linear $(2,2)$-categories arising in the context of representation theory. We have illustrated this to prove the termination of the linear $(3,2)$-polygraphs $\mathcal{KLR}$. The construction of such heuristic is the first step to get a linear basis of the 2-category from [11] which is a candidate categorification for a quantum group. We now need to develop new confluence criteria. The isotopy relations may bring non-confluent critical branchings, as it is the case for $\mathcal{KLR}$, and we want to develop a theory of rewriting modulo these isotopy relations, and to obtain new bases results in this context.

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A. THE LINEAR $(3,2)$-POLYGRAPHS $\mathcal{KLR}$

In this appendix, we describe explicitely the termination proof of a linear $(3,2)$-polygraph defined from a linear $(2,2)$-category arising in a process of categorification of quantum deformations of some Kac-Moody algebras, see [11] for more details. These Kac-Moody algebras have a weight lattice, that we will just see as a set X in here.

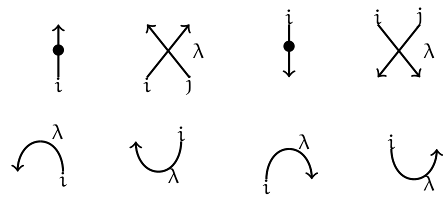
Each Kac-Moody algebra comes from the data of a Cartan datum and a root datum associated. Actually, the data of a Cartan datum is equivalent to the data of a given oriented graph, and we will consider this approach here. Let Γ be a graph whose set of vertices is denoted I, and $\mathbb{K}$ any field. We set $\mathcal{V} = \sum_{i \in I} \nu_i . i \in \mathbb{N}[I]$ an element of the free semi-group generated by I. We set $m := |\mathcal{V}| = \sum \mathcal{V}_i$.

For such an element $\mathcal{V}$, we define the set $\mathrm{SSeq}(\mathcal{V})$ as the set of sequences of vertices of Γ in which the vertex i appears exactly $\mathcal{V}_i$ times, and endowed with signs in $\{-,+\}$. For instance,

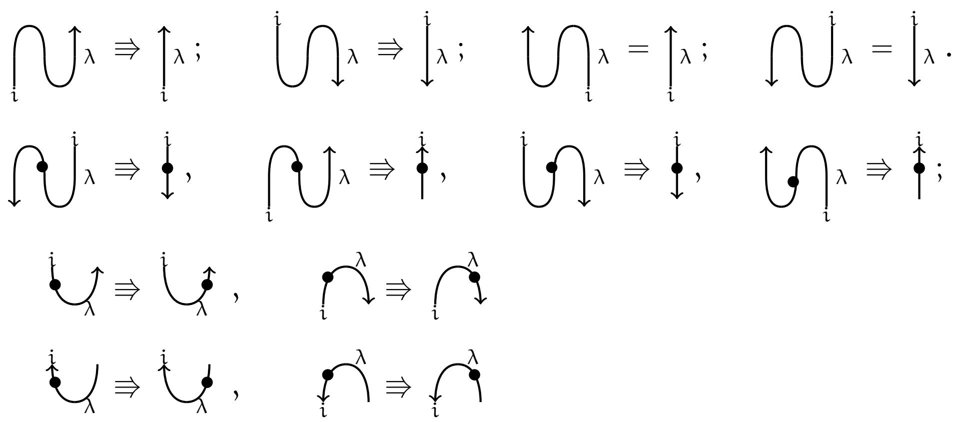
$$\mathrm{SSeq}(i+2j) = \{(+i,-j,-j),(+i,-j,+j),(+i,+j,-j),(+i,+j,+j),(-i,-j,-j),(-i,-j,+j),\dots\}$$

A.1 Definition. Let $\mathcal{KLR}$ be the linear $(3,2)$-polygraph defined by:

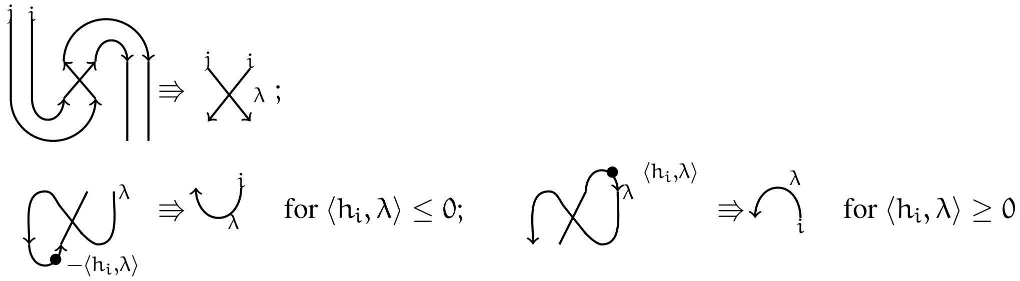
- $\mathcal{KLR}_0 = X$ the weight lattice of the Kac-Moody algebra;
- $\mathcal{KLR}_1 = \mathrm{SSeq} := \coprod_{\mathcal{V} \in \mathbb{N}[I]} \mathrm{SSeq}(\mathcal{V})$, that is signed sequences of vertices of Γ;
- $\mathcal{KLR}_2$ admits for generating 2-cells:



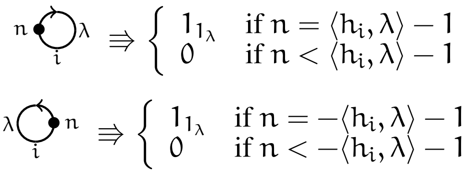
- $\mathcal{KLR}_3$ consists of the following 3-cells:
  1) The 3-cells of the given by the natural orientation of the relations of the KLR algebras in which we add an upward (resp. downward) orientation on each strand, see [10] for a precise description of these rules.
  2) The "isotopy" 3-cells: for any $i \in I$ and $\lambda \in X$



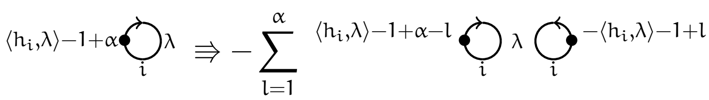
3) Some 3-cells coming from the definition of new generators: for any $i, j \in I$, $\lambda \in X$



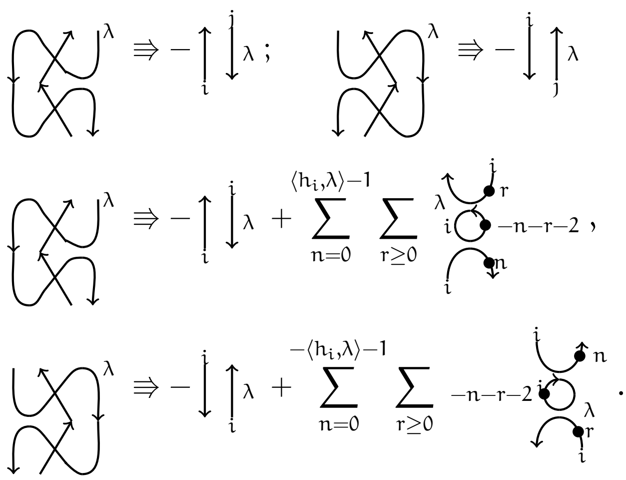
4) Some 3-cells for degree conditions on bubbles: for any $i \in I$, $\lambda \in X$



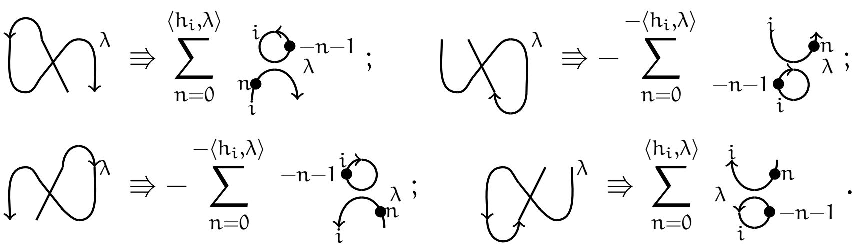
5) The "Infinite-Grassmannian" 3-cells: for any $i \in I$, $\lambda \in X$ and $\alpha > 0$,



6) Some invertibility 3-cells: for any $i, j \in I$ and $\lambda \in X$,



7) Some "$\mathfrak{sl}_2$" 3-cells: for any $i \in I$ and $\lambda \in X$,



The 2-cells in $\mathcal{KLR}$. Given two signed sequences $\mathcal{E}$ and $\mathcal{F}$ with respective lengths n and m, the 2-cells between $\mathcal{E}$ and $\mathcal{F}$ correspond to diagrams drawn in the strip of the plane $\mathbb{R} \times [0, 1]$. We place n points on the line $\mathbb{R} \times \{0\}$: $(1, 0), \ldots, (n, 0)$ and m points $\mathbb{R} \times \{1\}$: $(1, 1), \ldots, (m, 1)$. Each of this point is linked with another dot by an oriented strand using our generating 2-morphisms. The strands are oriented following the signs of the vertices, if a strand start from a vertex with a positive (resp. negative) sign, it will be upward (resp. downward) oriented.

Termination of $\mathcal{KLR}$. Let $\mathcal{E}$ and $\mathcal{F}$ be two elements of SSeq($\mathcal{V}$) with $\max(|\mathcal{E}|, |\mathcal{F}|) := m$. Then we define a function

$$\begin{array}{rccl} \Phi_{\mathcal{E},\mathcal{F}} := \Phi_m : & \mathcal{KLR}_2(\mathcal{E},\mathcal{F}) & \to & \mathbb{N}^{m+4} \times \mathbb{Z} \\ & D & \mapsto & (c(D), c_1(D), \ldots, c_m(D), ybg(D), I(D)l\text{-}dot(D), deg_b(D)) \end{array}$$

where:

- $c(D)$ is the number of crossings between strands in D;
- for $1 \leq k \leq m$, $c_k(D)$ is defined as above;
- $ybg(D)$ corresponds to the number of instances of the target of the Yang-Baxter 3-cell with an upward or downward orientation;
- $I(D)$ corresponds to the number of rightward caps and leftward cups that appear in D;
- l-dot(D) corresponds to the number of positively leftward dotted caps and cups as described above.
- $deg_b(D) := \begin{cases} \#\{\text{bubbles in D}\} + \sum\limits_{\pi \text{ clockwise oriented bubble in D}} deg(\pi) & \text{if D is a diagram with bubbles,} \\ 0 & \text{if D is a diagram without bubbles,} \\ -\infty & \text{if } D = 0. \end{cases}$

Then, for each pair of 1-cells $\mathcal{E}$ and $\mathcal{F}$ in $\mathcal{KLR}_1$, we define a partial order on $\mathcal{KLR}_2(\mathcal{E},\mathcal{F})$ in the following way:

$$D_1 \prec D_2 \Leftrightarrow |\mathcal{E}| = |\mathcal{F}| (:= m) \text{ and } \Phi_m(D_1) \leq_{lex} \Phi_m(D_2)$$

where $\leq_{lex}$ is the usual lexicographic order.

A.2 Lemma. *$\prec$ is a well-founded order.*

Proof. We first begin by stating that for any $m \in \mathbb{N}$, $0 \in \mathcal{KLR}_2(\mathcal{E},\mathcal{F})$ for any $\mathcal{E}, \mathcal{F} \in \mathcal{KLR}_1$ because of the structure of vector space for each space of 2-cells. Besides, $\Phi_m(0) = (\underbrace{0, \ldots, 0}_{m+4 \text{ terms}}, -\infty)$ is the lowest element in $\mathbb{N}^{m+4} \times \mathbb{Z}$.

Now, let's assume that there exist an infinite stricly decreasing sequence for $\prec$:

$$u_1 \prec u_2 \prec \ldots$$

Then, using the well-foundation of the lexicographic order on $\mathbb{N}$, there exists a rang k from which it is the last $\mathbb{Z}$-component that strictly decreases, that is to say

$$deg_b(u_k) < deg_b(u_{k+1}) < \ldots$$

Then necessarily, there exist a rank $l \geq k$ such that $deg_b(u_l) < 0$ and thus u_l has at least one clockise oriented bubble of degree < 0 in it. Then using the 3-cells for the degree conditions, this bubble reduces to 0, and so necessarily $u_{l+1} = 0$.

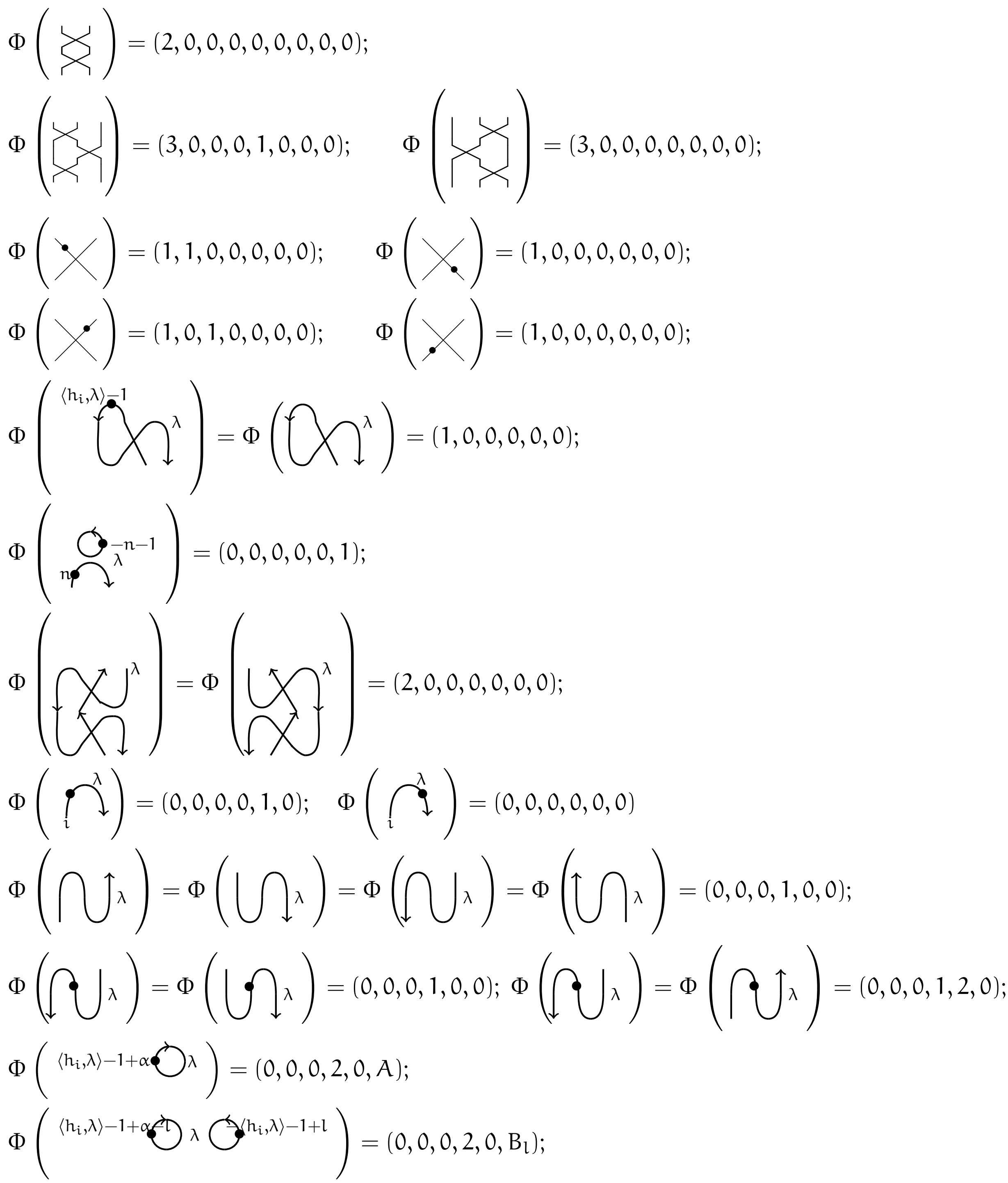
As $\Phi_m(0)$ is the lowest element for the lexicographic order on $\mathbb{N}^{m+4} \times \mathbb{Z}$, there can't exist u_{l+2} such that $u_{l+1} = 0 \prec u_{l+2}$. So $\prec$ is well-founded. ☐

A.3 Proposition. *The aforegiven order $\prec$ is a monomial order on the underlying 2-polygraph of $\mathcal{KLR}$ and satisfies the following condition:*

$$s(\alpha) \prec g \textit{ for any } g \in \textit{Supp}(t(\alpha))$$

Proof. $\prec$ is well founded according to A.2. The monomiality is clear because if we have $D_1 \prec D_2$, then plugging a context in D_1 and D_2 whenever it makes sense, we add a constant number of crossings, a constant number of instances of the right Yang-Baxter 3-cell, a constant number of (dotted or not) caps and cups, the same clockise oriented bubbles and the same number of new crossing below each dot.
It remains to check the condition on the 3-cells. All our 3-cells are between 2-cells with the same length 2 or 3, and one can notice that Φ does not depend on the orientation of the through strands. We have the following equalities for any labelling of the strands:

$$\Phi(0) = (0,0,0,0,0,0,0,0,-\infty); \qquad \Phi(1_{1_\lambda}) = (0,0,0,0,0,0,0,0);$$



where $A = 1 + \alpha i \cdot i$ and $B_l = 2 + (\alpha - l) i \cdot i$, so that $A > B$ because $l \geq 1$ and $i \cdot i = 2$. Besides, as Φ is clearly zero for identities or identities with dots, the required inequalities are satisfied. $\square$