On sets of terms with a given intersection type

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Dedicated to Corrado Bohm, the nonno of intersection types.

We show

(1) For each strongly normalizable lambda term M, with beta-eta normal form N, there exists an intersection type A such that in BCD we have $\vdash M : A$ and N is the unique beta-eta normal term s.t. $\vdash N : A$. A similar result holds for finite sets of strongly normalizable terms

(2) For each intersection type A if the set of all closed terms M such that in BCD $\vdash M : A$ is infinite then when closed under beta-eta conversion this set forms an adequate numeral system for untyped lambda calculus. In particular, all these terms are generated from a single 0 by the application of a successor S,

$$S(\ldots(S0)\ldots)$$

and by beta-eta conversion.

1 Introduction

Here we are interested in how much of the structure of a strongly normalizable lambda term is captured by its intersection types and how much all the terms of a given type have in common.

In this note we consider the theory BCD (Barendregt,Coppo, and Dezani) of intersection types without the element U_{top} ([1] pps 582-583) and the notion of an adequate numeral system for the untyped lambda calculus ([3] 6.4 pps 135-137).

2 Formal Theory of Type Assignment

We define the notion of an expression as follows. $a, b, c, \ldots$ are atomic expressions. If A, and B are expressions then so are $(A \to B)$ and $(A \wedge B)$. Even though we write infix notation we say that these expressions begin with $\to$ and $\wedge$ resp. A basis F is a map from a finite set of variables, $\text{dom}(F)$, to the set of types. Below we shall often conflate F with the finite set

$$\{x : F(x) | x : \text{dom}(F)\}.$$

The formal theory of type assignments BCD (Barendregt, Coppo, and Dezani) is defined by the following set of rules here presented sequentially. For basis F and terms X, Y

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$F, x : A \vdash x : A$					(axiom)
$F, x : A \vdash X : B$			$\Rightarrow$	$F \vdash \backslash xX : A \to B$	$(\to I)$
$F \vdash X : A \to B$	&	$F \vdash Y : A$	$\Rightarrow$	$F \vdash (XY) : B$	$(\to E)$
$F \vdash X : A$	&	$F \vdash X : B$	$\Rightarrow$	$F \vdash X : A \wedge B$	$(\wedge I)$
$F \vdash X : A$	&	$A \lfloor B$	$\Rightarrow$	$F \vdash X : B$	$(\lfloor)$

Here we note that the rule ([) is read "less that or equal to".

3 The Relational Theory of Types

The rule $(\lfloor)$ is governed by the free theory of a preorder;

$a \lfloor a$

$a \lfloor b \quad \& \quad b \lfloor c \Rightarrow a \lfloor c$

$a \wedge b \lfloor a$

$a \wedge b \lfloor b$

$c \lfloor a \quad \& \quad c \lfloor b \Rightarrow c \lfloor a \wedge b,$

and a contravariant-covariant operation $\to$,

$c \lfloor a \quad \& \; b \lfloor d \Rightarrow a \to b \lfloor c \to d$

satisfying the weak distributive law

$(c \to a) \wedge (c \to b) \lfloor c \to (a \wedge b).$

There is an equivalent equational theory.

4 The Algebraic Theory of Types

A semilattice with meet operation $\wedge$
$a \wedge (b \wedge c) \sim (a \wedge b) \wedge c$
$a \wedge b \sim b \wedge a$
$a \sim a \wedge a$
satisfying the distributive law
$c \to (a \wedge b) \sim (c \to a) \wedge (c \to b)$
and an absorption law
$a \to b \sim (a \to b) \wedge ((a \wedge c) \to b)$
where the quotient partial order can be recovered
$a \lfloor b \Leftrightarrow a \sim a \wedge b.$

5 Theory of Expressions and their Rewriting

With an equational presentation we can associate a set of rewrite rules. The one step rewrite of an expression A by the rule R to the expression B is denoted $A\ R\ B$. This is the replacement of exactly one occurrence of the left hand side of the rule as a subexpression of A, the redex, by the righthand side. Sets of rules can be combined by the regular operations + (union) and * (reflexive-transitive closure). We define rewrites

(asso.)	$A \wedge (B \wedge C)$	asso.	$(A \wedge B) \wedge C$
(asso.)	$(A \wedge B) \wedge C$	asso.	$A \wedge (B \wedge C)$
(comm.)	$A \wedge B$	comm.	$B \wedge A$
(idem.)	A	idem.	$A \wedge A$
(absp.)	$A \rightarrow B$	absp.	$(A \rightarrow B) \wedge ((A \wedge C) \rightarrow B)$
(dist.)	$A \rightarrow (B \wedge C)$	dist.	$(A \rightarrow B) \wedge (A \rightarrow C)$

and we set semi. = assoc. + comm., and slat. = semi. + idem. Let redo. = slat. + absp. + dist.. redo. generates the congruence on expressions induced by the algebraic theory.

We recall the following properties of the rewrite theory from ([5]).

(1) idem. can be restricted to atoms.

(2) comm. can be restricted to atoms and expressions beginning with $\rightarrow$.

(3) If A slat.* B then there exists C such that
$$A \text{ idem.} * C \text{ semi.} * B.$$

(4) Every dist. reduction terminates.

(5) dist. has the weak diamond property.

(6) redo. has the Church-Rosser property.

For each type expression A, the unique dist. normal form of A is denoted dnf(A).Each type expression A in dnf can be written

$$A_1 \wedge \ldots \wedge A_k$$

associatively with each $A_i =$

$$A_{i,1} \rightarrow (\ldots(A_{i,t(i)} \rightarrow a_i)\ldots).$$

Here a_i is the principal atom of A_i. The A_i are the components of A.

6 Formal Theories of Type Expressions

The formal theory of [simply mirrors the relational theory.

(1) Traditional:

Axioms;

$A \,[\, A$

$A \wedge B \,[\, A$

$A \wedge B \,[\, B$

$(A \rightarrow B) \wedge (A \rightarrow C) [A \rightarrow (B \wedge C)$

Rules;

$C \,[\, A \;\&\; C \,[B \;\Rightarrow C \,[\, A \wedge B$
$C \,[\, B \;\&\; B \,[A \;\Rightarrow C \,[\, A$
$A \,[\, C \;\&\; D \,[B \;\Rightarrow C \to D \,[\, A \to B$

(2) Equational:

We add the axioms for the algebraic theory of $\sim$ and the usual rules for $\sim$ being a congruence. Since in this theory $A \,[\, B$ is defined by $A \sim A \wedge B$, the BCD rule,

$F \vdash X : A \;\&\; A \,[\, B \quad \Rightarrow F \vdash X : B$
is replaced by
$F \vdash X : A \wedge B \quad \Rightarrow F \vdash X : A \qquad (\wedge E)$
and
$F \vdash X : A \wedge B \quad \Rightarrow F \vdash X : B \qquad (\wedge E)$

in the formal theory of type assignment. In addition, there is a useful proof theoretic variant.

1. Munich Version

The notions of positive, negative, and strictly positive are defined recursively by

A is positive and strictly positive in A.

If C is positive in B then C is positive in $A \to B$ and negative in $B \to A$.

If C is strictly positive in B then C is strictly positive in $A \to B$.

If C is positive in A or B then C is positive in $A \wedge B$.

If C is strictly positive in A or B then C is strictly positive in $A \wedge B$.

If C is negative in B then C is negative in $A \to B$ and positive in $B \to A$.

If C is negative in A or B then C is negative in $A \wedge B$.

A single occurrence of B as a subexpression of A will be indicated $A(B)$. An expression can be thought of as a rooted oriented binary tree with atoms at its leaves and either $\to$ or $\wedge$ at each internal vertex. For each subexpression B of A there is a unique path from the root of A to the root of B. If we remove this occurrence of B we have a context $A(.)$ where we could just as easily have thought of this as the replacement of B by a new atom p. The rules of the Munich version are

Axioms;

$A \,[\, B$	if A, B are slat. congruent
$A(D(B \wedge C)) \,[\, A(D(B) \wedge (D(C)))$	if $D(.)$ strictly positive
$A(B \wedge C) \,[\, A(B)$	if $A(.)$ is positive
$A(B) \,[\, A(B \wedge C)$	if $A(.)$ is negative

Rules;

$A \,[\, B \quad \& \quad B \,[\, C \Rightarrow A [C$

Lemma 6.1. *(Munich)*
If by the Traditional rules $A \,[\, B$ then by the Munich rules $A \,[\, B$

Proof. We verify that the Munich rules are closed under the Traditional rules by simulating the Traditional proofs by Munich proofs. First observe that if in Munich $A \,[\, B$ and $D(.)$ is positive then $D(A) \,[\, D(B)$ and if $D(.)$ is negative then $D(B) \,[\, D(A)$. Next observe that if we have $C \,[\, A$ and $C \,[\, B$ in Munich, then $C \wedge C \,[\, A \wedge B$, so by idem. $C \,[\, A \wedge B$. End of proof. □

This version is named in honor of Kurt Schutte.

7 Evaluating Types in the Tropical Semiring

The tropical semiring is the semiring of integers with 0,1,+,min, and max (and + and - infinity, but we will not need these; ([2]). With each type expression A we associate a tropical semiring polynomial $e(A)$. The definition is by recursion over subtype expressions of A and is not context free; it depends on the sign of the subtype expression. For each atom a pick a distinct variable x and set

$$\begin{array}{lcl} e(a) & = & x \\ e(C \to B) & = & 1 + e(C) + e(B) \\ e(C \wedge B) & = & \min(e(C), e(B)) \text{ if } C \wedge B \text{ is positive} \\ e(C \wedge B) & = & \max(e(C), e(B)) \text{ if } C \wedge B \text{ is negative.} \end{array}$$

The dual of e, denoted $\sim e$, is obtained from e by interchanging max and min.

Facts; (tropical)
For all natural number values of the variables x

(1) $e(A)$ is monotone.

(2) If A and B are slat. congruent then $e(A) = e(B)$.

(3) $e(A(D(B \wedge C))) = e(A(D(B) \wedge (D(C))))$ if $D(.)$ is strictly positive.

(4) $$\begin{array}{llll} e(A(B \wedge C)) & < \text{ or } = & e(A(B)) & \text{if } A(.) \text{ is positive} \\ e(A(B)) & < \text{ or } = & e(A(B \wedge C)) & \text{if } A(.) \text{ is negative} \end{array}$$

¿From these facts we conclude the

Lemma 7.1. *If A [B then $e(A) <$ or $= e(B)$*

Proof. By the Munich axioms and rules. End of proof. □

8 Proof Theory of BCD

Here we need to strengthen several of the derived rules for BCD stated in ([3]) chapter 14 to proof theoretic statements. For this it is convenient to write proofs in tree form. Indeed, we shall implicitly adopt the natural deduction form of the rules of BCD using the left hand side of sequences to indicate active assumptions. We denote proofs P, P', Q, Q' etc.

Lemma 8.1. *(dnf)*
Suppose that for each $x : \mathrm{dom}(F)$ $F(x)$ and A are in dnf, and $F \vdash X : A$ is provable. Then there is a proof where every type expression is in dnf.

Proof. Let P be such a proof. Our proof is by induction on P. We write $\mathrm{dnf}(F)$ for the basis F' such that $F'(x) = dnf(F(x))$.
Basis; $X = x$ and $P =$ the axiom $F \vdash x : A$. Then
$\mathrm{dnf}(F) \vdash x : dnf(A)$ is an axiom.

Induction step;

Case 1; P ends in the BCD rule [. Since B [C implies $\mathrm{dnf}(B)$ [$\mathrm{dnf}(C)$ this case is obvious.

Case 2: P ends in the BCD rule $\wedge I$. Since $\mathrm{dnf}(B \wedge V) = \mathrm{dnf}(B) \wedge \mathrm{dnf}(C)$ this case is obvious. Remark; the case $\wedge E$ is similar and could be included here.

Case 3; P ends in the BCD rule $\to E$. So we may suppose that
$A = B \to C, X = (UV)$, and P has the form

$$\frac{F \vdash \overset{P'}{U} : B \to C \quad F \vdash \overset{P''}{V} : B}{F \vdash (UV) : C}$$

Let $\mathrm{dnf}(C) = C_1 \wedge \ldots \wedge C_n$ where each C_i is dnf and does not begin with $\wedge$. By induction hypothesis there exist proofs Q', Q'' of

$$\mathrm{dnf}(F) \vdash U : (\mathrm{dnf}(B) \to C_1) \wedge \ldots \wedge (\mathrm{dnf}(B) \to C_n)$$

and $\mathrm{dnf}(F) \vdash V : \mathrm{dnf}(B)$ resp..Now for $i = 1, \ldots, n$, applications of $\wedge E$ to Q'' gives a proof of

$$\mathrm{dnf}(F) \vdash U : \mathrm{dnf}(B) \to C_i,$$

which when combined with Q'' by $\to E$ gives a proof Q_i of

$$\mathrm{dnf}(F) \vdash (UV) : C_i.$$

$n-1$ applications of $\wedge I$ to the Q_i gives the desired proof of

$$\mathrm{dnf}(F) | - (UV) : \mathrm{dnf}(C).$$

Case 4; P ends in the BCD rule $\to I$. Again we may suppose that $A = B \to C, X$ has the form uU and P has the form

$$P'$$

$$F, u : B \vdash v : C$$

$$F \vdash \backslash uV : B \to C$$

Now if $\mathrm{dnf}(C) = C_1 \wedge \ldots \wedge C_n$, where each C_i is dnf and does not begin with $\wedge$, then

$$\mathrm{dnf}(A) = \mathrm{dnf}(B) \to C_1 \wedge \ldots \wedge \mathrm{dnf}(B) \to C_n.$$

By induction hypothesis there exists a proof of

$$\mathrm{dnf}(F), u : \mathrm{dnf}(B) \vdash U : \mathrm{dnf}(C)$$

which after applications of $\wedge E$ yields a proof Q_i of

$$\mathrm{dnf}(F), u : \mathrm{dnf}(B) \vdash U : C_i.$$

Thus for $i = 1, \ldots,\ n$ we have proofs

$$Q_i$$

$$\mathrm{dnf}(F), u : \mathrm{dnf}(B) \vdash U : C_i$$

$$\mathrm{dnf}(F) \vdash \backslash uU : \mathrm{dnf}(B) \to C_i$$

These can be combined by $\wedge I$ for the desired result. End of proof. □

Lemma 8.2. *(predicate reduction)*

Suppose P is a proof of $F \vdash X : A$ where every type expression is in dnf. Then there exists a similar proof where every application of the BCD rule [is to a variable as the subject.

Proof. We first recall the criterion for [on dnfs verified in ([5]) section (19)

If

$A =$ $A_1 \wedge \ldots \wedge A_n$
where $A_i = A_{(i,1)} \to (\ldots (A_{(i,m(i))} \to a_i) \ldots)$
a_i is an atom, $m(i)$ may be 0,
and each $A_{(i,j)}$ is in distributive normal form
for $i = 1, \ldots, n$

$B =$ $B_1 \wedge \ldots \wedge B_k$
where $B_i = B_{(i,1)} \to (\ldots (B_{(i,l(i))} \to b_i) \ldots)$
b_i is an atom, $l(i)$ may be 0,
and each $B_{(i,j)}$ is in distributive normal form
for $i = 1, \ldots, k$

then

A [B iff for each $i = 1, \ldots, k$ there exists $j = 1, \ldots, n$
such that $b_i = a_j, l(i) = m(j)$ and
for $r = 1, \ldots, l(i)$ we have $B_{(i,r)}[A_{(j,r)}$.

Next we consider an application of the BCD rule ([) in P immediately following the application of a different rule, and we show how the ([) can be promoted (if you like, permuted).

Case 1; $P =$

$$\begin{array}{cc} P' & P'' \\ F \vdash U : C \to A & F \vdash V : C \\ \multicolumn{2}{c}{F \vdash (UV) : A} \\ \multicolumn{2}{c}{F \vdash (UV) : B.} \end{array}$$

Now for each B_i we have A [B_i so we have proofs

$P_i =$

$$\begin{array}{cc} P' & P'' \\ F \vdash U : C \to A & F \vdash V : C \\ F \vdash U : C \to B_i & F \vdash V : C \\ \multicolumn{2}{c}{F \vdash (UV) : B_i} \end{array}$$

which can be combined by $\wedge I$.

Case 2; $P =$

$$\begin{array}{c} P\prime \\ F, u : C \vdash U : D \end{array}$$

$$F \vdash \backslash uU : C \to D$$

$$F \vdash \backslash uU : B$$

Now for each B_i we have $B_{(i,1)}$ [C and

$$D \, [\, B_{(i,2)} \to (\ldots(B_{(i,l(i))} \to b_i)\ldots).$$

Replacing axioms $G, u : C \vdash u : C$ in P' by

$$G, u : B(i,1) \vdash u : B(i,1)$$

$$G, u : B(i,1)| - u : C$$

gives new proofs

$$P_{ii}$$

$$F, u : B_{(i,1)} \vdash U : D$$

$$F, u : B_{(i,1)} \vdash U : B_{(i,2)} \to (\ldots(B_{(i,l(i))} \to b_i)\ldots)$$

$$F \vdash \backslash uU : B_i$$

which can be combined by $\wedge I$.

Case 3; $P =$

$$F \overset{P'}{\vdash} X : C \quad F \overset{P''}{\vdash} X : D$$

$$F \vdash X : C \wedge D$$

$$F \vdash X : B$$

We may suppose $C = A_1 \wedge \ldots \wedge A_r$ and $D = A_{r+1} \wedge \ldots \wedge A_n$. By the criterion for [of dnfs, for each $i = 1, \ldots, k$ there exists $0 < f(i) < n+1$ such that $A_{f(i)}$ [B_i. So for each such i where $f(i) < r+1$ we have the proof $P_i =$

$$F \overset{P'}{\vdash} X : C$$

$$F \vdash X : B_i$$

and for each i such that $f(i) > r$ we have the proof $P_i =$

$$F \overset{P''}{\vdash} X : D$$

$$F \vdash X : B_i$$

and these can all be combined with $\wedge I$. End of proof.

□

A sequence of inferences

$$
\begin{array}{ll}
F \vdash x : A & \\
F \vdash x : A_1 \to (\ldots(A_k \to (B \to C)\ldots) \quad (\wedge E) & \\
F \vdash x : A_1 \to (\ldots(A_k \to (D \to E)\ldots) \quad ([) & F \vdash X_1 : A_1 \\
\quad F \vdash xX_1 : A_2 \to (\ldots(A_k \to (D \to E)\ldots) & \\
\quad\quad \vdots & \\
\quad F \vdash xX_1X_2\ldots X_k : (D \to E) \quad F \vdash X : D & \\
\quad\quad F \vdash xX_1X_2\ldots X_kX : E &
\end{array}
$$

is said to be "intemperate" and can be replaced by

$$
\begin{array}{ll}
F \vdash x : A & \\
F \vdash x : A_1 \to (\ldots(A_k \to (B \to C)\ldots) \quad (\wedge E) & \\
F \vdash x : A_1 \to (\ldots(A_k \to (B \to E)\ldots) \quad ([) & F \vdash X_1 : A_1 \\
\quad F \vdash xX_1 : A_2 \to (\ldots(A_k \to (B \to E)\ldots) & \\
\quad\quad \vdots & F \vdash X : D \\
F \vdash xX_1X_2\ldots X_k : (B \to E) & F \vdash X : B \quad ([) \\
\quad\quad F \vdash xX_1X_2\ldots X_kX : E &
\end{array}
$$

Theorem 8.3. *Suppose that for each x:dom(F) $F(x)$ and A are in dnf, and $F \vdash X : A$ is provable. Then there is a proof of $F \vdash X : A$ such that*

(1) every type expression is in dnf,

(2) every application of the BCD rule [is to a variable as the subject, and

(3) there are no intemperate sequences

Proof. we already know that proofs satisfying (1) and (2) exist for X. The proof is by induction on the length of X with a subsidiary induction on length of a proof P and (1) and (2). We suppose that $X =$

(a) $\backslash x_1 \ldots x_k.\ xX_1 \ldots X_l$ (head normal form), or

(b) $\backslash x_1 \ldots x_k.\ (\backslash xX_0)X_1 \ldots X_l$ (head redex)
and $A = A_1 \wedge \ldots \wedge A_m$, showing all components, and we distinguish several cases.

Case 1; P ends in $\wedge I$. By subsidiary induction hypothesis. Otherwise P ends in the BCD rule [, in which case we are done, or $\to I$, or $\to E$, Thus we can assume that $m = 1$ and A begins with $\to$;$A = B_1 \to (\ldots(B_n \to b)\ldots)$.
Case 2; Let $G = x_1 : B_1, \ldots, x_k : B_k$.
In case (a) P has the form

$$
\begin{array}{lll}
G \vdash X : C & & P_1 \\
G \vdash x : C_1 \to (\ldots(C_p \to c)\ldots) & G \vdash X_1 : C_1 & \\
& G \vdash xX_1 : C_2 \to (\ldots(C_p \to c)\ldots) & \\
& \vdots & P_1 \\
G \vdash xX_l \ldots X_{l-1} : C_1 \to (\ldots(C_p \to c)\ldots) & & G \vdash X_l : C_1 \\
& G \vdash xX_1 \ldots X_l : C_{1+1} \to (\ldots(C_p \to c)\ldots) & \\
G - \{x_k : B_k\} \vdash \backslash x_k.xX_1 \ldots X_l : B_k \to (C_{1+1} \to (\ldots(C_p \to c)\ldots) & & \\
& \vdots & \\
& F \vdash X : A. &
\end{array}
$$

Thus, $p = n$ and for $i = l+1 \ldots n$, $C_i = B_i$. Now suppose that $C = D_i \wedge \ldots \wedge D_q$ show all components. By the criterion for [on dnfs verified in [5] section (19) there exists some D_i such that D_i [$C_1 \to (\ldots(C_p \to c)\ldots)$ and $D_i = D_1 \to (\ldots(D_p \to c)\ldots)$ with, for $j = 1, \ldots, p$, C_j [D_j. Thus we can alter the proofs above to

$$
\begin{array}{cc}
P_j & \\
G \vdash X_j : C_j & \\
G \vdash X_j : D_j & ([)
\end{array}
$$

and apply the major induction hypothesis to them while we replace the [inference
$G \vdash x : C$
$G \vdash x : C_1 \to (\ldots(C_p \to c)\ldots)$

appropriately.
The case (b) follows form the main induction hypothesis. End of Proof. □

Definition 8.4 A BCD proof satisfying conditions (1),(2), and (3) is said to be "almost minimal" (am)
Definition 8.5 We define the notion of a oscillation in the Bohm tree of a beta normal term X as follows. An oscillation is a sequence of pairs of nodes which descend in the tree such that the node

$$\backslash x_1 \ldots x_k.x$$

is paired with one of the arguments of this occurrence of x which has a non-empty lambda prefix, and the next pair (if it exists) has the head variable of its first coordinate bound by this lambda prefix.

$$
\begin{array}{l}
\backslash x_1 \ldots x_k.x \\
\qquad \ldots \backslash \ldots \\
\quad \backslash y_1 \ldots y_l.y \\
\qquad\qquad / \\
\qquad\quad \cdot^{\cdot^{\cdot}} \\
\qquad\qquad \backslash \\
\quad \backslash z_1 \ldots z_m.y_i \\
\qquad\qquad \ldots \backslash \ldots \\
\qquad \backslash u_1 \ldots u_n.u
\end{array}
$$

Definition 8.6 We say that the closed term $X =$

$$\backslash x_1 \ldots x_k.x_i X_1 \ldots X_l$$

in beta normal form is of class n if

(i) every lambda prefix in the Bohm tree of X has at most n lambdas
(ii) every node in the Bohm tree of X has at most n immediate descendants
(iii) every oscillation in the Bohm tree of X has at most length n.

Lemma 8.4. *(class)*
Let X be in beta normal form. If $F \vdash X : A$ is provable in BCD then X then any oscillation beginning with x :dom(F) has length less than or equal to $\sim e(F(x))$ and other oscillations have length less than $e(A)$

Proof. w.l.o.g we may assume that A and all $F(x)$ are in dnf. The proof is by induction on an am proof P of $F \vdash X : A$

Basis; P is an axiom. Obvious.
Induction step; we distinguish several cases.
Case 1; P ends in $\wedge I$. Then $A = B \wedge C$ and $e(A) = \min\{e(B), e(C)\}$. By induction hypothesis applied to the premise of minimum e.
Case 2; P ends in $\rightarrow I$. Then $A = B \rightarrow C, X = \backslash uU$ and $P =$

$$P'$$
$$F, u : B \vdash U : C$$
$$F \vdash \backslash uU : B \rightarrow C.$$

This case follows immediately.
Case 3; P ends in $\rightarrow E$ Now consider the leftmost path of P proceeding up P'; i.e. we take left premises of $\rightarrow E$'s as often as possible, and then possibly the premise of an ([) inference with a variable subject, and end at an axiom for a variable x. This is all that is possible since P is am. Let the axiom for x be

$$F', x : D \vdash x : D$$

If the axiom for x is immediately followed by an ([) rule inference

$$F', x : D \vdash x : D$$
$$F'x : D \vdash x : E$$

note that $\sim e(E) <$ or = $\sim e(D)$ by tropical fact (4). Now consider one of the $\rightarrow$ inferences on the leftmost path of P.

$$\frac{F'', x : D \vdash xX_1 \ldots X_i : E' \rightarrow E'' \qquad F', x : D \vdash X_{i+1} : E'}{F', x : D \vdash xX_1 \ldots X_{i+1} : E''}$$

By induction hypothesis any oscillation in X_{i+1} beginning with x' : dom(F) has length less than or equal to $\sim e(F(x'))$ and other oscillations have length less than $e(E')$ less than $\sim e(D)$ since E' is negative in E. Such an oscillation extends to one beginning with the with the head variable x.
Case 4; P ends in [. Similar to case 3. End of proof. □

Corollary 8.5. *For each type A there exists an integer n s.t. for any closed beta normal M such that $\vdash M : A$ in BCD M has class n.*

Proof. by induction on the length of an am proof of $F \vdash X : A$ using the class lemma. End of proof. □

Lemma 8.6. *(thinning)*
Suppose that P is an am proof of $F, x : B \vdash X : A$ where X is in beta normal form and the principal atoms of B do not occur in either A or any $F(y)$ for $y : \mathrm{dom}(F)$. Then x does not occur in P.

Proof. By induction on P. End of proof. □

Defintion 8.10 An intersection type in dnf is said to be functional if each atom occurs at most twice and if twice then with opposite sign, and there is no + occurrence of $\wedge$. The type is co-functional if each atom occurs at most twice and if twice then with opposite sign, and there is no - occurrence of $\wedge$.
Definition 8.11 If A is dnf, say $A = A_1 \wedge \ldots \wedge A_k$ with $A_i = A_{i,1} \to (\ldots (A_{i,t(i)} \to a_i) \ldots)$ the set of hereditary components of A, $hc(A)$, is defined recursively by

$$\mathrm{hc}(A) = \{A_1, \ldots, A_k\} U\ U\ \mathrm{hc}(A_{i,j}).$$

Defintion 8.12 If A is dnf and S is a subset of $hc(A)$ then $A - S$ is obtained by deleting the members of S from A and the corresponding arrows if entire lhs's are deleted.
Lemma; (eta normal form)

Suppose that we have a functional type
$A = A_1 \to (\ldots (A_n \to a) \ldots)$ where
$A_{k+1} = C_1 \to (\ldots (C_p \to (B_{k+2} \to (\ldots (B_n \to a) \ldots))) \ldots)$,
and for $i + k = 1, \ldots, n$
$A_i \ [\ B_i$
Let Y be in beta eta normal form. Then if $k + 1 < i$, and
$x_1 A_1, \ldots, x_{k+1} : A_{k+1}$,
$y_{k+2} : A_{k+2}, \ldots, y_n : A_n \vdash Y : B_i$
we have $y = y_i$.

Proof. by induction on the length of an am proof P. let

$$\begin{aligned} F &= x_1 : A_i, \ldots, x_{k+1} : A_{k+1}, \\ &\quad\ y_{k+2} : A_{k+2}, \ldots, y_n : A_n \end{aligned}$$

and suppose that $y = \backslash z_1 \ldots z_m.\ z Z_1 \ldots Z_i$. W.l.o.g and may assume P does to end in $\wedge$ and B_i does not begin with $\wedge$. Suppose that Y is not y_i. We then consider the left most path of P. Beginning at the top this path consists of 0 or 1 application [, followed by 1 application of $\to E$ followed by m applications of $\to I$. Now write $B_i =$

$$D_1 \to (\ldots (D_q \to b) \ldots).$$

Now since $A_i \ [\ B_i$, by functionality, A_i has a unique component

$$E_i \to (\ldots (E_q \to b) \ldots)$$

with principal type b. This accounts for the two possible occurrences of b so $z = y_1$, and we have for $i = 1,\dots,q$ E_i [D_i
Le $G =$

$$F,\ z_1 : E_1,\dots,z_q : E_q.$$

Then the axiom at the top of the leftmost path is

$$G - \{z_{m+1} : E_{m+1},\dots,z_q : E_q\} \vdash y_i : A_i$$

and for $j = 1,\dots,l,\ G \vdash Z_j :\ D_j$. Now consider the type $A' =$
$A'_1 \to (\dots(A'_{n+q} \to b)\dots)$ where
$A'_j = A_j$ for $j < k+1$ and $k+2 < j < n+1$
$A'_{k+1} = C_1 \to (\dots(C_p \to (B_{k+2} \to (\dots(B_{i+1} \to (E_1 \to (\dots(E_q \to (B_{i+1} \to (\dots(B_n \to a)\dots)))\dots)))\dots)$
$A'_{n+j} = D_j$ for $j = 1,\dots,q$.
This is functional, so by induction hyothesis $Z_j = z_j$. But $m = 1$. End of proof. □

Construction;
Suppose that X is in beta normal form with free variables
$x_1,\dots,x_k$. We construct co-functional types $A_1,\dots,A_k$, and a functional type A such that if F is the basis such that $F(x_i) = A_i$ then $F \vdash X : A$. We construct the types by recursion; it will be convenient not to identify different free occurrences of each variable x_i until x_i becomes bound, so that the A_i do not have strictly positive occurrences of $\wedge$ until binding. Thus, in F, x_i may have several types not beginning with $\wedge$. This is only a convenience.

1. if $x = x_i$ then $A = A_i =$ the atom a_i
2. if $X = \backslash x_{k+1}.Y$ then let A_{k+1} be the $\wedge$ of all the types assigned to the different occurrences of x_{k+1} in Y; then the type for X is $A_{k+1} \to A$
3. if $X = x_i X_1 \dots X_l$ then we have already types $B_1 \dots, B_l$ such that $F \vdash X_j : B_j$. We add the new occurrence $x_i : B_1 \to (\dots(B_l \to b)\dots)$, for b a new atom, to F and set the type for $x = b$.

Lemma 8.7. *(uniqueness)*
Suppose that $A = A_1 -> (...(A_k -> B)...)$ *is functional and there exists a beta-eta normal* X *such that* $x_1 : A_1,\dots,x_k : A_k \vdash X : B$
then X *is unique.*

Proof. ; set $F = x_1 : A_1,\dots,x_k : A_k$; the proof is by induction on the length of an am proof P of $F \vdash X : B$. Write X as $z_1 \dots z_m z Z_1 ... Z_n$. Then $B =$

$$B_1 \to (\dots(B_l \to b)\dots)$$

where $m < l+1$. There are two cases which are determined by A alone.
Case 1; z is X_i. Let $G = F, z_1 : B_1,\dots,z_m : B_m$. Then, since P is am, A_i has the component

$$C = C_1 \to (\dots(C_n \to (B_{m+1} \to (\dots(B_l \to b)\dots)))\dots)$$

where $B(i)$ [$B'(i)$
and the conclusion of the [on the leftmost path of P

(if it exists, otherwise $A_i = C$) is $G \vdash z : C$. Now the lemma applies, and if

$x_1 : A_1, \ldots, x_{k+1} : A_{k+1}, z_1 : B_1, \ldots, z_m : B_m$,
$z_{m+1} : B'_1, \ldots, z_1 : B'_1 \vdash z : B'_i$
then since B_i [B'_i for $i = m+1, \ldots, l$ we have
$Z_i = z_i$. Thus, since X is eta normal, this case cannot happen for two distinct values of m. We have

$$G \vdash Z_j : C_j, \text{ for } j = 1, \ldots, n,$$

by shorter am proofs than P, and

$A_1 \to (\ldots(A_k \to (A_{k+2} \to ((B_1 \to (\ldots(B_m \to C_j)\ldots))))\ldots)$
is functional so our induction hppothesis applies.
Case 2; z is a z_i. Similar. End of proof. □

Theorem 8.8. *If M is strongly normalizable then there exists a functional type A such that if N is the beta eta normal form of M then $\vdash M : A$ and N is the unique beta eta normal form such that $\vdash N : A$.*

Proof. by the construction of a functional type above and the uniqueness lemma, the theorem follows for M already beta normal. We must show that this extends to all strongly normalizable M. To this end we will consider a standard reduction from a strongly normalizable Y to its beta normal form X. This is sufficient by eta postponement and the construction above. For one step $Y \to Z \twoheadrightarrow X$ we will have as an induction hypothesis $F \vdash Z : A$ where for each x:dom(F), $F(x)$ is co functional, A is functional,and if range$(F) =$
$B_1, \ldots, B_l$ then

$$B_1 \to (\ldots(B_l \to A)\ldots)$$

is functional. We will show that there exists G, B such that $G \vdash Y : B$, where for each x:dom$(G), G(x)$ is co functional, B is functional and if range$(G) = C_1, \ldots, C_m$ then $C_1 \to (\ldots(C_m \to B)\ldots)$ is functional.Indeed F, G, A, B will be related in the following way

1. dom(F) is contained in dom(G).
   Let $F = x_1 : B_1, \ldots, x_l : B_l$, $G = x_1 : C_1, \ldots, x_m : C_m$
2. There exists a partition of the atoms $R\ U\ T$ such that

(i) none of the atoms in R occur in the B_i or A

(ii) if an atom in P occurs in an hc of C_i or B then all the atoms of that hc belong to R

(iii) If S is the set of all hc's with atoms from R then for each $i, B_i = C_i - S$ and $B - S = A$

□

Remark 8.15 it is "almost true" that C_i [B_i and A [B. We say "almost true" because of the possibility of $\to$ deletion. To make the inequalities C_i[B_i and A [B true we will at every stage of the induction assume that we start the basis case with dom(F) large enough to include all the free variables in Y. We proceed now by induction on the size of the reduction tree of Y with a subsidiary induction on the length of Y. We distinguish a number of cases.

Case 1; Y begins with lambda. Say, $Y = \backslash z.Z$. Then $\backslash z.Z \to \backslash z.Z'$ in the standard reduction strategy and the main induction hypothesis applies to $\backslash z.Z'$. Thus $G \vdash \backslash z.Z' : B$, $B = B_1 \to B_2$, and $G, z : B_1 \vdash Z' : B_2$. Now the subsidiary induction hypothesis applies to Z and this gives the case.
Case 2; Y begins with a head variable. Say, $Y = yY_1 \ldots Y_k$. Now the standard reduction contracts a redex in one of the Y_i with result Y_i', say $i = t$; otherwise set $Y_j' = Y_j$. The main induction hypothesis applies to $Y' = yY_1' \ldots Y_k'$ so there exists co-functional F and functional A, with all the desired properties, such that $F \vdash Y' : A$. Take an am proof P of $F \vdash Y' : A$. Now since F is co-functional F(y) has a component of the form

$$B_1 \to (\ldots B_k \to A) \ldots)$$

such for $i = 1, \ldots, k$.

$$F \vdash Y_i' : B_i,$$

By the subsidiary induction hypothesis there exists G, C with all the desired properties *w.r.t.* F and B_t such that $G \vdash Y_t : C$. In particular there is a partition of atoms RUT as above. In particular, all the atoms in $F(x)$ and A lie in T. Now replace all the atoms in T by new atoms, but for notational purposes we will continue to write the results as G and C. Now define H by
$H(x) = F(x) \wedge G(x)$ if x is not y
$H(y) = B_1 \to (\ldots B_{t-1} \to (c \to (B_{t+1} \to (\ldots A \ldots))) \ldots)$
$\wedge G(y) \wedge$ the other components of y in F.
It is easy to see that H,A have the desired properties.
Case 3; Y begins with a head redex. In case the head redex is a lambda I redex the case follows from the subject expansion theorem for lambda I ([2]pg 620). Otherwise we have $Y = (\backslash zZ)Z_0Z_1 \ldots Z_k$ and the main induction hypothesis applies to $Y' = ZZ_1 \ldots Z_k$. The main induction hypothesis also applies to Z_0. Thus there exists F, G, A, B with the desired properties *s.t.* $F \vdash Y' : A$, and $G \vdash Z_0 : B$. We replace all the atoms in B and $G(x)$, for all x:dom(G), by new atoms, but for notational purposes we continue to write the results as G an B. Now inspection of the am proof of $F \vdash Y' : A$ shows that there exists C such that
$C = C_1 \to (\ldots C_k \to A)$,
$F \vdash Z : C$, and
$F \vdash Z_i : C_i$ for $i = 1, \ldots, k$.
Thus $F \vdash \backslash zZ : B \to C$. Now define H by
$H(x) = F(x) \wedge G(x)$
and $H \vdash Y : A$ with the desired properties. End of proof.

Corollary 8.9. *(finite sets)*
If $M_1, \ldots, M_m$ are strongly normalizable then there exists a type A such that if N_i is the beta eta normal form of M_i then $\vdash M_i : A$ and the N_i are the only beta eta normal forms N such that $\vdash N : A$.

Theorem 8.10. *(class n)*
If for each n there exists a type A such that if M is a beta eta normal form of class n then $\vdash M : A$

Proof. suppose that we are given a term X in beta normal form of class n. We shall perform certain operations on X which may increase its class to at most $3n$.
(1) Each occurrence of a variable in the initial lambda prefix should be eta expanded so its lambda prefix has length $2n$. In addition, the eta variables so introduced for the head occurrence of X should be similarly expaneded. The number of arguments of altered variable occurrences is now between n and $3n$. Oscillation could be increased to 1.
(2) We eta expand so that for any maximal subterm

$$\backslash x_1 \ldots x_k.\ x_iX_1 \ldots X_l$$

where x_i was not considered in (1),
we have $l = n$ so $k <$ or $= 3n$, or $l = 0$ and $k = 0$.
In the result only the newly introduced eta variables are to have $l = 0$. Oscillations may have increased by 1.

(3) Next we eta expand the new eta variables in X so that every maximal oscillation in the Bohm tree of X has the same length $n+1$.

We call this normal form the vers normal form of X. We shall also assume that in X no bould variable is bound twice and no bound varialbe is also free; this is just a convenience. If X is in vers normal form then any occurrence of a given variable in X begins a maximal oscillation of the same length by (3). We call this the rank of the variable. We define by recursion on rank an intersection type for each such variable which depens only on its rank. In the process, we define an intersection type for each subterm. Variables of maximum rank are treated as a special case.

We suppose that A has been defined for variables of rank k. If $k+1$ is not maximum set $T_t =$
$A \to (\ldots(A \to a)\ldots)$

|
t

Let $S = k_1, \ldots, k_n$ be any sequence of non-negative integers less than or equal to $3n$. Let T_s be the type

$$T_{k_1} \to (\ldots(T_{k_n} \to a)\ldots).$$

Finally the A for $k+1$ is the intersection of all these T_s. Now if $k+1$ is maximum let $s(t) = k_1, \ldots, k_t$ be any sequence of non-negative integers less than or equal to $3n$, for $t = n, \ldots, 3n$. Finally the A for $k+1$ is the intersection of all these $T(s(t))$. End of proof. □

9 Adequate Numeral Systems

A numeral system $d_0, d_1, \ldots$ is a sequence of closed terms such that there exist lambda terms S and Z satisfying
$Sd_n = d_{n+1}$

$$Zd_n = \begin{cases} K* & \text{if} \quad n = 0 \\ K & \text{if} \quad n > 0. \end{cases}$$

A numeral system is adequate if every partial recursive function is lambda definable on the system ([1] page 136). Here we recall a corollary to Theorem 3.1 of ([4]).

Theorem 9.1. *Theorem; Suppose that S is an infinite R.E. set of closed terms each of which has a beta normal form and S is closed under beta-eta conversion. Then S is an adequate numeral system if and only if the map that takes a term in S to the Gödel number of its beta-eta normal form is representable.*

Here, representable means that there exists a closed term M such that for each closed beta-eta normal form N in S,

$$MN = 'N'.$$

This will be used below.

10 New Normal Form

Suppose that we are given a term X in beta normal form of class n. We shall perform certain operations on X which may increase its class.

(1) We eta expand each lambda prefix in X to length $n+1$. In the result only the newly introduced eta variables have a prefix of length $< n+1$; namely, length = 0. In the result, the maximum number of arguments of any variable occurrence may have increased to $2n+1$. Oscillations may have increased by 1.

(2) Next we eta expand the new eta variables in X so that every maximal oscillation in the Bohm tree of X has the same length $n+1$.

We call this normal form the new normal form of X. Since class can be increased by 1 in the next definition we begin with $n-1$.

Next, we construct terms which will compute a bound on the applicative depth of a closed term of class $n-1$ put in new normal form. It will be convenient to construct these terms as simultaneous fixed points, however for fixed n they can simply be defined recursively. Indeed, since the length of oscillations and lambda prefixes is fixed at n our term can be defined recursively as if we are in the simple typed case with one exception. The number of arguments of a head variable can vary between 0 and $2n$. The term replacing the head variable must first compute the number of arguments of the original variable and then proceed to compute the depth recursively. This can be achieved by adding a suffix $1 \ldots 2n+1$ and having the term replacing the head variable compute which integer is in position $2n+1$; e.g. 0 yields $2n+1$, $2n+1$ yields 1 etc. These terms use the lambda calculus representations of the sg and pred functions; sg $0 = K*$, sg $(m+1) = K$, pred $0 = 0$, and pred $(m+1) = m$. They also use a term H which has specified values on the positive Church numerals and is easy to construct;
set

$$V = 2$$

$$H(k+1) = \backslash z_1 \ldots z_{2n-k} \backslash w_1 \ldots \backslash w_{2n+1}.$$

$$V z_1 \ldots z_{2n-k} w_1 \ldots w_{2n+1}$$

$$U = \backslash u_1 \ldots \backslash u_{2n+1} H(u_{2n+1}))u_1 \ldots u_{2n+1}$$

$$Gx = sgxV(\backslash x_1 \ldots x_{2n}.Ux(F(\text{predx})x_1)\ldots(F(predx)x_{2n}))$$

$$Fxy = y(Gx)\ldots n \text{ copies} \ldots (Gx)1\ldots 2n+1$$

Notation;
We write

$M(X,Y,s) \quad := XY\ldots s \text{ copies}\ldots Y$
$N(X,Y,s) \quad := M(X,Y,s)1\ldots 2n+1$
$R(X,Y,s) \quad := \backslash x_1\ldots x_s.X(Yx_1)\ldots(Yx_s).$
Computation;
Now if $0 < s$ and $X = \backslash x_1\ldots x_n.x_iX_1\ldots X_l$ is beta normal let $Y_i = \backslash x_1\ldots x_n.\ X_i$ then

$$G0 = V$$

$$Gs = \backslash x_1\ldots x_{2n}.U\ N(x_1,G(s-1),n)\ldots N(x_{2n},G(s-1),n)$$

$$F0X = V(M(Y_1,V,n)\ldots(M(Y_l,V,n))1\ldots 2n+1$$

$$FsX = Gs(M(Y_1,Gs,n)\ldots(M(Y_l,Gs,n)1\ldots 2n+1.$$

Lemma 10.1. *(depth)*
If X has class $n-1$ and is put in new normal form then FnX beta converts to a Church numeral m such that the depth of the Bohm tree of X is at most m.

11 Bohm-out

Fix n. We now describe an algorithm which given a closed beta eta normal form X of class n constructs the Gödel number of an eta expansion of X. The algorithm is the result of iterating a procedure at least depth of the Bohm tree of X times. The procedure can be realized as a normal lambda term and the iterations accomplished by the use of the previous lemma on depth.
p_i is (the Church numeral for) the ith prime

$P_i := \backslash x_1\ldots x_{2n+1}p_i$
$L_j := \backslash x_1\ldots x_{2n}\backslash a.ajx_1\ldots x_{2n}.$
For a positive integer s we define recursively the prime components of s to the the set of primes dividing s together with the prime components of the exponents of these primes in the prime power factorization of s.

We assume that X has been eta expanded so that for any subterm

$$\backslash x_1\ldots x_k.x_iX_1\ldots X_l$$

we have $l = n$ so $k <$ or $= 2n$, or $l = 0$ and $k = 0$. We suppose that we are currently working on such a subterm, recursing downwards, and that we have already substituted L_j for the the jth variable bound on the path in the Bohm tree to this subterm; say for $j = 1,\ldots,r$ and the substitution @. So,the term in front of us is

$$\backslash x_1\ldots x_k.L_j@X_1\ldots @X_l$$

or

$$\backslash x_1\ldots x_k.x_i@X_1\ldots @X_l$$

depending on i. Now apply the term in front of us to the sequence

$$P_1 \dots P_{6n}.$$

The result beta reduces to an integer s
We distinguish two cases

(i) the minimum element p_r of the prime components of s has $r <$ or $= 2n$.
In this case we have the second alternative for the term in front of us, and $i = r$. We now substitute L_{r+j} for x_j, for $j = 1, \dots, k$, thus expanding @, and recurse downwards on each of the $@X_t$ for $t = 1, \dots, n$. Observe that in this case $k = (t-1)/2$ such that p_t is the second smallest member of the set of prime components of s

(ii) the minimum element p_r of the prime components of s has $r > 2n$. In this case we have the first alternative for the term in front of us. In case $l = n$ we have $r = k + 2n + 1$ and for the second smallest we have $t = k + 2n + n$. In case $k = l = 0$ we have $r = 2n + 1$ and $t = 4n + 1$. Once k and l are known l can be computed by applying then term in front of us to the sequence $I \dots k + 2n - l$ copies $\dots I(\backslash x \backslash x_1 \dots x_{2n}.x)$. Now proceed as in (i).

We obtain the following

Theorem 11.1. *Let A be an intersection type and S be set of all closed terms M such that $\vdash M : A$ in BCD. Then the map that takes a term in S to the Gödel number of its beta-eta normal form is representable.*

12 An example.

An example of an adequate numeral system which is not the set of all closed terms of an intersection type is the set of Bohm-Berraducci numerals

$O := KK$
$S := C*$
$Z := C*(KK*)$
$P := \backslash xZxx(xI)$
since $C*(\dots(C*KK))\dots)$

do not have bounded oscillation. A better example has bounded oscillation. Let

$$F_{i,n} := \backslash f.\ f(\backslash x_1.f(\dots f(\backslash x_n.x_i)\dots))$$

Let S be a single valued infinite subset of the positive integer pairs (n,i) such that $i < n+1$. We distinguish two cases

(a) for infinitely many $n, i > n/2$

(b) for infinitely many $n, i < (n+1)/2$.

We consider the case (b) here. The case for (a) is almost identical.

Proposition 12.1. *Suppose that for $(n,i) : S$ we have $\vdash F_{i,n} : A$. Then for n sufficiently large depending only on A, we have for infinity many m*

$$\vdash F_{i,m} : A$$

Proof. we may suppose that A is in dnf. We first consider the case that A begins with $\rightarrow; A = B_0 \rightarrow C_0$. Consider an am proof of $\vdash F_{i,n} : A$ then we have
$f : B_0 \vdash f(\backslash x_1.f(\ldots f(\backslash x_n x_i)\ldots)) : C_0$

$f : D_1 \wedge ((B_1 \rightarrow C_1) \rightarrow C'_0), x_1 : B_1 \vdash$
$\quad f(\backslash x_2.f(\ldots f(\backslash x_n x_i)\ldots)) : C_1$
$\vdots$
$f : D_n \wedge ((B_1 \rightarrow C_1 \rightarrow C'_0) \wedge \ldots \wedge ((B_n) \rightarrow C_n) \rightarrow C'_{n-1}),$
$\quad x_1 : B_1, \ldots, x_n : B_n \vdash x_i : C_n$

where
$B_0 =$
$D_1 \wedge ((B_1 \rightarrow C_1) \rightarrow C'_0) \wedge \ldots \wedge ((B_i \rightarrow C_i) \rightarrow C'_{i-1}),$
modulo slat. $B_i[C_n$ and for $j = 1\ldots, n, C'_i[C_i)$.
We now define a directed graph on the components of B_0 which are of the form $(B \rightarrow C) \rightarrow D$. We make $(B \rightarrow C) \rightarrow D$ adjacent to $(B' \rightarrow C') \rightarrow D'$ provided D' [D. Thus an am proof of $\vdash F_{i,n} : A$ gives us a walk through this digraph of length n. Hence if $n/2$ is larger than the number of components of B_0 this walk contains a directed cycle in its second half. The corresponding section of $F_{i,n}$ can be repeated. The case for more than one component is similar by using least common multiples. End of proof. □

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