FIRST-ORDER HOMOTOPICAL LOGIC AND GROTHENDIECK FIBRATIONS

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1. Introduction

The purpose of this work is to investigate a "minimal" version of the homotopy-theoretic semantics familiar from homotopy type theory. Specifically, the homotopy-theoretic semantics have hitherto only been considered in connection with dependent type theory. However, the semantics already make sense and are interesting in the much simpler context of first-order logic. While some of the most interesting features of homotopy type theory (higher equality types, propositional truncation, universes) are absent in this context, what is perhaps the central feature ("equality-as-paths") remains, and so it seems worthwhile to isolate this feature and give it a thorough investigation.

Generally speaking, the notion of "propositions-as-types" which lies at the heart of dependent type theory is, first of all, a semantics for first-order logic.

Namely, it tells us how to associate to each sentence P of first-order logic a certain *set* $\tau(P)$. The definition is by recursion, having the familiar clauses; for example $\tau(P \wedge Q)$ is defined to be the product $\tau(P) \times \tau(Q)$.

The "base case" of the recursion concerns the atomic formulas which, for simplicity, we'll take just to be equalities $s = t$. Assuming we've already interpreted the terms s and t as elements a and b of some set, we then take $\tau(s = t)$ to be a one element set in case a and b are the same element, and empty otherwise.

Having thus associated a set $\tau(P)$ (depending on an interpretation of our first-order language) to each sentence P, we can easily go on to verify that the set $\tau(P)$ is inhabited if and only if P is validated in the usual sense by our interpretation.

Similarly, the basic idea behind the homotopical semantics for type theory (as described, for example, in [KL12]) gives us, first of all, a semantics for first-order logic. It is these semantics which we propose to study.

2. Locally bicartesian closed categories

The semantics in question associates, not a set, but a *space* (actually, a Kan simplicial set) to each sentence. The most natural way to carry out this definition is to first phrase the set-theoretic semantics in a "purely categorical" way – that is, replacing each instance of the word "set" with "object of the category **Set**" and only using such constructions as would make sense in an arbitrary (sufficiently nice) category **C**.

Here, "sufficiently nice" turns out to mean "locally bicartesian closed" – that is, each slice category $\mathbf{C}/X$ of **C** should have finite products and coproducts, as well as exponential objects.

The category **sSet** of simplicial sets, being a category of set-valued functors, is locally bicartesian closed. Hence, given any structure for a first-order language L (having, for simplicity, only function symbols and no relation symbols) in **sSet** – this being understood in the usual sense of categorical logic (see e.g., [MR77]) – we obtain, for each sentence of L, a simplicial set.

However, there is a twist. Whereas the general definition of semantics in a locally bicartesian closed category would have us interpret the atomic formula $u = v$ (u and v being variables) as a diagonal morphism $X \to X \times X$, we want – and this it the main novelty of the homotopical semantics – to interpret it as a path space fibration $X^I \to X \times X$. Making this one modification then gives us the desired interpretation of sentences as simplicial sets.

3. Grothendieck fibrations

Following the typical pattern in categorical logic (as initiated in [Law63]), we expect to be able to organize the language itself into the same sort of structure as the one giving rise to the semantics, so that the models are just morphisms from the language-structure to the semantics-structure.

Here, the relevant kind of structure is that of a Grothendieck fibration, which is defined to be a functor $p : \mathbf{E} \to \mathbf{B}$ satisfying certain properties that we will not recall here. Grothendieck fibrations are discussed extensively in the context of categorical logic in [Jac99], and are studied in the present context of semantics for first-order logic in [Mak93]. They were first introduced into categorical logic (though in a slightly different form) in [Law06].

In the fibration corresponding to a first-order language L, the morphisms of the "base-category" $\mathbf{B}$ correspond to the terms of L, and the objects of the "total category" $\mathbf{E}$ correspond to the formulas of L.

The fibration corresponding to the set-theoretic "propositions-as-types" semantics – or, in general, semantics in any locally bicartesian closed category $\mathbf{C}$ – is the codomain functor $\mathrm{cod} : \mathbf{C}^{\to} \to \mathbf{C}$ associating to each morphism of $\mathbf{C}$ its codomain.

In the case at hand, $\mathbf{C} = \mathbf{sSet}$, this isn't quite what we want – as described above, the fibration $\mathbf{sSet}^{\to} \to \mathbf{sSet}$ gives the "wrong notion of equality". However, somewhat surprisingly, this can easily be repaired; a slight modification of the above fibration – obtained by taking homotopy-classes of maps in the total category – produces a fibration in which the equality is given by the path-space, as desired.

4. Homotopy invariance

Having defined the semantics, we would now like to say something about it. The first thing that comes to mind is the following.

Given any two (usual, set-theoretic) structures for a language L, we have the easily-verified fact that, if the two structures are *isomorphic* – in the sense that there are bijections between the underlying sets of the structures, commuting with the interpretations of the function symbols – then any sentence of L which holds in one structure also holds in the other.

If we consider the set-theoretic "proof-as-types" semantics, this tells us that for the interpretations $\tau(P)$ and $\tau'(P)$ of a sentence P in the two structures, $\tau(P)$ is inhabited if and only if $\tau'(P)$ is – but of course, we also have the stronger property that $\tau(P)$ and $\tau'(P)$ are in fact isomorphic.

In the homotopy-theoretic semantics, we expect the operative notion to be not *isomorphism* but *homotopy equivalence* – which is the same as isomorphism but with "bijection" replaced with "homotopy equivalence" and "commuting" replaced with "commuting up to homotopy".

Thus, we expect that for two homotopy equivalent structures, the two interpretations of a given sentence are homotopy equivalent Kan complexes. We show that this is indeed the case, and that it follows from a corresponding general fact about Grothendieck fibrations.

References

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