Strong normalization of simple types through uniform intersection types

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A new proof of strong normalization for simple type assignment for λ-calculus is obtained, through a translation from this system to a system of uniform intersection types, which is equivalent to it as typability power and whose strong normalization property can be easily proved by induction on derivation.

1 Introduction

The simple type assignment system for λ-calculus comes from the *simple type theory*, introduced by Alonzo Church. It assigns types to λ-terms and enjoys the strong normalization property, i.e., the evaluation of a term which can be typed eventually stops, independently from the choice of the reduction strategy. This property allowed for the design of type assignment systems for real programming languages, like ML and Haskell, based on simple types, assuring the termination of programs. There are in the literature various proofs of such a property, following different approaches. Between the others, Roger Hindley, in [8] supplies a semantic proof, based on a λ-model, René David uses a completely syntactical approach [5], Federico Aschieri and Margherita Zorzi [1] obtain a proof as consequence of an analysis of non-strongly normalizing terms in an extended calculus, using the notion of perpetual strategy.

Here we supply a further proof, based on a proof-theoretical approach. Namely we introduce a restriction of intersection types, where intersection comes without idempotency, and moreover the intersection can be applied only to copies of the same type: we represent them as multisets having a singleton as support. The resulting system has interesting properties: it has the same typability power of simple types but it has a quantitative property, in the sense we can statically derive, from a derivation, some information about the size of the normal form of the subject. Moreover, the strong normalization property for it can be proved quite easily, by induction on derivation. Then we prove that every derivation in the simple type assignment system can be translated in a derivation in this system, with the same subject.

The translation is not simple: the original derivation is translated rule by rule, and every rule needs a rewriting of the derivations obtained so far, to update the cardinality of the multiset types. But we think this procedure is interesting in itself, since it shows the difficulty to transform a purely qualitative system into a quantitative one.

Intersection types were first introduced in [4] in order to increase the typability power of simple types, but immediately they turned out to be a very powerful tool to study the λ-calculus semantics [9]. The intersection connective traditionally enjoys commutativity, reflexivity and idempotency. In [7, 6] idempotency was breached in order to gain a quantitative interpretation of types: in fact, some properties which are undecidable for standard intersection types, like inhabitation, become decidable if

Submitted to:
ITRS 2018

intersection is not idempotent [3]. As far as we know, this is the first time that uniform intersection has been introduced.

2 Preliminaries

Syntax We will follow the syntax and the notation for λ-calculus as defined in [2]. *Terms* and *term contexts* of Λ are generated respectively by the grammars:

$$\begin{array}{lll} \mathtt{M},\mathtt{N},\mathtt{P},\mathtt{Q} & ::= & \mathtt{x} \mid \lambda\mathtt{x}.\mathtt{M} \mid \mathtt{MM} \\ \mathtt{C} & ::= & \square \mid \mathtt{x} \mid \lambda\mathtt{x}.\mathtt{C} \mid \mathtt{MC} \mid \mathtt{CM} \end{array}$$

where $\mathtt{x}$ ranges over a countable set $\mathtt{Var}$ of *variables* (denoted by $\mathtt{x},\mathtt{y},\mathtt{z},\ldots$), and $\square$ denotes the *hole* of the term context. As usual, we assume that λ-abstraction associates to the right, and has higher priority than application. So, we write $\lambda\mathtt{xyz}.\mathtt{xyz}$ for $\lambda\mathtt{x}.(\lambda\mathtt{y}.(\lambda\mathtt{z}.((\mathtt{xy})\mathtt{z})))$. The set of *free variables* of a term $\mathtt{M}$ is denoted by $\mathtt{FV}(\mathtt{M})$. We say that a term $\mathtt{M}$ is *closed* whenever $\mathtt{FV}(\mathtt{M}) = \emptyset$ and we denote by Λ^0 the set of all closed terms. Both terms and term contexts are considered up to α*-conversion*, i.e., modulo renaming of bound variables. Given a term context $\mathtt{C}$, we denote by $\mathtt{C}(\mathtt{M})$ the term obtained from $\mathtt{C}$ by filling the hole with $\mathtt{M}$, allowing the capture of free variables. An occurrence of a subterm $\mathtt{N}$ inside a term $\mathtt{M}$ is a context $\mathtt{C}$ such that $\mathtt{M} = \mathtt{C}(\mathtt{N})$.

The symbol $\equiv$ denotes the syntactic identity, modulo α-conversion.

Reduction The reduction relation $\rightarrow_\beta$ is the contextual closure of the rule:

$$(\lambda\mathtt{x}.\mathtt{M})\mathtt{N} \rightarrow \mathtt{M}[\mathtt{N}/\mathtt{x}]$$

where $\mathtt{M}[\mathtt{N}/\mathtt{x}]$ denotes the capture avoiding simultaneous substitution of $\mathtt{N}$ for all free occurrences of $\mathtt{x}$ in $\mathtt{M}$. $(\lambda\mathtt{x}.\mathtt{M})\mathtt{N}$ is called a *redex* and $\mathtt{M}[\mathtt{N}/\mathtt{x}]$ is its *reduct*. As usual, $\rightarrow_\beta^*$ denotes the reflexive and transitive closure of $\rightarrow_\beta$, and $=_\beta$ its reflexive, transitive and symmetric closure.

Normalization A term *is in normal form* if it has not occurrences of redexes, it *has normal form* (or it is *normalizing*) if it can be reduced to a normal form. A term is *strongly normalizing* if all the reduction sequences starting from it eventually terminate.

The simple type assignment system The set $\mathscr{T}$ of *types* is defined by the following grammar:

$$\mathtt{A},\mathtt{B},\mathtt{C} \quad ::= \quad \mathtt{a} \mid \mathtt{A} \rightarrow \mathtt{A}$$

where $\mathtt{a}$ ranges over a countable set of type constants. A *context* is a set of pairs $\mathtt{x} : \mathtt{A}$, where $\mathtt{x} \in \mathtt{Var}$ and $\mathtt{A} \in \mathscr{T}$. Contexts are ranged over by Γ,Δ; if $\mathtt{x} : \mathtt{A} \in \Gamma$, then $\Gamma(\mathtt{x}) = \mathtt{A}$; the domain of a context Γ is $\mathtt{dom}(\Gamma) = \{\mathtt{x} \mid \exists \mathtt{A}.\mathtt{x} : \mathtt{A} \in \Gamma\}$, $\Gamma \smile \Delta$ denotes that Γ and Δ agree, i.e., if $\mathtt{x} \in \mathtt{dom}(\Gamma) \cap \mathtt{dom}(\Delta)$, then $\Gamma(\mathtt{x}) = \Delta(\mathtt{x})$. Γ,Δ is short for $\Gamma \cup \Delta$ in case $\mathtt{dom}(\Gamma) \cap \mathtt{dom}(\Delta) = \emptyset$. Moreover $\mathtt{A} = \mathtt{B}$ if they are syntactically identical.

The simple type assignment system is a set of rules proving statements of the shape $\Gamma \vdash \mathtt{M} : \mathtt{A}$, where Γ is a context, $\mathtt{M}$ a term and $\mathtt{A}$ a type. The rules are shown in Figure 1. A *derivation* is a tree of rules, such that its leaves are applications of rule $(\mathtt{var})$, every node is an application of a rule whose premises are conclusion of its sons and its conclusion is one of the premises of its father, and the conclusion of the root

$$\frac{}{\mathtt{x : A \vdash x : A}}\ (\mathtt{var}) \qquad \frac{\Gamma \vdash \mathtt{M : A} \quad \mathtt{x} \notin \mathtt{dom}(\Gamma)}{\Gamma, \mathtt{x : B} \vdash \mathtt{M : A}}\ (\mathtt{weak})$$

$$\frac{\Gamma, \mathtt{x : A} \vdash \mathtt{M : B}}{\Gamma \vdash \lambda \mathtt{x.M : A} \to \mathtt{B}}\ (\to_{\mathtt{I}}) \qquad \frac{\Gamma \vdash \mathtt{M : B} \to \mathtt{A} \quad \Delta \vdash \mathtt{N : B} \quad \Gamma \smile \Delta}{\Gamma \cup \Delta \vdash \mathtt{MN : A}}\ (\to_{\mathtt{E}})$$

Figure 1: The simple type assignment system

is its conclusion. Derivations are ranged over by Π, Σ. $\Gamma \vdash \mathtt{M : A}$ is short for the existence of a derivation proving $\Gamma \vdash \mathtt{M : A}$, when we want to put in evidence a particular derivation Π with this conclusion we will write $\Pi \triangleright \Gamma \vdash \mathtt{M : A}$.

The system enjoys two important properties.

Theorem 1 *(subject reduction)* $\Gamma \vdash \mathtt{M : A}$ *and* $\mathtt{M} \to_\beta \mathtt{N}$ *imply* $\Gamma \vdash \mathtt{N : A}$.

Theorem 2 *(strong normalization)* $\Gamma \vdash \mathtt{M : A}$ *implies* $\mathtt{M}$ *is strongly normalizing.*

3 The uniform intersection types

In order to make easier the reading, we will use for the uniform intersection types the same notations as for simple types. We recall that a *multiset* is an unordered list of elements, whose support is the set of its elements. The union between two multisets, denoted by $\uplus$, takes into account the multiplicity of elements.

Definition 1 1. *The set* $\mathscr{T}_{\mathtt{i}}$ *of* uniform intersection types *is defined by the following grammar:*

$$\begin{array}{llll} \mathtt{A,B,C} & ::= & \mathtt{a} \mid \sigma \to \mathtt{A} & & (types) \\ \sigma, \tau, \rho & ::= & [\mathtt{A}]_n & (n \geq 1) & (multisets) \end{array}$$

where $\mathtt{a}$ *ranges over a countable set of type constants and* $[\mathtt{A}]_n$ *denotes a uniform multiset of types, with n elements, whose support is the singleton* $\{\mathtt{A}\}$.

2. *A* context *is a set of pairs* $\mathtt{x} : \sigma$*, where* $\mathtt{x} \in \mathtt{Var}$ *and* σ *is a multiset. Contexts are ranged over by* Γ, Δ.
3. *The uniform intersection type assignment system proves statements of the shape* $\Gamma \vdash_{\mathtt{i}} \mathtt{M : A}$*, where* Γ *is a context,* $\mathtt{M}$ *a term and* $\mathtt{A} \in \mathscr{T}_{\mathtt{i}}$*. The rules of the systems are shown in Figure 2.*

Notation

We extend to contexts all the notations introduced in the previous section, i.e.: if $\mathtt{x} : \sigma \in \Gamma$, then $\Gamma(\mathtt{x}) = \sigma$, $\mathtt{dom}(\Gamma) = \{\mathtt{x} \mid \exists \sigma.\mathtt{x} : \sigma \in \Gamma\}$. But in this setting $\Gamma \smile_{\mathtt{i}} \Delta$ denotes that, if $\mathtt{x} \in \mathtt{dom}(\Gamma) \cap \mathtt{dom}(\Delta)$, then $\Gamma(\mathtt{x})$ and $\Delta(\mathtt{x})$ are multisets with the same support. Moreover $\Gamma \uplus \Delta$ denotes the context such that $\Gamma \uplus \Delta(\mathtt{x}) = \Gamma(\mathtt{x}) \uplus \Delta(\mathtt{x})$. $|\sigma|$ denotes the cardinality of σ.

Rule $(\mathtt{weak})$ is necessary to obtain the subject reduction property, as shown in the next example.

Example 1 *Consider the following derivation:*

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathtt{y} : [\mathtt{A}] \vdash_{\mathtt{i}} \mathtt{y : A}}\ (\mathtt{var})}{\mathtt{y} : [\mathtt{A}] \vdash_{\mathtt{i}} \lambda \mathtt{x.y} : [\mathtt{A}] \to \mathtt{A}}\ (\to_{\mathtt{I}}) \qquad \dfrac{}{\mathtt{y} : [\mathtt{A}] \vdash_{\mathtt{i}} \mathtt{y : A}}\ (\mathtt{var})}{\mathtt{y} : [\mathtt{A}]_2 \vdash_{\mathtt{i}} (\lambda \mathtt{x.y})\mathtt{y : A}}\ (\to_{\mathtt{E}})}{\vdash_{\mathtt{i}} \lambda \mathtt{y}.(\lambda \mathtt{x.y})\mathtt{y} : [\mathtt{A}]_2 \to \mathtt{A}}\ (\to_{\mathtt{I}})$$

$$\frac{}{\mathtt{x}:[\mathtt{A}]\vdash_{\mathtt{i}}\mathtt{x}:\mathtt{A}}\ (\mathtt{var}) \qquad \frac{\Gamma\vdash_{\mathtt{i}}\mathtt{M}:\mathtt{A}\quad \{\mathtt{x}:\sigma\}\smile_{\mathtt{i}}\Gamma}{\Gamma\uplus\mathtt{x}:\sigma\vdash_{\mathtt{i}}\mathtt{M}:\mathtt{A}}\ (\mathtt{weak})$$

$$\frac{\Gamma,\mathtt{x}:\sigma\vdash_{\mathtt{i}}\mathtt{M}:\mathtt{B}}{\Gamma\vdash_{\mathtt{i}}\lambda\mathtt{x}.\mathtt{M}:\sigma\to\mathtt{B}}\ (\to_{\mathtt{I}}) \qquad \frac{\Gamma\vdash\mathtt{M}:[\mathtt{B}]_n\to\mathtt{A}\quad \Delta\vdash_{\mathtt{i}}\mathtt{N}:\mathtt{B}\quad \Gamma\smile_{\mathtt{i}}\Delta}{\Gamma\uplus(\Delta\uplus..\uplus\Delta)_n\vdash_{\mathtt{i}}\mathtt{MN}:\mathtt{A}}\ (\to_{\mathtt{E}})$$

$(\Delta\uplus..\uplus\Delta)_n$, is short for the multiset union of n copies of the context Δ.

Figure 2: The uniform intersection type assignment system

$\lambda\mathtt{y}.(\lambda\mathtt{x}.\mathtt{y})\mathtt{y}\to_\beta\lambda\mathtt{y}.\mathtt{y}$, *but to derive* $\vdash_{\mathtt{i}}\lambda\mathtt{y}.\mathtt{y}:[\mathtt{A}]_2\to\mathtt{A}$ *it is necessary to use the weakening rule.*

Moreover the system enjoys the strong normalization property. In order to prove it, we need first a substitution property, and a measure of its complexity.

Definition 2 *The measure of a derivation* Π *(denoted by* $\mathtt{meas}(\Pi)$*), is defined by induction in the following way:*
if Π *ends with an application of rule* $(\mathtt{var})$*, then* $\mathtt{meas}(\Pi)=1$*;*
if Π *ends with an application of rule* $(\to_{\mathtt{I}})$*, and* Σ *is its premise, then* $\mathtt{meas}(\Pi)=\mathtt{meas}(\Sigma)$*;*
if Π *ends with an application of rule* $(\mathtt{weak})$*, and* Σ *is its premise, then* $\mathtt{meas}(\Pi)=\mathtt{meas}(\Sigma)$*;*
if Π *ends with an application of rule* $(\to_{\mathtt{E}})$*, with* Σ_1 *and* Σ_2 *as major and minor premise respectively, and* $\Sigma_1\triangleright\Gamma\vdash\mathtt{M}:[\mathtt{A}]_n\to\mathtt{B}$*, then* $\mathtt{meas}(\Pi)=\mathtt{meas}(\Sigma_1)+n\times\mathtt{meas}(\Sigma_{2)}$.

Lemma 1 *If* $\Pi\triangleright\Gamma,\mathtt{x}:[\mathtt{A}]_n\vdash_{\mathtt{i}}\mathtt{M}:\mathtt{B}$ *and* $\Sigma\triangleright\Delta\vdash_{\mathtt{i}}\mathtt{N}:\mathtt{A}$*, then* $\Pi[\Sigma/\mathtt{x}]\triangleright\Gamma\uplus(\Delta\uplus...\uplus\Delta)_n\vdash_{\mathtt{i}}\mathtt{M}[\mathtt{N}/\mathtt{x}]$*, where* $\mathtt{meas}(\Pi[\Sigma/\mathtt{x}])<\mathtt{meas}(\Pi)+n\times\mathtt{meas}(\Sigma)$.

Proof. By induction on Π. If Π is:

$$\frac{}{\mathtt{y}:[\mathtt{A}]\vdash_{\mathtt{i}}\mathtt{y}:\mathtt{A}}\ (\mathtt{var})$$

If $\mathtt{x}=\mathtt{y}$ then $\Pi[\Sigma/\mathtt{x}]=\Sigma$, otherwise $\Pi[\Sigma/\mathtt{x}]=\Pi$. In both cases the result is obvious. If the last rule of Π is $(\to_{\mathtt{I}})$, the proof follows by induction. If Π is:

$$\frac{\Pi_1\triangleright\Gamma_1,\mathtt{x}:[\mathtt{A}]_m\vdash_{\mathtt{i}}\mathtt{P}:[\mathtt{C}]_s\to\mathtt{B}\quad \Pi_2\triangleright\Gamma_2,\mathtt{x}:[\mathtt{A}]_p\vdash_{\mathtt{i}}\mathtt{Q}:\mathtt{C}}{\Gamma_1\uplus(\Gamma_2\uplus...\uplus\Gamma_2)_s,\mathtt{x}:[\mathtt{A}]_{m+p\times s}\vdash_{\mathtt{i}}\mathtt{PQ}:\mathtt{B}}\ (\to_{\mathtt{E}})$$

where $n=m+p\times s$. Then, by induction, there are: $\Pi_1[\Sigma/\mathtt{x}]\triangleright\Gamma_1\uplus(\Delta\uplus...\uplus\Delta)_m\vdash_{\mathtt{i}}\mathtt{P}[\mathtt{N}/\mathtt{x}]:[\mathtt{C}]_s\to\mathtt{B}$ and $\Pi_2[\Sigma/\mathtt{x}]\triangleright\Gamma_2\uplus(\Delta\uplus...\uplus\Delta)_p\vdash_{\mathtt{i}}\mathtt{Q}[\mathtt{N}/\mathtt{x}]:\mathtt{C}$, such that: $\mathtt{meas}(\Pi_1[\Sigma/\mathtt{x}])<\mathtt{meas}(\Pi_1)+m\times\mathtt{meas}(\Sigma)$ and $\mathtt{meas}(\Pi_2[\Sigma/\mathtt{x}])<\mathtt{meas}(\Pi_2)+p\times\mathtt{meas}(\Sigma)$. Then by rule $(\to_{\mathtt{E}})$, with premises $\Pi_1[\Sigma/\mathtt{x}]$ and $\Pi_2[\Sigma/\mathtt{x}]$ we obtain: $\Pi[\Sigma/\mathtt{x}]\triangleright\Gamma_1\uplus(\Delta\uplus...\uplus\Delta)_m\uplus\Gamma_2\uplus(\Delta\uplus...\uplus\Delta)_p\vdash_{\mathtt{i}}\mathtt{PQ}[\mathtt{N}/\mathtt{x}]:\mathtt{B}$, where: $\mathtt{meas}(\Pi[\Sigma/\mathtt{x}])=\mathtt{meas}(\Pi_1[\Sigma/\mathtt{x}])+s\times\mathtt{meas}(\Pi_2[\Sigma/\mathtt{x}])<\mathtt{meas}(\Pi_1)+m\times\mathtt{meas}(\Sigma)+s\times(\mathtt{meas}(\Pi_2)+p\times\mathtt{meas}(\Sigma))=\mathtt{meas}(\Pi_1)+s\times\mathtt{meas}(\Pi_2)+(m+s\times p)\times\mathtt{meas}(\Sigma)=\mathtt{meas}(\Pi)+n\times\mathtt{meas}(\Sigma)$.
In case the last rule of Π is $(\mathtt{weak})$ the proof is obvious.

Property 1
1. $\Pi\triangleright\Gamma\vdash_{\mathtt{i}}\lambda\mathtt{x}.\mathtt{M}:\mathtt{A}$ *implies* $\mathtt{A}=\mathtt{B}\to\mathtt{C}$ *and there is* $\Pi'\triangleright\Gamma\vdash_{\mathtt{i}}\lambda\mathtt{x}.\mathtt{M}:\mathtt{A}$ *such that its last applied rule is* $(\to_{\mathtt{I}})$ *and* $\mathtt{meas}(\Pi)=\mathtt{meas}(\Pi')$.
2. $\Pi\triangleright\Gamma\vdash_{\mathtt{i}}\mathtt{MN}:\mathtt{A}$ *implies and there is* $\Pi'\triangleright\Gamma\vdash_{\mathtt{i}}\mathtt{MN}:\mathtt{A}$ *such that its last applied rule is* $(\to_{\mathtt{E}})$ *and* $\mathtt{meas}(\Pi)=\mathtt{meas}(\Pi'$.

Proof. In both cases, the only other possible rule is $(\mathtt{weak})$, which commutes with both the rules.

The quantitative properties of the system allow for a very easy proof of strong normalization.

Lemma 2 *If* $\Pi \triangleright \Gamma \vdash_{\mathtt{i}} \mathtt{M} : \mathtt{A}$ *and* $\mathtt{M} \to_\beta \mathtt{M}'$*, then* $\Pi' \triangleright \Gamma \vdash_{\mathtt{i}} \mathtt{M}' : \mathtt{A}$ *and* $\mathtt{meas}(\Pi') < \mathtt{meas}(\Pi)$.

Proof. $\mathtt{M} \to_\beta \mathtt{M}'$ means that $\mathtt{M} = \mathtt{C}((\lambda \mathtt{x.P})\mathtt{Q})$ and $\mathtt{M}' = \mathtt{C}(\mathtt{P}[\mathtt{Q}/\mathtt{x})$. The proof is by induction on $\mathtt{C}$. Let $\mathtt{C} = \Box$. Then, by Property 1, Π has the following shape:

$$\dfrac{\dfrac{\Theta \triangleright \Gamma, \mathtt{x} : [\mathtt{B}]_n \vdash_{\mathtt{i}} \mathtt{P} : \mathtt{A}}{\Gamma \vdash_{\mathtt{i}} \lambda \mathtt{x.P} : [\mathtt{B}]_n \to \mathtt{A}}\,(\to_{\mathtt{I}}) \qquad \Sigma \triangleright \Delta \vdash_{\mathtt{i}} \mathtt{Q} : \mathtt{B}}{\Gamma \uplus (\Delta \uplus \ldots \uplus \Delta)_n \vdash_{\mathtt{i}} (\lambda \mathtt{x.P})\mathtt{Q} : \mathtt{A}}\,(\to_{\mathtt{E}})$$

Let Π' be the derivation obtained by replacing Π by $\Theta[\Sigma/\mathtt{x}]$, then arranging the subjects. Then $\mathtt{meas}(\Pi') = \mathtt{meas}(\Theta[\Sigma/\mathtt{x}]) < \mathtt{meas}(\Pi)$ by induction. The induction cases are straighforward.

The measure of a derivation $\Pi \triangleright \Gamma \vdash_{\mathtt{i}} \mathtt{M} : \mathtt{A}$ is an upper bound to the number of variable occurrences (both bound and free) in the normal form of the subject $\mathtt{M}$.

Example 2 *Consider the following derivation* Π*:*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathtt{y} : [[\mathtt{A}] \to [\mathtt{A}] \to \mathtt{A}] \vdash_{\mathtt{i}} \mathtt{y} : [\mathtt{A}] \to [\mathtt{A}] \to \mathtt{A}}\,(\mathtt{var}) \quad \dfrac{}{\mathtt{x} : [\mathtt{A}] \vdash_{\mathtt{i}} \mathtt{x} : \mathtt{A}}\,(\mathtt{var})}{\mathtt{y} : [[\mathtt{A}] \to [\mathtt{A}] \to \mathtt{A}], \mathtt{x} : [\mathtt{A}] \vdash_{\mathtt{i}} \mathtt{yx} : [\mathtt{A}] \to \mathtt{A}}\,(\to_{\mathtt{E}}) \quad \dfrac{}{\mathtt{x} : [\mathtt{A}] \vdash_{\mathtt{i}} \mathtt{x} : \mathtt{A}}\,(\mathtt{var})}{\mathtt{y} : [[\mathtt{A}] \to [\mathtt{A}] \to \mathtt{A}], \mathtt{x} : [\mathtt{A}]_2 \vdash_{\mathtt{i}} \mathtt{yxx} : \mathtt{A}}\,(\to_{\mathtt{E}})}{\mathtt{y} : [[\mathtt{A}] \to [\mathtt{A}] \to \mathtt{A}] \vdash_{\mathtt{i}} \lambda \mathtt{x.yxx} : [\mathtt{A}]_2 \to \mathtt{A}}\,(\to_{\mathtt{I}}) \qquad \Sigma \quad \Sigma}{\mathtt{y} : [[\mathtt{A}] \to [\mathtt{A}] \to \mathtt{A}], \mathtt{z} : [\mathtt{A}]_2 \vdash_{\mathtt{i}} (\lambda \mathtt{x.yxx})\mathtt{z} : \mathtt{A}}\,(\to_{\mathtt{E}})$$

where Σ *is the derivation:*

$$\dfrac{}{\mathtt{z} : [\mathtt{A}] \vdash_{\mathtt{i}} \mathtt{z} : \mathtt{A}}\,(\mathtt{var})$$

Then $\mathtt{meas}(\Pi) = 5$. $(\lambda \mathtt{x.yxx})\mathtt{z} \to_\beta \mathtt{yzz}$*, which is typed by the following derivation* Π' *:*

$$\dfrac{\dfrac{\dfrac{}{\mathtt{y} : [[\mathtt{A}] \to [\mathtt{A}] \to \mathtt{A}] \vdash_{\mathtt{i}} \mathtt{y} : [\mathtt{A}] \to [\mathtt{A}] \to \mathtt{A}}\,(\mathtt{var}) \quad \dfrac{}{\mathtt{z} : [\mathtt{A}] \vdash_{\mathtt{i}} \mathtt{z} : \mathtt{A}}\,(\mathtt{var})}{\mathtt{y} : [[\mathtt{A}] \to [\mathtt{A}] \to \mathtt{A}], \mathtt{z} : [\mathtt{A}] \vdash_{\mathtt{i}} \mathtt{yz} : [\mathtt{A}] \to \mathtt{A}}\,(\to_{\mathtt{E}}) \quad \dfrac{}{\mathtt{z} : [\mathtt{A}] \vdash_{\mathtt{i}} \mathtt{z} : \mathtt{A}}\,(\mathtt{var})}{\mathtt{y} : [[\mathtt{A}] \to [\mathtt{A}] \to \mathtt{A}], \mathtt{z} : [\mathtt{A}]_2 \vdash_{\mathtt{i}} \mathtt{yzz} : \mathtt{A}}\,(\to_{\mathtt{E}})$$

and $\mathtt{meas}(\Pi') = 3$.

Theorem 3 *Let* $\Pi \triangleright \Gamma \vdash_{\mathtt{i}} \mathtt{M} : \mathtt{A}$. *Then* $\mathtt{M}$ *is strongly normalizing.*

Proof. The proof is a corollary of Lemma 2.

4 Strong normalization of simple types

The proof of strong normalization of simple types assignment system is based on a translation from it to uniform intersection type assignment system. First, we will define a translation from intersection types to simple types, which simply erases the multisets in $\mathscr{T}_{\mathtt{i}}$.

Definition 3 *1. The translation $(.)^\circ$ from $\mathscr{T}_{\mathtt{i}}$ to $\mathscr{T}$ is defined by induction on the size of types in the following way:*

$$(\mathtt{a})^\circ = \mathtt{a};$$
$$(\sigma \to \mathtt{A})^\circ = (\sigma)^\circ \to (\mathtt{A})^\circ;$$
$$([\mathtt{A}]_n)^\circ = (\mathtt{A})^\circ.$$

2. The translation $(.)^\circ$ can be extended to contexts in the following way:
$(\Gamma)^\circ = \{\mathtt{x} : \mathtt{A} \mid \mathtt{x} : \sigma \in \Gamma, \mathtt{A} = (\sigma)^\circ\}$.

Definition 4 *On $\mathscr{T}_{\mathtt{i}}$ we define two relations.*

1. *$\preceq \subseteq \mathscr{T}_{\mathtt{i}} \times \mathscr{T}_{\mathtt{i}}$ is defined in the following way:*
   (a) $\mathtt{a} \preceq \mathtt{a}'$ *if* $\mathtt{a} = \mathtt{a}'$*;*
   (b) $[\mathtt{A}]_n \preceq [\mathtt{A}]_m$ *if* $n \leq m$*;*
   (c) $\sigma \to \mathtt{A} \preceq \tau \to \mathtt{B}$ *if* $\sigma \preceq \tau$ *and* $\mathtt{A} \preceq \mathtt{B}$.
2. *$\simeq \subseteq \mathscr{T}_{\mathtt{i}} \times \mathscr{T}_{\mathtt{i}}$ is defined in the following way:*
   (a) $\mathtt{a} \simeq \mathtt{a}'$ *if* $\mathtt{a} = \mathtt{a}'$*;*
   (b) $[\mathtt{A}]_n \simeq [\mathtt{A}]_m$*, for any* n, m*;*
   (c) $\sigma \to \mathtt{A} \simeq \tau \to \mathtt{B}$ *if* $\sigma \simeq \tau$ *and* $\mathtt{A} \simeq \mathtt{B}$.

Property 2 $(\mathtt{A})^\circ = (\mathtt{B})^\circ$ $((\sigma)^\circ = (\tau)^\circ)$ *implies* $\mathtt{A} \simeq \mathtt{B}$ $(\sigma \simeq \tau)$*. Moreover, the class $S_{\mathtt{B}} = \{\mathtt{A} \mid (\mathtt{A})^0 = \mathtt{B}\}$ has a minimum element, with respect to the number of symbol occurrences in a type.*

Proof. Easy, by induction on types. The minimum element of $S_{\mathtt{B}}$ is a type $\mathtt{A} \in \mathscr{T}_{\mathtt{i}}$ where all multisets have cardinality 1.

Definition 5 *The operation* $\mathtt{merge} : \mathscr{T}_{\mathtt{i}} \times \mathscr{T}_{\mathtt{i}} \to \mathscr{T}_{\mathtt{i}}$ *is defined as follows:*
$\mathtt{merge}(\mathtt{A},\mathtt{B}) = \mathtt{if}\ \mathtt{A} \not\simeq \mathtt{B}\ \mathtt{then\ undefined,\ else}\ :$
$\mathtt{merge}(\mathtt{a},\mathtt{a}) = \mathtt{a};$
$\mathtt{merge}([\mathtt{A}]_{\mathtt{n}} \to \mathtt{B}, [\mathtt{A}']_{\mathtt{m}} \to \mathtt{B}') = [\mathtt{merge}(\mathtt{A},\mathtt{A}')]_{\mathtt{max(n,m)}} \to \mathtt{merge}(\mathtt{B},\mathtt{B}')$

Definition 6 *1. A type context is obtained by adjoining a new constant $\Box$ (the hole) to the syntax of types:*

$$\mathtt{TC} \ ::= \ \Box \mid \mathtt{a} \mid [\mathtt{TC}]_n \to \mathtt{A} \mid [\mathtt{A}]_n \to \mathtt{TC} \qquad (n \geq 1)$$

$\mathtt{TC}(\mathtt{A})$ *denotes the result of replacing the hole $\Box$ in it by* $\mathtt{A}$*; note that filling the context* $[\mathtt{TC}]_n \to \mathtt{A}$ *with* $\mathtt{B}$ *produces the type* $[\mathtt{TC}(\mathtt{B})]_n \to \mathtt{A}$*. An occurrence of* $\mathtt{A}$ *inside a type* $\mathtt{B}$ *is the type context* $\mathtt{TC}$ *such that* $\mathtt{TC}(\mathtt{A}) = \mathtt{B}$*, so all the copies inside the same multisets are considered as a single occurrence: let* $\mathtt{B}$ *be* $[\mathtt{C}]_n \to \mathtt{A}$*, then there is a unique occurrence of* $\mathtt{C}$ *in it, namely* $[\Box]_n \to \mathtt{A}$.

Rules of Figure 2 can be trivially extended to the new syntax. In what follows, we will deal with occurrences of subtypes. In order to distinguish between different occurrences of the same (sub)type inside a derivation, we will use integer indexes. A more formal definition, using derivation contexts, could be possible, but it would be a very difficult syntax to work with.

Definition 7 *Let $\Pi \triangleright \Gamma \vdash_{\mathtt{i}} \mathtt{M} : \mathtt{A}$. An o-set of Π is a set of occurrences of the same type in it, and $O(\Pi)$ denotes the set of o-sets of Π. The definition of $O(\Pi)$ and the partial relation $\leq$ between o-sets are given by induction on Π in the following way:*
Let Π be:

$$\frac{}{\mathtt{x} : [\mathtt{TC}(\mathtt{A}_1)] \vdash_{\mathtt{i}} \mathtt{x} : \mathtt{TC}(\mathtt{A}_2)}\ (\mathtt{var})$$

then, for every context $\mathtt{TC}$, $\{\mathtt{A}_1,\mathtt{A}_2\} \in O(\Pi)$.

Let Π *end with an application of the rule* $(\rightarrow_{\mathtt{I}})$. *There are two cases, according to the possible type contexts into consideration.*

$$\frac{\Theta \rhd \Gamma, \mathtt{x} : [\mathtt{TC}(\mathtt{A}_1)]_n \vdash_{\mathtt{i}} \mathtt{M} : \mathtt{TC}'(\mathtt{B}_1)}{\Gamma \vdash_{\mathtt{i}} \lambda \mathtt{x}.\mathtt{M} : [\mathtt{TC}(\mathtt{A}_2)]_n \rightarrow \mathtt{TC}'(\mathtt{B}_2) = ([\mathtt{TC}(\mathtt{A})]_n \rightarrow \mathtt{TC}'(\mathtt{B}))_1} \; (\rightarrow_{\mathtt{I}})$$

Then, for every context $\mathtt{TC}$, $\mathscr{A} \in O(\Theta)$ *and* $\mathtt{A}_1 \in \mathscr{A}$ *implies* $\mathscr{A} \cup \{\mathtt{A}_2\} \in O(\Pi)$, $\mathscr{B} \in O(\Theta)$ *and* $\mathtt{B}_1 \in \mathscr{B}$ *implies* $\mathscr{B} \cup \{\mathtt{B}_2\} \in O(\Pi)$. *Moreover* $\{([\mathtt{TC}(\mathtt{A})]_n \rightarrow \mathtt{TC}'(\mathtt{B}))_1\} \in O(\Pi)$. *Every other o-set of* Θ *is also an o-set of* Π.

Let Π *be:*

$$\frac{\Theta \rhd \Gamma, \mathtt{x} : [\mathtt{C}_1]_n \vdash_{\mathtt{i}} \mathtt{M} : \mathtt{B}_1}{\Gamma \vdash_{\mathtt{i}} \lambda \mathtt{x}.\mathtt{M} : ([\mathtt{C}]_n \rightarrow \mathtt{B})_1} \; (\rightarrow_{\mathtt{I}})$$

Then, $\{([\mathtt{C}]_n \rightarrow \mathtt{B})_1\} \in O(\Pi)$, *and , for every* $\mathscr{C}, \mathscr{B}$ *such that* $\mathtt{C}_1 \in \mathscr{C} \in O(\Theta)$ *and* $\mathtt{B}_1 \in \mathscr{B} \in O(\Theta)$ $\mathscr{C}, \mathscr{B} \in O(\Pi)$, *and* $\{([\mathtt{C}]_n \rightarrow \mathtt{B})_1\} \leq \mathscr{C}$ *and* $\{([\mathtt{C}]_n \rightarrow \mathtt{B})_1\} \leq \mathscr{B}$. *Every other o-set of* Θ *is also an o-set of* Π.

Let Π *ends with an application of the rule* $(\rightarrow_{\mathtt{E}})$. *Also here there are two cases, according to the possible type contexts.*

$$\frac{\Theta_1 \rhd \Gamma \vdash_{\mathtt{i}} \mathtt{M} : [\mathtt{TC}'(\mathtt{B}_1)]_n \rightarrow \mathtt{TC}(\mathtt{A}_1) \quad \Theta_2 \rhd \Delta \vdash_{\mathtt{i}} \mathtt{N} : \mathtt{TC}'(\mathtt{B}_2) \quad \Gamma \smile_{\mathtt{i}} \Delta}{\Gamma \uplus (\Delta \uplus \ldots \uplus \Delta)_n \vdash_{\mathtt{i}} \mathtt{MN} : \mathtt{TC}(\mathtt{A}_2)} \; (\rightarrow_{\mathtt{E}})$$

then $\mathtt{B}_1 \in \mathscr{B}_1 \in O(\Theta_1)$ *and* $\mathtt{B}_2 \in \mathscr{B}_2 \in O(\Theta_2)$ *imply* $\mathscr{B}_1 \cup \mathscr{B}_2 \in O(\Pi)$. *If* $\mathtt{A}_1 \in \mathscr{A} \in O(\Theta_1)$, *then* $\mathscr{A} \cup \{\mathtt{A}_2\} \in O(\Pi)$. *Moreover, for every* $\mathtt{x} \in \mathtt{dom}(\Gamma) \cap \mathtt{dom}(\Delta)$, *if* $\mathtt{x} : [\mathtt{TC}''(\mathtt{C}_1)]_n \in \Gamma$ *and* $\mathtt{x} : [\mathtt{TC}''(\mathtt{C}_2)]_m \in \Delta$, *if* $\mathtt{C}_1 \in \mathscr{C}_1 \in O(\Theta_1)$ *and* $\mathtt{C}_2 \in \mathscr{C}_2 \in O(\Theta_2)$, *then* $\mathscr{C}_1 \cup \mathscr{C}_2 \in O(\Pi)$. *Every other o-set of* Θ_1 *and* Θ_2 *is also an o-set of* Π.

Let Π *be:*

$$\frac{\Theta_1 \rhd \Gamma \vdash_{\mathtt{i}} \mathtt{M} : ([\mathtt{B}_1)]_n \rightarrow \mathtt{A}_1)_1 \quad \Theta_2 \rhd \Delta \vdash_{\mathtt{i}} \mathtt{N} : \mathtt{B}_2 \quad \Gamma \smile_{\mathtt{i}} \Delta}{\Gamma \uplus (\Delta \uplus \ldots \uplus \Delta)_n \vdash_{\mathtt{i}} \mathtt{MN} : \mathtt{A}_2} \; (\rightarrow_{\mathtt{E}})$$

Then, by the previous point, $\mathtt{A}_1 \in \mathscr{A} \in O(\Theta_1)$ *implies* $\mathscr{A} \cup \{\mathtt{A}_2\} \in O(\Pi)$, *and* $\mathtt{B}_1 \in \mathscr{B} \in O(\Theta_1)$ *implies* $\mathscr{B} \cup \{\mathtt{B}_2\} \in O(\Pi)$. *Moreover* $([\mathtt{B}_1)]_n \rightarrow \mathtt{A}_1)_1 \in \mathscr{E} \in O(\Theta_1)$ *implies* $\mathscr{E} \in O(\Pi)$. *So* $\mathscr{A} \cup \{\mathtt{A}_2\} \leq \mathscr{B} \cup \{\mathtt{B}_2\}$ *and also* $\mathscr{A} \cup \{\mathtt{A}_2\} \leq \mathscr{C}_1 \cup \mathscr{C}_2$, *for every* $\mathtt{x} \in \mathtt{dom}(\Gamma) \cap \mathtt{dom}(\Delta)$.

Let Π *be:*

$$\frac{\Gamma \vdash_{\mathtt{i}} \mathtt{M} : \mathtt{A} \quad \Gamma \smile_{\mathtt{i}} \{\mathtt{x} : [\mathtt{TC}(\mathtt{B})]_n\}}{\Theta \rhd \Gamma \uplus \mathtt{x} : [\mathtt{TC}(\mathtt{B}_1)]_n \vdash_{\mathtt{i}} \mathtt{M} : \mathtt{A}} \; (\mathtt{weak})$$

If $\mathtt{x} \notin \mathtt{dom}(\Gamma)$, *then* $\{\mathtt{B}_1\} \in O(\Pi)$, *otherwise there is* $\mathtt{x} : [\mathtt{TC}(\mathtt{B}_i)]_m$ *in* Γ, *and every o-set of* $O(\Theta)$ *belongs to* $O(\Pi)$.

Morally an *o*-set of Π collects all the (sub)types occurrences that are copies of the same logical (sub)formula and the $\leq$ relation connects the conclusion with the premises of a rule.

Example 3 *Consider the following derivation:*

$$\frac{}{\mathtt{x} : [[\mathtt{a}] \rightarrow \mathtt{a}] \vdash_{\mathtt{i}} \mathtt{x} : [\mathtt{a}] \rightarrow \mathtt{a}} \; (\mathtt{var})$$

Let us number the subtype occurrences of $\mathtt{a}$*:*

$$\frac{}{\mathtt{x} : [[\mathtt{a}_1] \to \mathtt{a}_2] \vdash_{\mathtt{i}} \mathtt{x} : [\mathtt{a}_3] \to \mathtt{a}_4} \ (\mathtt{var})$$

and the type occurrences of $[\mathtt{a}] \to \mathtt{a}$*:*

$$\frac{}{\mathtt{x} : [([\mathtt{a}] \to \mathtt{a})_1] \vdash_{\mathtt{i}} \mathtt{x} : ([\mathtt{a}] \to \mathtt{a})_2} \ (\mathtt{var})$$

then $\{\mathtt{a}_1, \mathtt{a}_3\}$, $\{\mathtt{a}_2, \mathtt{a}_4\}$, $\{([\mathtt{a}] \to \mathtt{a})_1, ([\mathtt{a}] \to \mathtt{a})_2\}$ *belong to* $O(\Pi)$.

Let us define a rewriting operation on derivations, $R(\Pi, \mathscr{A}, \mathtt{B})$, where $\mathscr{A}$ denotes an o-set of $O(\Pi)$ containing occurrences of $\mathtt{A}$ such that $\mathtt{A} \preceq \mathtt{B}$. Roughly speaking, this operation consists in replacing all occurrences of $\mathtt{A}$ in $\mathscr{A}$ and in all the o-sets $\geq \mathscr{A}$ by $\mathtt{B}$. We will prove at the same time that this operation preserves typability, i.e., $\Pi \triangleright \Gamma \vdash_{\mathtt{i}} \mathtt{M} : \mathtt{A}$ implies $R(\Pi, \mathscr{A}, \mathtt{B}) \triangleright \Gamma' \vdash \mathtt{M} : \mathtt{A}'$.

The definition is by induction on Π.

Let Π be:

$$\frac{}{\mathtt{x} : [\mathtt{TC}(\mathtt{A}_1)] \vdash_{\mathtt{i}} \mathtt{x} : \mathtt{TC}(\mathtt{A}_2)} \ (\mathtt{var})$$

If $\mathscr{A} = \{\mathtt{A}_1, \mathtt{A}_2\}$, $R(\Pi, \mathscr{A}, \mathtt{B})$ is:

$$\frac{}{\mathtt{x} : [\mathtt{TC}(\mathtt{B})] \vdash_{\mathtt{i}} \mathtt{x} : \mathtt{TC}(\mathtt{B})} \ (\mathtt{var})$$

Otherwise, $R(\Pi, \mathscr{A}, \mathtt{B}) = \Pi$. Clearly $R(\Pi, \mathscr{A}, \mathtt{B})$ is a correct derivation for $\mathtt{x}$.

Let $\mathtt{A}_1 \in \mathscr{A}$ and let Π be:

$$\frac{\Theta \triangleright \Gamma, \mathtt{x} : [\mathtt{TC}(\mathtt{A}_1)]_n \vdash_{\mathtt{i}} \mathtt{M} : \mathtt{TC}'(\mathtt{C}_1)}{\Gamma \vdash_{\mathtt{i}} \lambda \mathtt{x}.\mathtt{M} : [\mathtt{TC}(\mathtt{A}_2)]_n \to \mathtt{TC}'(\mathtt{C}_2)} \ (\to_{\mathtt{I}})$$

By induction there is $R(\Theta, \mathscr{A}, \mathtt{B})$, proving

$$\Gamma, \mathtt{x} : [\mathtt{TC}(\mathtt{B})]_n \vdash_{\mathtt{i}} \mathtt{M} : \mathtt{C}'$$

so $R(\Pi, \mathscr{A}, \mathtt{B})$ is obtained from it by rule $(\to_{\mathtt{I}})$. Let $\mathtt{C}_1 \in \mathscr{A}$; by induction there is $R(\Theta, \mathscr{A}, \mathtt{B})$, proving

$$\Gamma, \mathtt{x} : [\mathtt{A}']_n \vdash_{\mathtt{i}} \mathtt{M} : \mathtt{TC}'(\mathtt{B})$$

and the result follows by rule $(\to_{\mathtt{I}})$.

Let $\mathtt{A}_2 \in \mathscr{A}$. Since, by definition, $\mathtt{A}_1$ and $\mathtt{A}_2$ both are in are in $\mathscr{A}$, as $\mathtt{C}_1$ and $\mathtt{C}_2$, we can apply the same reasoning. Let $([\mathtt{A}']_n \to \mathtt{C}')_1 \in \mathscr{A}$ and let Π be:

$$\frac{\Theta \triangleright \Gamma, \mathtt{x} : [\mathtt{A}'_1]_n \vdash_{\mathtt{i}} \mathtt{M} : \mathtt{C}'_1}{\Gamma \vdash_{\mathtt{i}} \lambda \mathtt{x}.\mathtt{M} : ([\mathtt{A}']_n \to \mathtt{C}')_1} \ (\to_{\mathtt{I}})$$

Then $\mathtt{B} = [\mathtt{A}'']_m \to \mathtt{C}''$, where $\mathtt{A}' \preceq \mathtt{A}''$, $\mathtt{C}' \preceq \mathtt{C}''$ and $n \leq m$.

Then let $\mathscr{A}_1$ and $\mathscr{C}_1$ be such that $\mathtt{A}_1 \in \mathscr{A}_1 \in O(\Theta)$, and $\mathtt{C}_1 \in \mathscr{C}_1 \in O(\Theta)$. By definition $\mathscr{A} \leq \mathscr{A}_1$ and $\mathscr{A} \leq \mathscr{A}_2$, so we need to close the substitution w.r.t. $\mathscr{A}_1$ and $\mathscr{A}_2$. By induction $R(R(\Theta, \mathscr{A}_1, \mathtt{A}''), \mathscr{A}_2, \mathtt{C}'') \triangleright \Gamma', \mathtt{x} : [\mathtt{A}'']_n \vdash_{\mathtt{i}} \mathtt{M} : \mathtt{C}''$, and $R(\Pi, ([\mathtt{A}']_n \to \mathtt{C}')_1$ is the derivation:

$$\frac{\dfrac{\Gamma', \mathtt{x} : [\mathtt{A}'']_n \vdash_{\mathtt{i}} \mathtt{M} : \mathtt{C}''}{\Gamma', \mathtt{x} : [\mathtt{A}'']_m \vdash_{\mathtt{i}} \mathtt{M} : \mathtt{C}''} \ (\mathtt{weak})}{\Gamma' \vdash_{\mathtt{i}} \lambda \mathtt{x}.\mathtt{M} : [\mathtt{A}'']_m \to \mathtt{C}''} \ (\to_{\mathtt{I}})$$

Otherwise $R(\Pi, \mathtt{A}_1, \mathtt{B}) = \Pi$.

Let Π be:

$$\frac{\Theta_1 \triangleright \Gamma \vdash_{\mathtt{i}} \mathtt{M} : [\mathtt{TC}'(\mathtt{C}_1)]_n \to \mathtt{TC}(\mathtt{A}_1) \quad \Theta_2 \triangleright \Delta \vdash_{\mathtt{i}} \mathtt{N} : \mathtt{TC}'(\mathtt{C}_2) \quad \Gamma \smile_{\mathtt{i}} \Delta}{\Gamma \uplus (\Delta \uplus \ldots \uplus \Delta)_n \vdash_{\mathtt{i}} \mathtt{MN} : \mathtt{TC}(\mathtt{A}_2)} (\to_{\mathtt{E}})$$

Let $\mathtt{C}_1 \in \mathscr{A}$. Remember that, by definition, this implies $\mathtt{C}_2 \in \mathscr{A}$. Then by induction $R(\Theta_1, \mathscr{A}, \mathtt{B}) \triangleright \Gamma' \vdash_{\mathtt{i}} \mathtt{M} : [\mathtt{TC}(\mathtt{B})]_n \to \mathtt{A}'$ and $R(\Theta_2, \mathscr{A}, \mathtt{B}) \triangleright \Delta' \vdash_{\mathtt{i}} \mathtt{N} : \mathtt{TC}(\mathtt{B})$. Moreover, by definition of the relation $\leq$ between *o*-sets, $\Gamma' \smile_{\mathtt{i}} \Delta'$. So the result follows by rule $(\to_{\mathtt{E}})$. The case where $\{\mathtt{A}_1, \mathtt{A}_2\} \subseteq \mathscr{A}$ follows by induction.

Let Π be:

$$\frac{\Theta_1 \triangleright \Gamma \vdash_{\mathtt{i}} \mathtt{M} : ([\mathtt{B}_1]_n \to \mathtt{A}_1)_1 \quad \Theta_2 \triangleright \Delta \vdash_{\mathtt{i}} \mathtt{N} : \mathtt{B}_2 \quad \Gamma \smile_{\mathtt{i}} \Delta}{\Gamma \uplus (\Delta \uplus \ldots \uplus \Delta)_n \vdash_{\mathtt{i}} \mathtt{MN} : \mathtt{A}_2} (\to_{\mathtt{E}})$$

and $([\mathtt{B}_1]_n \to \mathtt{A}_1)_1 \in \mathscr{A}$. Then $\mathtt{B} = [\mathtt{B}']_m \to \mathtt{A}'$, where $\mathtt{B} \preceq \mathtt{B}'$ and $\mathtt{A} \preceq \mathtt{A}'$. By induction $R(\Theta_1, \mathscr{A}, \mathtt{B}) \triangleright \Gamma' \vdash_{\mathtt{i}} \mathtt{M} : [\mathtt{C}']_m \to \mathtt{A}'$ and $R(\Theta_2, \mathscr{A}, \mathtt{B}) \triangleright \Delta' \vdash_{\mathtt{i}} \mathtt{N} : \mathtt{C}'$. By definition $\Gamma' \smile_{\mathtt{i}} \Delta'$. So the result follows by rule $(\to_{\mathtt{E}})$.

Property 3 *Let $\Pi \triangleright \Gamma \vdash_{\mathtt{i}} \mathtt{M} : \mathtt{A}$. Then, for every o-set $\mathscr{A}$ of occurrences of $\mathtt{A}$, for every $\mathtt{B}$ such that $\mathtt{A} \preceq \mathtt{B}$, $R(\Pi, \mathscr{A}, \mathtt{B}) \triangleright \Gamma' \vdash_{\mathtt{i}} \mathtt{M} : \mathtt{A}'$, for some Γ' and $\mathtt{A}'$.*

Proof. The proof follows from the definition of $R(\Pi, \mathscr{A}, \mathtt{B})$.

Lemma 3 *Let $\Gamma \vdash \mathtt{M} : \mathtt{A}$. There are Γ^* and $\mathtt{A}^*$ such that $\Gamma^* \vdash_{\mathtt{i}} \mathtt{M} : \mathtt{A}^*$, and moreover:*

1. $(\mathtt{A}^*)^\circ = \mathtt{A}$;
2. $(\Gamma)^* = \{\mathtt{x} : \sigma \mid \mathtt{x} : (\sigma)^\circ \in \Gamma\}$.

Proof. By induction on the derivation Π proving $\Gamma \vdash \mathtt{M} : \mathtt{A}$. Let Π be:

$$\frac{}{\mathtt{x} : \mathtt{A} \vdash \mathtt{x} : \mathtt{A}} (\mathtt{var})$$

Between all types $\mathtt{A}' \in \mathscr{T}_{\mathtt{i}}$ such that $(\mathtt{A}')^\circ = \mathtt{A}$, let $\mathtt{A}^*$ be the minimum type w.r.t. $\simeq$ such that $((\mathtt{A}^*))^\circ = \mathtt{A}$. The desired derivation is:

$$\frac{}{\mathtt{x} : [\mathtt{A}^*] \vdash_{\mathtt{i}} \mathtt{x} : \mathtt{A}^*} (\mathtt{var})$$

Let Π be:

$$\frac{\Theta \triangleright \Gamma, \mathtt{x} : \mathtt{A} \vdash \mathtt{M} : \mathtt{B}}{\Gamma \vdash \lambda \mathtt{x}.\mathtt{M} : \mathtt{A} \to \mathtt{B}} (\to_{\mathtt{I}})$$

By induction there is a derivation $(\Theta)^* \triangleright (\Delta)^* \vdash_{\mathtt{i}} \mathtt{M} : (\mathtt{B})^*$, satisfying 1 and 2. So, by 2, $(\Delta)^* = (\Gamma)^*, \mathtt{x} : \sigma$ where $(\sigma)^\circ = \mathtt{A}$, and, by 1, $((\mathtt{B})^*)^\circ = \mathtt{B}$. So $\sigma = [(\mathtt{A})^*]_n$. Then the desired derivation is:

$$\frac{(\Theta)^* \triangleright (\Gamma)^*, \mathtt{x} : [(\mathtt{A})^*]_n \vdash_{\mathtt{i}} \mathtt{M} : (\mathtt{B})^*}{(\Gamma)^* \vdash_{\mathtt{i}} \lambda \mathtt{x}.\mathtt{M} : [(\mathtt{A})^*]_n \to (\mathtt{B})^*} (\to_{\mathtt{I}})$$

and it is easy to check that both the conditions 1 and 2 are satisfied.

Let Π be:

$$\frac{\Theta_1 \triangleright \Gamma \vdash \mathtt{M} : \mathtt{B} \to \mathtt{A} \quad \Theta_2 \triangleright \Delta \vdash \mathtt{N} : \mathtt{B} \quad \Gamma \smile_{\mathtt{i}} \Delta}{\Gamma \cup \Delta \vdash \mathtt{MN} : \mathtt{A}} (\to_{\mathtt{E}})$$

By induction, there are derivations: $(\Theta_1)^* \triangleright (\Gamma)^* \vdash_{\mathtt{i}} \mathtt{M} : (\mathtt{B} \to \mathtt{A})^*$ and $(\Theta_2)^* \triangleright (\Delta)^* \vdash_{\mathtt{i}} \mathtt{N} : (\mathtt{B})^*$, both satisfying conditions 1 and 2. Since $((\mathtt{B} \to \mathtt{A})^*)^\circ = \mathtt{B} \to \mathtt{A}$, by 1 and definition of $(.)^\circ$, $((\mathtt{B} \to \mathtt{A})^*)^\circ = (\sigma)^\circ \to (\mathtt{A}')^\circ$, for some σ and $\mathtt{A}'$ such that $(\sigma)^\circ = \mathtt{B}$ and $(\mathtt{A}')^\circ = \mathtt{A}$. So, by 1, $\sigma = [\mathtt{B}']_n$, for some n, and some $\mathtt{B}'$ such that $(\mathtt{B}')^\circ = \mathtt{B}$. In the derivation Θ_2, let $(\mathtt{B})^* = \mathtt{B}''$: by 1, $(\mathtt{B}'')^\circ = \mathtt{B}$, so $\mathtt{B}'' \simeq \mathtt{B}'$. Moreover, by 2, $(\Gamma)^* \smile_{\mathtt{i}} (\Delta)^*$. Let $\mathscr{B} \in O(\Pi)$ contain the occurrence of $\mathtt{B}$ in the type conclusion of Θ_1. Remember that, by construction, $\mathscr{B}$ contains also the occurrence of $\mathtt{B}$ in the type conclusion of Θ_2. Then the desired derivation is $R(\Pi, \mathscr{B}, \mathtt{merge}(\mathtt{B}', \mathtt{B}''))$.

Theorem 4 $\Gamma \vdash \mathtt{M} : \mathtt{A}$ *implies* $\mathtt{M}$ *is strongly normalizing.*

Proof. By Lemma 3, $\mathtt{M}$ is typable in the system of uniform intersection types. Then the result follows from Theorem 3.

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