Cubical Assemblies and the Independence of the Propositional Resizing Axiom

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We construct a model of cubical type theory with a univalent and impredicative universe in a category of cubical assemblies. We show that the cubical assembly model does not satisfy the propositional resizing axiom.

We say a universe $\mathcal{U}$ in dependent type theory is *impredicative* if it is closed under arbitrary dependent products: for an arbitrary type A and a function $B : A \to \mathcal{U}$, the dependent product type $\prod_{x:A} B(x)$ belongs to $\mathcal{U}$. An interesting use of such an impredicative universe in homotopy/cubical type theory is the *impredicative encoding* of a higher inductive type [Shu11] as well as ordinary inductive types. For example, in cubical type theory [CCHM18] with an impredicative universe, the unit circle is encoded as

$$S^1 := \prod_{X:\mathcal{U}} \prod_{x:X} \mathsf{Path}(X, x, x) \to X : \mathcal{U}$$

together with a base point $b := \lambda Xxp.x : S^1$, a loop $l := \langle i \rangle.\lambda Xxp.pi : \mathsf{Path}(S^1, b, b)$ and a recursor $r := \lambda Xxps.sXxp : \prod_{X:\mathcal{U}} \prod_{x:X} \mathsf{Path}(X, x, x) \to (S^1 \to X)$. Although the impredicative encoding of a higher inductive type does not satisfy the induction principle in general, some truncated higher inductive types have refinements of the encodings satisfying the induction principle [AFS18].

The first goal of this talk is to present a model of cubical type theory with a univalent and impredicative universe. Since an impredicative universe is modeled in the category of assemblies [LM91] where the impredicative universe classifies *modest families*, our strategy is to construct a model of type theory in the category of cubical objects in assemblies which we will call *cubical assemblies*. There has been a nice set of axioms given by Orton and Pitts [OP16] for modelling cubical type theory without universes of fibrant types in an elementary topos equipped with an interval object $\mathbb{I}$. We will almost entirely follow them, but the category of cubical assemblies is not an elementary topos. So our contribution is to show that the construction given by Orton and Pitts works in a non-topos setting. For constructing the universe of fibrant types, we can use the right adjoint to the exponential functor $(-)^{\mathbb{I}}$ in the same way as Licata, Orton, Pitts and Spitters [LOPS18].

Voevodsky [Voe12] has proposed the *propositional resizing axiom* [Uni13, Section 3.5] which asserts that every homotopy proposition is equivalent to some homotopy proposition in the smallest universe. The propositional resizing axiom can be seen as a form of impredicativity for homotopy propositions. Since the universe in the cubical assembly model is impredicative, one might expect that the cubical assembly model satisfies the propositional resizing axiom. Indeed, for a homotopy proposition A, there is a natural candidate A^* for propositional resizing defined as $A^* := \prod_{X:\mathsf{hProp}}(A \to X) \to X$ together with a function $\eta_A := \lambda aXh.ha : A \to A^*$, where hProp is the universe of homotopy propositions in $\mathcal{U}$. However, the propositional resizing axiom fails in the cubical assembly model. We construct a concrete counterexample to propositional resizing. Note that a homotopy proposition A admits propositional resizing if and only if the function $\eta_A : A \to A^*$ is an equivalence, so it suffices to give a homotopy proposition $\Gamma \vdash A$ such that A does not have an inhabitant but A^* does.

The key fact is that in the category of assemblies, well-supported *uniform* objects are left orthogonal to modest objects [Oos08]. In other words, for any well-supported uniform object A and modest object X, the function $\lambda xa.x : X \to (A \to X)$ is an isomorphism. Uniform objects and modest objects in an internal presheaf category are defined pointwise, and well-supported uniform presheaves are left orthogonal to modest presheaves. Here a presheaf is well-supported if the unique morphism into the terminal presheaf is regular epi, which does not imply the existence of a section. Hence, in order to give a counterexample to propositional resizing, it suffices to find a homotopy proposition $\Gamma \vdash A$ in the cubical assembly model, that is, a presheaf over the category of elements $\int \Gamma$ equipped with the structures of homotopy proposition, that is moreover well-supported and uniform but does not have a section.

For any family A of assemblies over Γ, the codiscrete cubical assembly $\Delta\Gamma \vdash \nabla A$ over the discrete cubical assembly $\Delta\Gamma$ is always a homotopy proposition, and this construction preserves well-supportedness and uniformity. If ∇A has a section, then so does A. Thus, if A is a well-supported and uniform family of assemblies that does not have a section, then the homotopy proposition $\Delta\Gamma \vdash \nabla A$ is a counterexample to propositional resizing. Finally we construct such a family A of assemblies by hand.

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