Towards a geometric model theory of type theory

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Models of univalent type theory have been sought in combinatorial models for higher homotopy types. This was first done using simplicial sets [LK16] and subsequently for cubical sets [BCH14] [Coh+18], which enjoys several advantages. In fact, a variety of different notions of cubical sets has been proposed specific to the interpretation of univalent type theory. Much of this industry has been inspired by an interpretation of cubical sets as 'nominal sets' with two distinct restriction operations, corresponding to the projections to 0 and 1 [Pit15]. This interpretation extends to other variants of cubical sets. It was an observation of the author that the topos of simplicial sets also admits such a representation, on the condition that names are given by an infinite interval.

Concretely, these toposes admit a representation as a topos of 'finitely supported M-sets', for M a 'monoid of substitutions' [Pit15]. In the simplest case of cubical sets [BCH14], such a monoid can be given by the $\{0, 1, \neq\}$-preserving endomorphisms $\mathbb{D} \cup \{0, 1\} \to \mathbb{D} \cup \{0, 1\}$, where $\mathbb{D} = \{x, y, z, \dots, \}$ is a set of *names*. For simplicial sets, an example is the monoid of $\{\leq, 0, 1\}$-preserving endomorphisms $\mathbb{Q} \cap [0, 1] \to \mathbb{Q} \cap [0, 1]$.

The present work, which is the content of the author's PhD thesis, consists in developing a theory of such toposes with the aim of structuring the range of examples from simplicial sets to the variety of cubical sets and beyond using geometric logic. This is based on an interpretation of M as the monoid of endomorphisms of a 'saturated' model of the underlying geometric theory. This notion of saturation must be formulated for *geometric* logic and soon abandons the province of classical model theory. A prominent feature is a new version of the Fraïssé-Hrushovski construction that enables the construction of models that are sufficiently saturated. This theory is then applied to study the known models of type theory. With the tentative notion of 'mould inclusion' the author manages to generalize open horns and boxes to arbitrary toposes of finitely supported M-sets. This is the starting point of work in progress that sets out to stratify models of (univalent) type theory within geometric model theory.

An advantage of using these saturated models is externalization of the internal logic. For a geometric formula $\varphi(\vec{a})$, an element x of a finitely supported M-set X supported by the tuple $\vec{a}$ of elements of the underlying model can be restricted to an element $x(\varphi(\vec{a}))$ as long as $\varphi(\vec{a})$ is consistent, for then there exists an endomorphism in M 'forcing' $\varphi(\vec{a})$. This notation has already been used naively, for instance in [BCH14], in the source/target maps $x(a_i = 0), x(a_i = 1)$.

A canonical site for a topos of finitely supported M-sets is the site of *orbits*. For a finite tuple of elements $\vec{a}$ in the model, its orbit is the finitely supported M-set:

$$O(\vec{a}) = \{f(\vec{a}) \,|\, f : M\} \tag{1}$$

Between orbits one can distinguish *face* and *degeneracy* maps. A degeneracy map is a projection $O(x, \vec{y}) \to O(\vec{y})$, and a face map is a section thereof. The coverage is generated by degeneracy maps. Since morphisms of orbits correspond to endomorphisms forcing a certain formula to hold, a face map $O(\vec{y}) \to O(x, \vec{y})$ corresponds to a formula $\delta(x, \vec{y})$ defining x uniquely in terms of $\vec{y}$. An example of this is $x = 0, 1$ in cubical sets and $x = y_i, y_{i+1}$

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in simplicial sets, where $y_i < x < y_{i+1}$. One obtains a convenient calculus for face and degeneracy maps in this way.

In the presence of enough face maps, the topos collapses to a presheaf topos. This is the case for simplicial sets and cubical sets. In this situation a *mould inclusion* can be defined as a subobject $\iota(\vec{y}) \vee \delta(x, \vec{y}) \rightarrowtail O(x, \vec{y})$ where δ is a face map and ι defines a subobject of $O(\vec{y})$. The intuition is that δ is a flat face of shape $O(\vec{y})$, and $\iota(\vec{y})$ describes the outline of an upstanding wall extending in the direction of x. In the case that $\iota(\vec{y})$ describes the entire border of $O(\vec{y})$, one obtains the familiar horns for simplicial sets, and boxes for cubical sets.

The connection to homotopy type theory is made by taking mould inclusions as a class of generating trivial cofibrations. These are similar in spirit to those in [GS17], however we emphasize their new logical interpretation. Moreover, they have representable codomain, although for simplicial sets it appears that one must abandon this for a more general type of mould inclusion if one wants to work constructively (this is based on recent joint work with Benno van den Berg). This new setting nevertheless provides a way of understanding the differences between cubical and simplicial models when it comes to dependent products, universes, and is suggestive of a host of new examples.

References

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