Algebraic models of dependent type theory

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There are many approaches to the categorical semantics of dependent type theory, with some notions being very closely tied to the syntax of type theory (e.g. Cartmell's *contextual categories* [3] and Voevodsky's *C-systems* [7]) and some more closely related to homotopy theory (e.g. Joyal's *tribes* [6]).

This talk is an overview of my PhD thesis, advised by Steve Awodey, which explores the notion of a *natural model* of dependent type theory.

A natural model is an essentially algebraic object consisting of a map of presheaves $p : \dot{\mathcal{U}} \to \mathcal{U}$ over a small category $\mathbb{C}$ together with data witnessing *representability* of p. Thinking of $\mathbb{C}$ as the category of contexts and substitutions of dependent type theory, the map p can be thought of as a context-indexed function sending terms-in-context to their unique type-in-context; context extension is then modelled by representability of p ([1, §1], [4, Appendix]).

Type constructors and polynomial monads. The first contribution of the thesis is to relate the type constructors of dependent type theory with the algebraic notions of *monads* and their *algebras* via the machinery of polynomials and polynomial functors [5].

Necessary and sufficient conditions under which a natural model $(\mathbb{C}, p)$ admits a unit type $\mathbf{1}$, dependent sum types $\sum_{x:A} B(x)$ and dependent product types $\prod_{x:A} B(x)$, can be succinctly expressed in terms of the existence of certain pullback squares in the category $\mathcal{E} = [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ of presheaves over $\mathbb{C}$ [1, §2]. By considering maps of presheaves as *polynomials* in $\mathcal{E}$, we can express these pullback squares as *cartesian morphisms* of polynomials as follows:

natural model supports...	unit type	dependent sums	dependent products
$\Leftrightarrow$ *there exist cartesian...*	$\eta : 1 \Rightarrow p$	$\mu : p \cdot p \Rightarrow p$	$\alpha : p(p) \Rightarrow p$

It is natural to ask whether η and μ give rise to a *monad* structure on p, and whether α gives p the structure of a *p-algebra*. Unfortunately, the answer is 'not quite', since this would require equations like $\mathbf{1} \times A = A$ to hold strictly, but they do not. Instead, we obtain the weaker notions of a *pseudomonad* and a *pseudoalgebra*.

Specifically, the bicategory $\mathbf{Poly}_{\mathcal{E}}^{\mathrm{cart}}$ can be equipped with the structure of a *tricategory* $\mathfrak{Poly}_{\mathcal{E}}^{\mathrm{cart}}$ such that a natural model $(\mathbb{C}, p)$ admits a unit type and dependent sum types if and only if p is a pseudomonad in $\mathfrak{Poly}_{\mathcal{E}}^{\mathrm{cart}}$, and $(\mathbb{C}, p)$ additionally admits dependent product types if and only if p is a p-psuedoalgebra. Proving this is the content of [2].

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Categories of natural models. Next, we provide a functorial description of morphisms of natural models, assembling natural models into a category **NM**. We prove that this is equivalent to the category $\mathbf{Mod}(\mathbb{T})$ of models of an essentially algebraic theory. It is known (e.g. [1]) that the latter coincides with the category **CwF** of categories with families, and hence:

(i) The category **NM** of natural models is equivalent to the categories of categories with families; of categories with attributes; and of discrete comprehension categories.

(ii) Contextual categories are precisely those natural models whose contexts are generated by a finite set of basic types.

Initial natural models. The *initiality conjecture* asserts that the syntax of dependent type theory itself has the structure of a model, which is the initial object of the category of all models.

We prove partial results in this direction in the setting of natural models: we construct the *free natural model* $(\mathbb{C}_\sigma, p_\sigma)$ on some signatures σ for dependent type theory and, for these signatures, prove that interpretations of σ in a suitably structured natural model $(\mathbb{C}, p)$ correspond naturally with homomorphisms $(\mathbb{C}_\sigma, p_\sigma) \to (\mathbb{C}, p)$. We also describe how to freely equip an arbitrary natural model with additional type theoretic structure.

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[1] Steve Awodey. Natural models of homotopy type theory. *Mathematical Structures in Computer Science*, 2016.

[2] Steve Awodey and Clive Newstead. Polynomial pseudomonads and dependent type theory. `arXiv:1802.00997`, 2018.

[3] John Cartmell. Generalised algebraic theories and contextual categories. *Annals of Pure and Applied Logic*, 32:209–243, 1986.

[4] Marcelo Fiore. Discrete generalised polynomial functors. *Automata, Languages, and Programming*, pages 214–226, 2012.

[5] Nicola Gambino and Joachim Kock. Polynomial functors and polynomial monads. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 154, pages 153–192. Cambridge Univ Press, 2013.

[6] André Joyal. Notes on clans and tribes. `arXiv:1710.10238`, 2017.

[7] Vladimir Voevodsky. A C-system defined by a universe category. `arXiv:1409.7925`, 2014.