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UNIFICATION BASED ON GENERALIZED EMBEDDING

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Abstract. In this paper we introduce a concept of a minimal and complete set of E-unifiers $\lambda U\Sigma_E(\Gamma)$ for a unification problem Γ, based on homeomorphic embedding modulo an equational theory E. We propose a definitional framework based on notions and definitions of standard unification theory of first order terms extended to the (homeomorphic) embedding order modulo E. The main result is that the set $\lambda U\Sigma_E(\Gamma)$ always exists for a finite signature Σ and it is always finite.

1. Introduction

Ordering is a well established concept in mathematics and it plays an important role in many areas of computer science too. *Quasi orderings* (*qo*) and most noteably *well founded quasi orderings* (*wfo)* and *well quasi orderings (wqo)* in particular are of great general interest, see [13]. Probably the most popular application within our own field is the use of certain quasi orders and well quasi orders on first order terms to prove the termination of rewriting rules, see [3, 4].

In the theory of E-unification of terms based on a signature Σ and an equational theory E, the set $\mathcal{U}\Sigma_E(\Gamma)$ denotes the set of all E-unifiers of a unification problem Γ. Of great interest is now to find a complete and minimal subset of $\mathcal{U}\Sigma_E(\Gamma)$, denoted as $\mu\mathcal{U}\Sigma_E(\Gamma)$, from which all other E-unifiers can be obtained.

Equality on terms induced by the equational theory E will be denoted as $=_E$ and the *E-subsumption order on terms* is denoted as $\leq_E$. So, if there are two unifiers τ and σ for the terms s and t, such that $s\tau =_E t\tau$ and $s\sigma =_E t\sigma$ and there is a substitution λ, such that $\tau =_E \sigma\lambda$, then τ is an instance of σ, or σ *subsumes* τ, denoted $\sigma \leq_E \tau$. This led to the notion of *a most general E-unifier* (mgu), that is an E-unifier, which is not an instance of any other E-unifier. The set of most general unifiers is denoted as $\mu\mathcal{U}\Sigma_E(\Gamma)$ and every E-unifier is E-subsumed by some element of $\mu\mathcal{U}\Sigma_E(\Gamma)$, that is, it can be obtained by instantiaton in an automated reasoning process, such as *resolution* [17]. Often we shall drop the E in E-unifiers if it is understood from the context.

To illustrate the role of orderings in E-unification, consider the equational theory A for free semigroups with the axiom of associativity for terms built over a binary function symbol f with $A = \{f(x, f(y, z)) = f(f(x, y), z)\}$. This is also known as the word (or string) algebra and the notation is that of words (strings), where we just drop the function symbol f and have concatenation of symbols.
For example the string unification problem $\Gamma_1 = \{ax =^? xa\}$ has most general unifiers of the form $\sigma_n = \{x \mapsto a^n : n \geqq 1\}$. Because the σ_n are ground substitutions, they are incomparable with respect to the subsumption order, so $\mu\mathcal{U}\Sigma_A(\Gamma_1) = \{\sigma_n : n \geqq 1\}$ is an infinite set and therefore Γ is of unification type *infinitary*. Furthermore, since the subsumption order is not a well quasi order, there are equational theories such that the set of mgus does not exist (see[2][18][9]).

In order to address these problems, we proposed the *encompassment* of terms (see e.g.[11]) to be generalized to the notion of encompassment of unifers and introduced the notion of an

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essential unifier. We say σ is encompassed by τ, $\sigma \sqsubseteq_E \tau$, iff each domain variable x of τ is also a domain variable of σ and $x\tau$ has an instance of $x\sigma$ as a subterm (modulo E). E-unifiers, which do not encompass any other unifier are then called *essential* unifiers and the complete set of essential unifiers is denoted as $e\mathcal{U}\Sigma_E(\Gamma)$ for a unification problem Γ. If $\mu\mathcal{U}\Sigma_E(\Gamma)$ exists, we have $e\mathcal{U}\Sigma_E(\Gamma) \subseteq \mu\mathcal{U}\Sigma_E(\Gamma)$, that is, the encompassment order generalizes the subsumption order and there are even cases where an E-unification problem with an infinite set of mgus reduces to a finite set of essential unifiers [10, 18]. Moreover it can happen, that an equational theory E, for which $\mu\mathcal{U}\Sigma_E(\Gamma)$ does not exist may have a minimal and complete set of essential unifiers $e\mathcal{U}\Sigma_E(\Gamma)$.

For example the unification type of Γ_1 from above changes drastically using the encompassment order: the *essential* unifier $\sigma_1 = \{x \mapsto a\}$ encompasses all the other most general unifiers $\sigma_n = \{x \mapsto a^n\}$, $n > 1$, because $\sigma_1 \sqsubseteq_A \sigma_n$, $n > 1$. More precisely encompassment allows the decomposition $\sigma_n = \lambda_n \sigma_1$, where $\lambda_n = \{x \mapsto a^n x\}$, $n \geqq 0$. So the minimal and complete set of essential unifiers for Γ_1 is $e\mathcal{U}\Sigma_E(\Gamma_1) = \{\sigma_1\}$, that is, it is unitary instead of infinitary as it is under the subsumption ordering.

Nevertheless there are still essentially infinitary string unification problems, as the following example shows. Let $\Gamma_2 = \{xby =^? ayayb\}$ be the string unification problem, which has $eU\Sigma_A(\mathit{\Gamma}_2) = \{\{x \mapsto ab^n a, y \mapsto b^n\} : n > 0\}$ as its minimal and complete set of essential unifiers. The unifiers are incomparable with respect to encompassment, because $ab^n a$ can not be a substring of $ab^m a$ for $m \neq n$. Forthermore, as the encompassment order on unifying substitutions is not a wqo, unfortunately again, there are theories with a solvable unification problem Γ, for which $e\mathcal{U}\Sigma_E(\Gamma)$ does not exist (see [2][8][18]).

This paper deals with a third approach, the extension of the well known *homeomorphic embedding of terms to a homeomorphic embedding modulo E of terms* (also used in [1] with a different definition) and to *a homeomorphic embedding modulo E of substitution*s, called E-embedding of terms or substitutions respectively. Informally, the homeomorphic embedding of terms is understood as follows: Let s and t be terms, then s is syntactically embedded into t, denoted as $s \trianglelefteq t$ iff s=t, or $s \trianglelefteq t_i$ for $t = f(t_1, ..., t_n)$ and some i or $s_i \trianglelefteq t_i$ for $s = f(s_1, ..., s_n)$ and all i. For example $f(x, b) \trianglelefteq \boldsymbol{f}(g(a, \boldsymbol{x}), f(x, \boldsymbol{b}))$ and also $f(x, b) \trianglelefteq \boldsymbol{f}(f(a, h(\boldsymbol{x})), f(\boldsymbol{b}, a))$ and $f(a, x) \trianglelefteq \boldsymbol{f}(g(\mathbf{a}, b), \boldsymbol{x})$, but $f(a, b) \ntrianglelefteq \boldsymbol{f}(g(\boldsymbol{a}, b), x)$.

The E-embedding order for terms, denoted as $\trianglelefteq_E$, will be lifted to an E-embedding order for substitutions similar to the encompassment order in [18]. We define $\sigma \trianglelefteq_E \tau$ iff each domain variable x of τ is also a domain variable of σ and $x\tau$ homeomorphically E-embeds $x\sigma$, that is if $\tau = \{x_i \mapsto t_i\}$ and $\sigma = \{x_i \mapsto s_i\}$, $1 \leqq i \leqq n$, then $\sigma \trianglelefteq_E \tau$ iff $s_i \trianglelefteq_E t_i$. To illustrate the effect of this E-embedding order, take $\mathit{\Gamma}_2$ from above as an example, where E is the equational theory A for strings. In this case $aba \trianglelefteq_A ab......ba$ and $b \trianglelefteq_A b.....b$, hence with $\sigma_1 = \{x \mapsto aba, y \mapsto b\}$ we have $\sigma_1 \trianglelefteq_A \sigma_n$ for all $n > 1$. Consequently σ_1 is the only unifier and the set of embedment free unifiers for Γ_2 is $\lambda\mathcal{U}\Sigma_A(\Gamma_2) = \{\sigma_1\}$ and it is finite. In fact it can be shown that the theory is even *unitary* instead of infinitary under the subsumption and encompassment order [8, 9].

But in order to generalize the encompassment order for terms to the embedment order for unification problems, we need a more general notion of embedment. This is achieved by defining, that a term s is *E-instance-embedded* into a term t iff an instance of s, say $s\lambda$, is E-embedded into t, which we call *$\lambda-$embedded modulo E* or *λ_E-embedding*. This is denoted as $s \lessdot_E t$. Furthermore E-unifiers, which have no E-instance embedded unifier are called *embedment free E-unifiers* or *free λ_E-unifiers* and the complete set of free λ_E-unifiers is denoted as $\lambda U\Sigma_E(\Gamma)$ for a unification problem Γ.

In the following we introduce the concept of a minimal and complete set of E-unifiers based on λ_E-embedding and propose a definitional framework based on notions and definitions of standard unification theory extended to the (homeomorphic) E-embedding order.

2. Notions and Notation

Notation and basic definitions in unification theory are well known and have found their way into many and diverse academic fields and most monographs and textbooks on automated reasoning have sections on unification. In the full paper to be published in a journal we unify the various presentations of the necessary concepts for unification towards a concise notation which serves our purpose and we show how the additional concepts for ordering E-unifiers based on homeomorphic embedding can be built upon these definitions. These sections as well as several proofs and details are deleted in this workshop paper.

Our main interest is to show that the set of free λ_E-unifiers always exists and the main technique to show this result is based on orderings, in particular on well quasi orderings.

Definition 1. A *quasi order* is a relation that is *reflexive* and *transitive.*

A term t is an *instance* of a term s, denoted $s \leq t$, if $s\sigma = t$ for some substitution σ

$$s \leq t \iff \exists \sigma : s\sigma = t$$

We also say *s (syntactically) subsumes t* and this relation is a quasi order (or preorder as it is sometimes called). We call it the *subsumption order* on terms.
A term t (syntactically) *encompasses* a term s, denoted $s \sqsubseteq t$, if an instance of s is a subterm of t. With **Sub(t)**, the set of all subterms of t, we have:

$$s \sqsubseteq t \iff \exists \sigma : s\sigma \in \mathbf{Sub}(t)$$

Encompassment conveys the notion that s appears in t with a context "above" and a substitution instance "below". We say *t encompasses s* or *s* is *encompassed* by *t* and $\sqsubseteq$ is the *encompassment order*. In particular $s \sqsubset t$ is called *strict encompassment*, if $s\sigma$ is a proper subterm of t.

A term s is *homeomorphically embedded* into t iff s can be obtained from t by erasing some parts in t. We abbreviate "homeomorphical embedding" just to "embedding". Embedment conveys the notion that the structure of s and some corresponding symbols appear within t. A term s is *instance-embedded* into t, we also say it is λ*-embedded* into t, iff an instance of s, i.e. $s\lambda$, is embedded into t. This is the main notion of this paper, which we will generalize to embedment of substitutions later on.[1]
Using $s \trianglelefteq t$ to denote that s is a subterm of t, we have the following *orders on terms,* extended to equality modulo E for the congruences induced by the equations in E. *:*

Definition 2. (ordering terms modulo E)

(1) A term s is an *E-subterm of t, denoted* $s \preceq_E t$, iff there is an $s' =_E s$ and a term $t' =_E t$ such that $s' \preceq t'$.
(2) A term s *E-subsumes t,* $s \leqslant_E t$, iff there exists a substitution σ with $s\sigma =_E t$.
(3) A term s is *E-encompassed by t*, $s \sqsubseteq_E t$ iff there is a substitution σ such that $s\sigma \preceq_E t$.
(4) A term s is *E-embedded* into a term t, denoted $s \trianglelefteq_E t$, if $s =_E t$, or there is a term $s' =_E s$ and a term $t' =_E t$ such that s' is syntactically embedded into t':

$$s \trianglelefteq_E t \iff \begin{cases} s =_E t, \; or \\ t = f(t_1, \ldots, t_n) \; and \; \exists s' \in [s]_E \; and \; \exists t'_i \in [t_i]_E : \; s' \trianglelefteq t'_i, \; or \\ t = f(t_1, \ldots t_n), s = f(s_1, \ldots s_n), \\ \quad and \; \forall i : \; s'_i \trianglelefteq t'_i, \; where \; s' \in [s_i]_E, \; t' \in [t_i]_E. \end{cases}$$

(5) A term s is *E-instance-embedded* into t, denoted $s \lessdot_E t$, if an instance of s is E-embedded into t , that is $s\lambda \trianglelefteq_E t$ for a substitution λ. We say s is λ_E-embedded.

Theorem 3. *The E-embedment order,* $\trianglelefteq_E$*, is a quasi order on terms.*

Proof. Let r, s, t be terms.
reflexivity: $r \trianglelefteq_E r$ is obvious, because terms embed themselves.
transitivity: $r \trianglelefteq_E s \trianglelefteq_E t \Longrightarrow r \trianglelefteq_E t$.
By Definition 2.(4) $r \trianglelefteq_E s \Longrightarrow \exists r' \in [r]_E \; and \; \exists s' \in [s]_E : \; r' \trianglelefteq s'$ and
$s \trianglelefteq_E t \Longrightarrow \exists s'' \in [s]_E$ and $\exists t'' \in [t]_E : \; s'' \trianglelefteq t''$.
Now $r' \trianglelefteq s' =_E s'' \; and \, s'' \trianglelefteq t'' \Longrightarrow r' \trianglelefteq_E t''$.
Hence $r =_E r' \trianglelefteq_E t'' =_E t \Longrightarrow r \trianglelefteq_E t$.

□

Definition 4. (ordering of substitutions modulo E restricted to a set of variables)
Let V be some set of variables.

[1]Signs and notation are still not uniform in all related fields; our notation is used more often in the literature on automated theorem proving and unification theory, whereas term rewriting systems usually prefer notational conventions as proposed in [5] and [6].

(1) A substitution σ is a *sub-substitution modulo E* of τ, denoted as $\sigma \leq_E^V \tau$, if $\mathbf{Dom}(\sigma) = \mathbf{Dom}(\tau)$ and these are variables in V and for all x in this domain $x\sigma$ is an E-subterm of $x\tau$, i.e. $x\sigma \leq_E x\tau$.
(2) A substitution σ *E-subsumes* a substitution τ *restricted to* V, denoted as $\sigma \leq_E^V \tau$, if there exists a substitution λ such that $\sigma\lambda =_E^V \tau$. The relation $\leq_E^V$ is called the *E-subsumption order for substitutions* restricted to V.
We denote *E-subsumption equivalence* as $\sigma \sim_E^V \tau$, if $\sigma \leq_E^V \tau$ *and* $\tau \leq_E^V \sigma$.
(3) A substitution σ is *E-encompassed* by τ *restricted to* V, denoted $\sigma \sqsubseteq_E^V \tau$, if there exists λ, such that $(\sigma\lambda)\mid_V$ is a sub-substitution of τ modulo E restricted to V. We denote *E-encompassment equivalence* as $\sigma \approx_E^V \tau$ if $\sigma \sqsubseteq_E^V \tau$ *and* $\tau \sqsubseteq_E^V \sigma$.
(4) A substitution σ is *E-embedded* into a substitution τ, denoted as $\sigma \trianglelefteq_E^V \tau$, iff $\mathbf{Dom}(\sigma) = \mathbf{Dom}(\tau)$ and for all x in this domain we have $x\sigma \trianglelefteq_E^V x\tau$.
(5) A substitution σ is λ_E*-embedded* into a substitution τ, denoted as $\sigma \preccurlyeq_E^V \tau$, iff $\mathbf{Dom}(\sigma) = \mathbf{Dom}(\tau)$ and there is a substitution λ, such that $\forall x \in V : x(\sigma\lambda)\mid_V$ is E-embedded into $x\tau$.

The encompassment and embedment order on terms are well known as quasi orderings, but the *modulo E* extension to substitutions requires verification.

Theorem 5. *The E-encompassment order is a quasi order on substitutions.*

For a proof see [18] and the early proof in [10].

Theorem 6. *The E-embedment order is a quasi order on substitutions.*

Proof. This is shown by lifting Theorem 3 for terms componentwise to substitutions.

□

Theorem 7. *The* λ_E*-embedment order* $\preccurlyeq_E$ *is a quasi order on terms.*

Proof. *Let* r, s, t *be terms:*
reflexivity: is obvious because every term λ-embeds itself.
transitivity: we show $r \preccurlyeq_E s \preccurlyeq_E t$ implies $r \preccurlyeq_E t$.
By Definition 2.(5) we have:
$r \preccurlyeq_E s$ implies $\exists\sigma : r\sigma \trianglelefteq_E s$ and
$s \preccurlyeq_E t$ implies $\exists\tau : s\tau \trianglelefteq_E t$. Furthermore:
$r\sigma \trianglelefteq_E s \Longrightarrow \exists\widetilde{r} \in [r\sigma]_E$ and $\exists s' \in [s]_E : \widetilde{r} \trianglelefteq s'$ and
$s\tau \trianglelefteq_E t \Longrightarrow \exists\widetilde{s} \in [s\tau]_E$ and $\exists t' \in [t]_E : \widetilde{s} \trianglelefteq t'$.
Since it can be shown that $\trianglelefteq$ is substitution-composable from the right we have
$\widetilde{r} \trianglelefteq s' \Longrightarrow \widetilde{r}\tau \trianglelefteq s'\tau$ and $s'\tau =_E \tilde{s} \trianglelefteq t' \Longrightarrow s'\tau \trianglelefteq_E t'$.
Now: $\widetilde{r}\tau \trianglelefteq s'\tau \trianglelefteq_E t'$ and transitivity of $\trianglelefteq_E$ implies $\widetilde{r}\tau \trianglelefteq_E t'$
and $r(\sigma\tau) \trianglelefteq_E s'\tau \trianglelefteq_E t' =_E t$ and transitivity of $\trianglelefteq_E$
implies $r(\sigma\tau) \trianglelefteq_E t$. But this means $r \preccurlyeq_E t$.

□

Theorem 8. *The* λ_E*-embedment order is a quasi order on substitutions.*

Proof. Similar to Theorem 7 by lifting it componentwise to substitutions.

□

Our interest in this paragraph is on quasi orderings and the next definition lists some well known notions, see [12, 16].

Definition 9. Let $\leq$ be a quasi ordering on a set S, then:

(1) An infinite sequence of elements of S, $a_1, a_2, a_3, ...$ is called a $\leq$*−chain* if $a_i \leq a_{i+1}$ for all $i \geq 1$. The sequence $a_1, a_2, a_3, ...$ is said to *contain a chain* if it has a subsequence that is a chain.
(2) The infinite sequence $a_1, a_2, a_3, ...$ is called an *antichain* if neither $a_i \leq a_j$ nor $a_j \leq a_i$, for all $1 \leq i, j$ and i≠j.
(3) The quasi ordering $\leq$ is *well-founded* (wfo) if it contains no infinite strictly descending <-chain; that is, there is no infinite sequence $a_1, a_2, a_3, ...$ of elements of S such that $a_i > a_{i+1}$ for every i *in* $\mathbb{N}$.

(4) A *well-quasi-ordering on S* (wqo), $\leq$, is a quasi-ordering which is well-founded and it has no infinite antichains in S with respect to $\leq$.

The following ***Tree Theorem*** due to Kruskal states that the set of finite trees over a well-quasi-ordered set of labels is itself well-quasi-ordered under homeomorphic embedding. He uses a notation where T(Y) denotes the collection of all (structured) trees over an alphabet Y.

Theorem 10. ***The Tree Theorem***.
If Y is well quasi ordered then T(Y) is well quasi ordered too.

Proof. See Joseph B. Kruskal [12] and the more elegant proof by Crispin Nash-William [16] □

The following theorem is a consequence of the tree theorem for the set of first order terms $T(F,X)$, built over a finite set of function symbols F and a finite set of variable symbols X. Hereby we refer to the work of Jean H. Gallier and M. Leuschel [7, 15]. They discuss the proof that "Given a finite alphabet $\Sigma = F \cup X$ which is well quasi ordered (in our case by equality) then $\trianglelefteq$ is also a well quasi order on $T(F, X)$". The next theorem is a genaralisation to "modulo E".

Theorem 11. *Let E be an equational theory. The E-embedding relation , $\trianglelefteq_E$, is a well quasi order on the set of terms built over a* ***finite*** *signature.*

Proof. (Sketch)
(i) $\trianglelefteq_E$ is *well founded*.
If not, then there exists an infinite strictly descending $\triangleright_E$ −chain: $t_1 \triangleright_E t_2 \triangleright_E t_3 \triangleright_E ....$
which has the more detailed form: $t'_1 \triangleright t'_2 =_E t''_2 \triangleright t'_3 =_E t''_3 \triangleright t'_4 ....$ Now take the following infinite sub sequence of terms t'_1, t'_2, t'_3, ... Because of Theorem 10 there are two indices i, j, i<j such that t_i is embedded into t_j, hence the chain above can not be infinite. Thus $\trianglelefteq_E$ is well founded.
(ii) There are *no antichains* with respect to $\trianglelefteq_E$.
Otherwise there is an $\trianglelefteq_E$-antichain $s_1, s_2, s_3, ...$ with respect to $\trianglelefteq_E$ and it can be shown that in this case there exists a corresponding infinite sequence of terms s'_1, s'_2,s'_3, ... , where $s'_i \in [s_i]_E, i \geq 1$, which are incomparable and this again contradicts Kruskal's theorem. □

E-unification of first order terms is based on an *infinite* set of variable symbols and it is well known, that the embedding order of terms with an infinite set of variable symbols is not a well quasi order, since we have the antichain $x_1, x_2, x_3, ..$ Of course the same is the case then for embedment modulo E.
But fortunately *well foundedness* of the embedding ordering is still valid, since the number of symbols decreases in a strictly descending $\triangleright$ −chain .

Theorem 12. *Let E be an equational theory. E-embedding, $\trianglelefteq_E$, is a well founded quasi order on the set of terms.*

Proof. The proof is based on the fact that E-equivalent terms do not have new variable symbols. □

The next Theorem is similar and shows that λ_E-embedding is well foundned too.

Theorem 13. *Let E be an equational theory. λ_E−embedding, $\leqslant_E$, is a well founded quasi order on the set of terms.*

3. Ordering E-unifiers under homeomorphic embedding

We shall now look at unification under λ_E-embedding, which is our main interest in this paper, and we start with a recapitulation of the standard notions of E-unification.

3.1. **E-Unification.** Let E be an equational theory and let Σ be the signature of the term algebra. An *E-unification problem* is a finite set of equations

$$\Gamma = \{s_1 =^?_E t_1, \ldots, s_n =^?_E t_n\}$$

An *E-unifier* for Γ is a substitution σ such that

$$s_1\sigma =_E t_1\sigma, \ldots, s_n\sigma =_E t_n\sigma$$

The set of all E-unifiers of Γ is denoted $\mathcal{U}\Sigma_E(\Gamma)$. A *complete* set of E-unifiers $c\mathcal{U}\Sigma_E(\Gamma)$ for Γ is a set of E-unifiers, such that for every E-unifier τ there exists $\sigma \in c\mathcal{U}\Sigma_E(\Gamma)$ with $\sigma \leq_E \tau$. The set $\mu\mathcal{U}\Sigma_E(\Gamma)$ is called a *minimal complete set* of E-unifiers for Γ, if it is complete and for all distinct elements σ and σ' in $\mu\mathcal{U}\Sigma_E(\Gamma)$ if $\sigma \leq_E \sigma'$ then $\sigma =_E \sigma'$.

3.2. **E-Unifiers ordered by Homeomorphic Embedding.** This paper is based on the observation that certain solutions *embed* the instances of other solutions. This then leads to the notion of *(embedment-) free E-unifiers,* where free E-unifiers are the elements of our new minimal and complete set of E-unifiers, which we denote as $\lambda\mathcal{U}\Sigma_E(\Gamma)$.

Definition 14. Let E be an equational theory, Γ a solvable E-unification problem and let $U\Sigma_E(\Gamma)$ be the set of E-unifiers for Γ. If an E-unifier σ in $U\Sigma_E(\Gamma)$ does *not* have any instance E-embedded unifier (any λ_E-embedded unifier), then σ is called a *free λ_E-unifier*. The minimal and complete set of free λ_E-unifiers will be denoted as $\lambda U\Sigma_E(\Gamma)$.

Theorem 15. *For first order terms built over a* ***finite*** *signature Σ and a solvable unification problem Γ and an equational theory E: The set of free λ-unifiers, $\lambda U\Sigma_E(\Gamma)$, exists and it is minimal, complete and finite.*

Proof. The proof is based on Theorem 11

□

It is well known that the set of terms can not be well quasi ordered since we usually have an infinite set of variables and they form an antichain. That is, we can not use Theorem 11. But it may be possible for an automated deduction system to set a limit to the number of variables involved in the search for a proof and the above theorem would be still useful.

Unfortunately we do not know if the relation $\leqslant_E$ is wqo[2]. But we know it is wellfounded and hence we have:

Theorem 16. *For a signature Σ, a solvable unification problem Γ and an equational theory E: The set of free λ_E-unifiers, $\lambda U\Sigma_E(\Gamma)$, exists and is minimal and complete. But it is not necessarily finite .*

Finally there is a standard trick used in logic programming [14] [15] as well as in termination research for term rewriting systems [5], namely to disregard the name of a variable and simply view all variables as the same entity. This observation led to the notion of *pure embedding,* which we abbreviate to *π-embedding* in the following and it will be denoted as $s \trianglelefteq^\pi t$. As before we generalize embedding to instance embedding or π-embedding by saying a term s is *π-embedded* into a term t, if it is λ-embedded and in addition the names of the variable symbols are ignored. It is defined as follows:

Definition 17. (*Pure E-embedding*)

(1) A term s is *π_E-embedded* into a term t, denoted $s \trianglelefteq^\pi_E t$, if s and t are variables, or $s =_E t$ or there is a term $s'_i =_E s_i$ and a term $t'_i =_E t_i$ such that s'_i is π_E- embedded into t'_i :

$$s \trianglelefteq^\pi_E t \iff \begin{cases} s =_E t, \ \textit{or s and t are variables or} \\ t = f(t_1, \ldots, t_n) \ \textit{and} \ \exists s' \in [s]_E \ \textit{and} \ \exists t'_i \in [t_i]_E : s' \trianglelefteq^\pi_E t'_i \ \textit{or} \\ t = f(t_1, \ldots t_n) \ \textit{and} \ s = f(s_1, \ldots s_n) \\ \qquad \textit{and} \ \forall i : s'_i \trianglelefteq^\pi_E t'_i, \ \textit{where} \ s' \in [s_i]_E, \ t' \in [t_i]_E. \end{cases}$$

(2) A term s is instance π_E-embedded into a term t, denoted $s \leqslant^\pi_E t$, if an instance of s is π_E-embedded into t , that is $s\lambda \trianglelefteq^\pi_E t$ for a substitution λ.

[2]We have not been able to prove it (yet) nor to disprove it.

(3) A substitution σ is *instance* π_E*−embedded* into a substitution τ for a set of variables V, denoted as $\sigma \leqslant_E^{\pi V} \tau$, iff $\mathbf{Dom}(\sigma) = \mathbf{Dom}(\tau)$ and there is a substitution λ, such that $\forall x \in V : x(\sigma\lambda) |_V$ is π-embedded into $x\tau$.

Since instance π-embedding is a special case of λ-embedding we have (using theorem 11) that π-embedding, $s \trianglelefteq^\pi t$, is a well quasi order on the set of terms and it is now easy to show that π-embedding is a well quasi order on the set of substitutions as well.

Theorem 18. *π_E-embedding is a well quasi order on the set of substitutions.*

The final step is now to extend π_E-embedding of substitutions to instance π_E-embedding.

Theorem 19. *Instance π_E-embedding is a well quasi order on the set of substitutions.*

E-unifiers which do not contain any instance π_E-embedded unifiers are called *free π_E unifiers* and this set is denoted as $\pi U\Sigma_E(\Gamma)$.

Our main result now follows from these theorems, but note the completeness proof is more complex than ususal, because we need a generator to compute all unifiers from $\pi U\Sigma_E(\Gamma)$.

Theorem 20. *For first order terms built over a signature Σ, a solvable unification problem Γ and an equational theory E: The set of free π_E-unifiers, $\pi U\Sigma_E(\Gamma)$, exists and is minimal, complete and finite.*

4. Conclusion

These results do not imply that we have a general way of **efficiently** generating $\lambda U\Sigma_E(\Gamma)$ nor $\pi U\Sigma_E(\Gamma)$, which is unlikely to be found in general. We need to look for an appropriate algorithm for each specific theory E, just as in standard unification theory and this has not been done yet.

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