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About the unification type of topological logics over Euclidean spaces

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1 Introduction

Topological logics (TLs) are formalisms for reasoning about topological relations (contact, connectedness, etc) between regions [5, 13, 14, 15]. Their languages are obtained from the language of Boolean algebras by the addition of predicates representing these relations. Interpreted over mereotopological spaces, the formulas of these languages describe configurations of concrete objects. Recently, the validity problem determined by different classes of mereotopological spaces has been intensively investigated [8, 9, 10].
In this paper, we introduce a new inference problem for TLs, the unifiability problem, which extends the validity problem by allowing one to replace variables by terms before testing for validity. For example, within the context of the mereotopology of all regular closed polygons of the real plane, the formula $C(p, q) \rightarrow x \not\equiv 0 \wedge x \leq p \cup q$, read "if p is in contact with q then x is nonempty and x is contained in $p \cup q$", is not valid but can be made valid after replacing x either by $p \cup (q \cap x)$, or by $q \cup (p \cap x)$.
There is a wide variety of situations where unifiability problems arise. Suppose the formula $\varphi(p_1, \ldots, p_m)$ describes a given geographic configuration of constant regions $p_1, \ldots, p_m$ and the formula $\psi(x_1, \ldots, x_n)$ represents a desirable geographic property of variable regions $x_1, \ldots, x_n$. It may happen that $\varphi(p_1, \ldots, p_m) \rightarrow \psi(x_1, \ldots, x_n)$ is not valid in the considered geographic environment. Hence, one may ask whether there are n-tuples $(a_1, \ldots, a_n)$ of terms such that $\varphi(p_1, \ldots, p_m) \rightarrow \psi(a_1, \ldots, a_n)$ is valid in this environment. Moreover, one may be interested to obtain, if possible, the most general n-tuples $(a_1, \ldots, a_n)$ of terms such that $\varphi(p_1, \ldots, p_m) \rightarrow \psi(a_1, \ldots, a_n)$ is valid.
In this paper, we adapt to the problem of unifiability with constants in TLs (interpreted over the mereotopology of all regular closed polygons of the real plane) the line of reasoning developed by Balbiani and Gencer [4] within the simpler context of the problem of unifiability without constants in Boolean Region Connection Calculus (interpreted over Kripke models). This adaptation is far from obvious. Our main result is that, within the context of the mereotopology of all regular closed polygons of the real plane, unifiable formulas always have finite complete sets of unifiers.

2 Syntax

Terms Let CON be a countable set of *constants* (p, q, etc) and VAR be a countable set of *variables* (x, y, etc). Let $(p_1, p_2, \ldots)$ be an enumeration of CON without repetitions and $(x_1, x_2, \ldots)$ be an enumeration of VAR without repetitions. An *atom* is either a constant, or a variable. The *Boolean terms* (a, b, etc) are defined by the rule

- $a, b ::= p \mid x \mid 0 \mid a^{\star} \mid (a \cup b)$.

The other Boolean constructs for terms (for instance, 1 and $\cap$) are defined as usual. Reading terms as regions, the constructs 0, * and $\cup$ should be regarded as the empty region, the complement operation and the union operation. As a result, the constructs 1 and $\cap$ should be regarded as the full region and the intersection operation. For all $m, n \geq 0$, let $TER_{m,n}$ be the set of all terms whose constants form a subset of $\{p_1, \ldots, p_m\}$ and whose variables form a subset of $\{x_1, \ldots, x_n\}$. Let TER be the set of all terms.

Formulas The *formulas* (φ, ψ, etc) are defined by the rule

- $\varphi, \psi ::= C(a, b) \mid a \equiv b \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi)$.

Here, a and b are terms whereas C is the predicate of contact and $\equiv$ is the predicate of equality. We use the notation $a \leq b$ for $a \cup b \equiv b$. For $C(a, b)$ and $a \equiv b$, we propose the readings "a is in contact with b" and "a is equal to b". As a result, for $a \leq b$, we propose the reading "a is contained in b". The other connectives for formulas (for instance, $\top$ and $\wedge$) are defined as usual. A formula is *equational* iff $\equiv$ is the only predicate possibly occurring in it. Let FOR be the set of all formulas and FOR^{eq} be the set of all equational formulas. Note that FOR and FOR^{eq} are denoted $\mathcal{C}$ and $\mathcal{B}$ in [8, 9, 10].

3 Semantics

Topological spaces A *topological space* is a structure of the form (X, τ) where X is a nonempty set and τ is a set of subsets of X such that the following conditions hold:

- $\emptyset$ is in τ,
- X is in τ,
- if $\{A_i : i \in I\}$ is a finite subset of τ then $\bigcap\{A_i : i \in I\}$ is in τ,
- if $\{A_i : i \in I\}$ is a subset of τ then $\bigcup\{A_i : i \in I\}$ is in τ.

The subsets of X in τ are called *open sets* whereas their complements are called *closed sets.* In this paper, we will interest with the topological space $(\mathbb{R}^2, \tau_{\mathbb{R}^2})$, i.e. the real plane $\mathbb{R}^2$ together with its ordinary topology $\tau_{\mathbb{R}^2}$.

Regular closed subsets Let (X, τ) be a topological space. Let Int_τ and Cl_τ denote the *interior operator* and the *closure operator* in (X, τ). A subset A of X is *regular closed* iff $Cl_\tau(Int_\tau(A)) = A$. Regular closed subsets of X will also be called *regions.* It is well-known that the set $RC(X, \tau)$ of all regular closed subsets of X forms a Boolean algebra $(RC(X, \tau), 0_X, \star_X, \cup_X)$ where for all $A, B \in RC(X, \tau)$:

- $0_X = \emptyset$,
- $A^{\star_X} = Cl_\tau(X \setminus A)$,
- $A \cup_X B = A \cup B$.

As a result, for all $A, B \in RC(X, \tau)$, $1_X = X$ and $A \cap_X B = Cl_\tau(Int_\tau(A \cap B))$. Since regions are regular closed subsets of X, therefore two regions are *in contact* iff they have a nonempty intersection. For this reason, we define the relation $C^{(X,\tau)}$ on $RC(X, \tau)$ by

- $C^{(X,\tau)}(A, B)$ iff $A \cap B \neq \emptyset$.

The following conditions hold for all $A, B, A', B' \in RC(X, \tau)$:

- if $C^{(X,\tau)}(A,B)$ and $A \subseteq A'$ then $C^{(X,\tau)}(A',B)$,
- if $C^{(X,\tau)}(A,B)$ and $B \subseteq B'$ then $C^{(X,\tau)}(A,B')$,
- if $C^{(X,\tau)}(A \cup A',B)$ then either $C^{(X,\tau)}(A,B)$, or $C^{(X,\tau)}(A',B)$,
- if $C^{(X,\tau)}(A,B \cup B')$ then either $C^{(X,\tau)}(A,B)$, or $C^{(X,\tau)}(A,B')$,
- if $C^{(X,\tau)}(A,B)$ then $A \neq \emptyset$ and $B \neq \emptyset$,
- if $A \neq \emptyset$ then $C^{(X,\tau)}(A,A)$,
- if $C^{(X,\tau)}(A,B)$ then $C^{(X,\tau)}(B,A)$.

Mereotopologies Let (X,τ) be a topological space. A *mereotopology* over (X,τ) is a Boolean subalgebra M of $RC(X,\tau)$ such that for all $P \in X$ and for all $A \in \tau$, if $P \in A$ then there exists $B \in M$ such that $P \in B$ and $B \subseteq A$. A *mereotopological space* over (X,τ) is a structure (X,τ,M) where M is a mereotopology over (X,τ) [12]. Over the topological space $(\mathbb{R}^2,\tau_{\mathbb{R}^2})$, several mereotopologies can be considered. One can consider the mereotopology consisting of the set $RC(\mathbb{R}^2)$ of all regular closed subsets of $\mathbb{R}^2$. Nevertheless, as regions are supposed to be parts of the real plane occupied by concrete objects, it is clear that some of the regular closed subsets of $\mathbb{R}^2$ cannot count as regions. For this reason, one can consider the more concrete mereotopology consisting of the set $RCS(\mathbb{R}^2)$ of all regular closed semi-algebraic subsets of $\mathbb{R}^2$, i.e. those regular closed subsets of $\mathbb{R}^2$ definable by a first-order formula in the language of arithmetic interpreted over $\mathbb{R}$. The main property of this mereotopology is that any of its elements is a finite union of semi-algebraic cells, i.e. semi-algebraic subsets of $\mathbb{R}^2$ homeomorphic to a closed disc. But $RCS(\mathbb{R}^2)$ is not the only candidate for a region-based model of space. A simpler candidate is the mereotopology consisting of the set $RCP(\mathbb{R}^2)$ of all regular closed polygons of $\mathbb{R}^2$, i.e. those regular closed subsets of $\mathbb{R}^2$ definable by a finite union of finite intersections of closed half-planes. Although this mereotopology may seem overly simple, its study from the perspective of the unifiability problem will turn out to be relatively interesting.

Models Let (X,τ,M) be a mereotopological space. A *valuation* on (X,τ,M) is a map associating with every atom a regular closed subset of X in M. Given a valuation $\mathcal{V}$ on (X,τ,M), we define:

- $\bar{\mathcal{V}}(p) = \mathcal{V}(p)$,
- $\bar{\mathcal{V}}(x) = \mathcal{V}(x)$,
- $\bar{\mathcal{V}}(0) = \emptyset$,
- $\bar{\mathcal{V}}(a^\star) = Cl_\tau(X \setminus \bar{\mathcal{V}}(a))$,
- $\bar{\mathcal{V}}(a \cup b) = \bar{\mathcal{V}}(a) \cup \bar{\mathcal{V}}(b)$.

As a result, $\bar{\mathcal{V}}(1) = X$ and $\bar{\mathcal{V}}(a \cap b) = Cl_\tau(Int_\tau(\bar{\mathcal{V}}(a) \cap \bar{\mathcal{V}}(b)))$. Thus, $\mathcal{V}$ interprets every term as a regular closed subset of X in M. A *model* on (X,τ,M) is a structure $\mathcal{M} = (X,\tau,M,\mathcal{V})$ where $\mathcal{V}$ is a valuation on (X,τ,M). The connectives $\perp$, $\neg$ and $\vee$ being classically interpreted, the *satisfiability* of $\varphi \in FOR$ in $\mathcal{M}$ (in symbols $\mathcal{M} \models \varphi$) is defined as follows:

- $\mathcal{M} \models C(a,b)$ iff $C^{(X,\tau)}(\bar{\mathcal{V}}(a),\bar{\mathcal{V}}(b))$,
- $\mathcal{M} \models a \equiv b$ iff $\bar{\mathcal{V}}(a) = \bar{\mathcal{V}}(b)$.

As a result, $\mathcal{M} \models a \leq b$ iff $\bar{\mathcal{V}}(a) \subseteq \bar{\mathcal{V}}(b)$.

Validity Let (X, τ, M) be a mereotopological space. A formula φ is *valid* in (X, τ, M) iff for all valuations $\mathcal{V}$ on (X, τ, M), $(X, \tau, M, \mathcal{V}) \models \varphi$. Let $\mathcal{C}$ be a class of mereotopological spaces. A formula φ is $\mathcal{C}$*-valid* iff for all mereotopological spaces (X, τ, M) in $\mathcal{C}$, φ is valid in (X, τ, M). The $\mathcal{C}$*-validity problem* consists in determining whether a given formula is $\mathcal{C}$-valid. In this paper, we will be interested in the polygon-based mereotopological space $(\mathbb{R}^2, \tau_{\mathbb{R}^2}, RCP(\mathbb{R}^2))$ over $(\mathbb{R}^2, \tau_{\mathbb{R}^2})$. As a result, when we write "valid", we mean "valid in the mereotopological space $(\mathbb{R}^2, \tau_{\mathbb{R}^2}, RCP(\mathbb{R}^2))$".

Proposition 1. *For all $\varphi \in FOR^{eq}$, the following are equivalent:* (**1**) *φ is valid;* (**2**) *for all finite Boolean algebras $\mathcal{B}$ and for all valuations $\mathcal{V}_{\mathcal{B}}$ on $\mathcal{B}$, $(\mathcal{B}, \mathcal{V}_{\mathcal{B}}) \models \varphi$;* (**3**) *for all Boolean algebras $\mathcal{B}$ and for all valuations $\mathcal{V}_{\mathcal{B}}$ on $\mathcal{B}$, $(\mathcal{B}, \mathcal{V}_{\mathcal{B}}) \models \varphi$.*

4 Unification

Substitutions A *substitution* is a function $\sigma : VAR \longrightarrow TER$ which moves at most finitely many variables. The *domain* of a substitution σ (in symbols *dom*(σ)) is the set of variables σ moves. Given a substitution σ, let $\bar{\sigma} : TER \cup FOR \longrightarrow TER \cup FOR$ be the endomorphism such that for all variables x, $\bar{\sigma}(x) = \sigma(x)$. The *composition* of the substitutions σ and τ is the substitution $\sigma \circ \tau$ such that for all $x \in VAR$, $(\sigma \circ \tau)(x) = \bar{\tau}(\sigma(x))$. For all $m, n \geq 0$, let $\Sigma_{m,n}$ be the set of all substitutions σ such that $dom(\sigma) \subseteq \{x_1, \ldots, x_n\}$ and for all positive integers $i \leq n$, $\sigma(x_i)$ is in $TER_{m,n}$. A substitution σ is *equivalent* to a substitution τ (in symbols $\sigma \simeq \tau$) iff for all variables x, $\sigma(x) \equiv \tau(x)$ is valid. Obviously, $\simeq$ is reflexive, symmetric and transitive on the set of all substitutions. A substitution σ is *more general* than a substitution τ (in symbols $\sigma \preceq \tau$) iff there exists a substitution υ such that $\sigma \circ \upsilon \simeq \tau$. Obviously, $\preceq$ is reflexive and transitive on the set of all substitutions. Moreover, it contains $\simeq$. A set of substitutions is *small* iff it contains finitely many non-pairwise equivalent substitutions modulo $\simeq$.

Proposition 2. *For all $m, n \geq 0$, $\Sigma_{m,n}$ is small.*

Unifiable formulas A formula φ is *unifiable* iff there exists a substitution σ such that $\bar{\sigma}(\varphi)$ is valid. In that case, we say that σ is a *unifier* of φ. The *unifiability problem* (in symbols $\mathcal{UNIF}$) consists in determining whether a given formula is unifiable [3]. A set of unifiers of $\varphi \in FOR$ is *complete* iff for all unifiers σ of φ, there exists a unifier τ of φ in that set such that $\tau \preceq \sigma$. An important question in unification theory is [6]: when a formula is unifiable, has it a minimal complete set of unifiers? When the answer is "yes", how large is this set?

Unification types A unifiable formula φ is *finitary* iff there exists a finite complete set of unifiers of φ but there exists no with cardinality 1. A unifiable formula φ is *unitary* iff there exists a unifier σ of φ such that for all unifiers τ of φ, $\sigma \preceq \tau$. In that case, we say that σ is a *most general unifier* of φ.

Proposition 3. *For all unifiable $\varphi \in FOR$, the following are equivalent:* (**1**) *φ is either finitary, or unitary;* (**2**) *there exists a small set Σ of substitutions such that for all unifiers σ of φ, there exists a unifier τ of φ in Σ such that $\tau \preceq \sigma$.*

Proposition 4. *Let $\varphi \in FOR$, $n \geq 2$ and $\sigma_1, \ldots, \sigma_n$ be substitutions. If the following hold then φ is finitary:* (**1**) *for all positive integers $i \leq n$, σ_i is a unifier of φ;* (**2**) *for all positive integers $i, j \leq n$, if $i \neq j$ then $\sigma_i \not\preceq \sigma_j$;* (**3**) *$\sigma_1, \ldots, \sigma_n$ form a complete set of unifiers of φ.*

For all a in TER, when we write "a^0", we mean "$a^\star$" and when we write "a^1", we mean "a".

5 Examples

For some formulas, if they are unifiable then they are finitary. Luckily, in many cases, this can be easily proved. For example, let us consider the formula

$$\varphi_{01} \quad := \quad x \equiv 0 \vee x \equiv 1.$$

Let σ_0 and σ_1 be the substitutions such that $\sigma_0(x) = 0$, $\sigma_1(x) = 1$ and for all variables y, if $x \neq y$ then $\sigma_0(y) = y$ and $\sigma_1(y) = y$.

Proposition 5.

- *σ_0 and σ_1 are unifiers of φ_{01},*
- *neither $\sigma_0 \preceq \sigma_1$, nor $\sigma_1 \preceq \sigma_0$,*
- *σ_0 and σ_1 form a complete set of unifiers of φ_{01},*
- *φ_{01} is finitary.*

Unfortunately, there are unifiable formulas for which the proof that they are finitary can be more involved. For example, let us consider the formula

$$\varphi_{pq} \quad := \quad C(p, q) \rightarrow x \not\equiv 0 \wedge x \leq p \cup q.$$

Let σ_p and σ_q be the substitutions such that $\sigma_p(x) = p \cup (q \cap x)$, $\sigma_q(x) = q \cup (p \cap x)$ and for all variables y, if $x \neq y$ then $\sigma_p(y) = y$ and $\sigma_q(y) = y$.

Proposition 6.

- *σ_p and σ_q are unifiers of φ_{pq},*
- *if $p \neq q$ then neither $\sigma_p \preceq \sigma_q$, nor $\sigma_q \preceq \sigma_p$,*
- *if $p \neq q$ then σ_p and σ_q form a complete set of unifiers of φ_{pq},*
- *if $p \neq q$ then φ_{pq} is finitary.*

6 Monomials

The purpose of this section is to introduce definitions and properties about terms. These definitions and properties are purely Boolean. They will be used later in Sections 7 and 8. From now on, when we write "**CPL**", we mean "Classical Propositional Logic". Let $k, m, n \geq 0$ be such that $n \leq k$. An *m-vector* is a map $\vec{s}$ associating with every positive integer $i \leq m$ an element $\vec{s}(i)$ of $\{0, 1\}$. A *(k, m, n)-correspondence* is a map f associating with every m-vector $\vec{s}$ a surjective function $f_{\vec{s}} : \{0, 1\}^k \longrightarrow \{0, 1\}^n$. An *$n$-monomial* is a term of the form $x_1^{\beta_1} \cap \ldots \cap x_n^{\beta_n}$ where $(\beta_1, \ldots, \beta_n) \in \{0, 1\}^n$. For all m-vectors $\vec{s}$, considering a term a in $TER_{m,n}$ as a formula in **CPL**, let $mon_{\vec{s}}(n, a)$ be the set of all n-monomials $x_1^{\beta_1} \cap \ldots \cap x_n^{\beta_n}$ such that a is a tautological consequence of $p_1^{\vec{s}(1)} \cap \ldots \cap p_m^{\vec{s}(m)} \cap x_1^{\beta_1} \cap \ldots \cap x_n^{\beta_n}$.

Proposition 7. *Let $a \in TER_{m,n}$. Considered as formulas in* **CPL**, *the terms a and $\bigcup\{p_1^{\vec{s}(1)} \cap \ldots \cap p_m^{\vec{s}(m)} \cap x_1^{\alpha_1} \cap \ldots \cap x_n^{\alpha_n} : \vec{s}$ is an m-vector and $x_1^{\alpha_1} \cap \ldots \cap x_n^{\alpha_n} \in mon_{\vec{s}}(n, a)\}$ are equivalent.*

For all positive integers $i \leq n$, let $\pi_i : \{0, 1\}^n \longrightarrow \{0, 1\}$ be the function such that for all $(\beta_1, \ldots, \beta_n) \in \{0, 1\}^n$, $\pi_i(\beta_1, \ldots, \beta_n) = \beta_i$. Let f be a (k, m, n)-correspondence. For all m-vectors $\vec{s}$, for all $(\beta_1, \ldots, \beta_n) \in \{0, 1\}^n$ and for all positive integers $i \leq n$, let $f_{\vec{s}}^{-1}(\beta_1, \ldots, \beta_n)$ be the set of all $(\alpha_1, \ldots, \alpha_k) \in \{0, 1\}^k$ such that $f_{\vec{s}}(\alpha_1, \ldots, \alpha_k) = (\beta_1, \ldots, \beta_n)$, $\Delta_{\vec{s},i}$ be the set of all $(\alpha_1, \ldots, \alpha_k) \in \{0, 1\}^k$ such that $\pi_i(f_{\vec{s}}(\alpha_1, \ldots, \alpha_k)) = 1$ and $c_{\vec{s},i}$ be the term $\bigcup\{x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k} : (\alpha_1, \ldots, \alpha_k) \in \Delta_{\vec{s},i}\}$. Remark that $\Delta_{\vec{s},i}$ and $c_{\vec{s},i}$ depend on f — more precisely, on $f_{\vec{s}}$ — too.

Proposition 8. *For all m-vectors $\vec{s}$ and for all $(\beta_1, \ldots, \beta_n) \in \{0,1\}^n$, considered as formulas in* **CPL***, the terms* $\bigcup\{x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k} : (\alpha_1, \ldots, \alpha_k) \in f_{\vec{s}}^{-1}(\beta_1, \ldots, \beta_n)\}$ *and* $c_{\vec{s},1}^{\beta_1} \cap \ldots \cap c_{\vec{s},n}^{\beta_n}$ *are equivalent.*

7 Tuples of terms

Let $k, m, n \geq 0$ be such that $n \leq k$. Let $(a_1, \ldots, a_n) \in TER_{m,k}^n$. For all m-vectors $\vec{s}$, we define on $\{0,1\}^k$ the equivalence relation $\sim_{(a_1,\ldots,a_n)}^{k,\vec{s}}$ by $(\alpha_1, \ldots, \alpha_k) \sim_{(a_1,\ldots,a_n)}^{k,\vec{s}} (\alpha_1', \ldots, \alpha_k')$ iff for all positive integers $i \leq n$, $x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k} \in mon_{\vec{s}}(k, a_i)$ iff $x_1^{\alpha_1'} \cap \ldots \cap x_k^{\alpha_k'} \in mon_{\vec{s}}(k, a_i)$.

Proposition 9. *For all m-vectors $\vec{s}$, $\sim_{(a_1,\ldots,a_n)}^{k,\vec{s}}$ has at most 2^n equivalence classes on $\{0,1\}^k$.*

Proposition 10. *There exists a (k,m,n)-correspondence f such that for all m-vectors $\vec{s}$ and for all $(\alpha_1, \ldots, \alpha_k), (\alpha_1', \ldots, \alpha_k') \in \{0,1\}^k$, if $f_{\vec{s}}(\alpha_1, \ldots, \alpha_k) = f_{\vec{s}}(\alpha_1', \ldots, \alpha_k')$ then $(\alpha_1, \ldots, \alpha_k) \sim_{(a_1,\ldots,a_n)}^{k,\vec{s}} (\alpha_1', \ldots, \alpha_k')$.*

A (k,m,n)-correspondence f is *balanced* iff for all m-vectors $\vec{s}$ and for all $(\alpha_1, \ldots, \alpha_k), (\alpha_1', \ldots, \alpha_k') \in \{0,1\}^k$, if $f_{\vec{s}}(\alpha_1, \ldots, \alpha_k) = f_{\vec{s}}(\alpha_1', \ldots, \alpha_k')$ then $(\alpha_1, \ldots, \alpha_k) \sim_{(a_1,\ldots,a_n)}^{k,\vec{s}} (\alpha_1', \ldots, \alpha_k')$. By Proposition 10, let f be a balanced (k,m,n)-correspondence. For all m-vectors $\vec{s}$, by means of f — more precisely, of $f_{\vec{s}}$ —, we define the n-tuple $(b_{\vec{s},1}, \ldots, b_{\vec{s},n})$ of terms by setting for all positive integers $i \leq n$, $b_{\vec{s},i} = \bigcup\{x_1^{\beta_1} \cap \ldots \cap x_n^{\beta_n} : x_1^{\alpha_1} \cap \ldots \cap x_k^{\alpha_k} \in mon_{\vec{s}}(k, a_i)$ and $f_{\vec{s}}(\alpha_1, \ldots, \alpha_k) = (\beta_1, \ldots, \beta_n)\}$. An n-tuple $(b_1, \ldots, b_n) \in TER_{m,n}^n$ of terms is *properly obtained* from $(a_1, \ldots, a_n)$ iff for all positive integers $i \leq n$, $b_i = \bigcup\{p_1^{\vec{s}(1)} \cap \ldots \cap p_m^{\vec{s}(m)} \cap b_{\vec{s},i} : \vec{s}$ is an m-vector$\}$. For all m-vectors $\vec{s}$, for all $(\beta_1, \ldots, \beta_n) \in \{0,1\}^n$ and for all positive integers $i \leq n$, let $f_{\vec{s}}^{-1}(\beta_1, \ldots, \beta_n)$, $\Delta_{\vec{s},i}$ and $c_{\vec{s},i}$ be as in Section 6. A substitution υ is *properly obtained* from $(a_1, \ldots, a_n)$ iff for all variables y, if $y \notin \{x_1, \ldots, x_n\}$ then $\upsilon(y) = y$ and for all positive integers $i \leq n$, $\upsilon(x_i) = \bigcup\{p_1^{\vec{s}(1)} \cap \ldots \cap p_m^{\vec{s}(m)} \cap c_{\vec{s},i} : \vec{s}$ is an m-vector$\}$.

Proposition 11. *Let $(b_1, \ldots, b_n) \in TER_{m,n}^n$ and υ be a substitution. If $(b_1, \ldots, b_n)$ and υ are properly obtained from $(a_1, \ldots, a_n)$ then for all positive integers $i \leq n$, considered as formulas in* **CPL***, the terms a_i and $\bar{\upsilon}(b_i)$ are equivalent.*

Proposition 12. *Let σ be the substitution such that for all variables y, if $y \notin \{x_1, \ldots, x_n\}$ then $\sigma(y) = y$ and for all positive integers $i \leq n$, $\sigma(x_i) = a_i$. Let $(b_1, \ldots, b_n) \in TER_{m,n}^n$ and τ be the substitution such that for all variables y, if $y \notin \{x_1, \ldots, x_n\}$ then $\tau(y) = y$ and for all positive integers $i \leq n$, $\tau(x_i) = b_i$. Let υ be a substitution. If $(b_1, \ldots, b_n)$ and υ are properly obtained from $(a_1, \ldots, a_n)$ then $\tau \circ \upsilon \simeq \sigma$.*

Proposition 13. *Let $(b_1, \ldots, b_n) \in TER_{m,n}^n$. If $(b_1, \ldots, b_n)$ is properly obtained from $(a_1, \ldots, a_n)$ then for all valuations $\mathcal{V}$ on $RCP(\mathbb{R}^2)$, there exists a valuation $\mathcal{V}'$ on $RCP(\mathbb{R}^2)$ such that for all positive integers $i \leq n$, $\bar{\mathcal{V}}(b_i) = \bar{\mathcal{V}}'(a_i)$.*

Proposition 14. *Let σ be the substitution such that for all variables y, if $y \notin \{x_1, \ldots, x_n\}$ then $\sigma(y) = y$ and for all positive integers $i \leq n$, $\sigma(x_i) = a_i$. Let $\varphi \in FOR$. Let $(b_1, \ldots, b_n) \in TER_{m,n}^n$ and τ be the substitution such that for all variables y, if $y \notin \{x_1, \ldots, x_n\}$ then $\tau(y) = y$ and for all positive integers $i \leq n$, $\tau(x_i) = b_i$. If $(b_1, \ldots, b_n)$ is properly obtained from $(a_1, \ldots, a_n)$ then σ is a unifier of φ only if τ is a unifier of φ.*

8 Unification type

Now, we are ready to prove the main results of this paper.

Proposition 15. *Let* $\varphi \in FOR$. *Let* $m, n \geq 0$ *be such that* φ*'s constants form a subset of* $\{p_1, \ldots, p_m\}$ *and* φ*'s variables form a subset of* $\{x_1, \ldots, x_n\}$. *For all unifiers* σ *of* φ, *there exists a unifier* τ *of* φ *in* $\Sigma_{m,n}$ *such that* $\tau \preceq \sigma$.

Proof. Let σ be a unifier of φ. Let σ' be the substitution defined by $\sigma'(x_i) = \sigma(x_i)$ for all $i = 1 \ldots n$ and $\sigma'(y) = y$ for all y not in $\{x_1, \ldots, x_n\}$. Obviously, σ' is a unifier of φ too. Now, it may happen that for some $i \in \{1, \ldots, n\}$, $\sigma'(x_i)$ contains extra constants which do not appear in φ. If it is, then let $q_1, \ldots, q_l$ be the list of these extra constants. Take new variables $z_1, \ldots, z_l$ and define σ'' by uniformly replacing in $\sigma'(x_1), \ldots, \sigma'(x_n)$ each occurrence of $q_1, \ldots, q_l$ by, respectively, $z_1, \ldots, z_l$. Obviously, σ'' is a unifier of φ too. As a result, for all constants q, if $q \notin \{p_1, \ldots, p_m\}$ then for all positive integers $i \leq n$, q does not occur in $\sigma''(x_i)$ and for all variables y, if $y \notin \{x_1, \ldots, x_n\}$ then $\sigma''(y) = y$. Let $k \geq 0$ and $(a_1, \ldots, a_n) \in TER^n_{m,k}$ be such that $n \leq k$ and for all positive integers $i \leq n$, $\sigma''(x_i) = a_i$. For all m-vectors $\vec{s}$, let $\sim^{k,\vec{s}}_{(a_1,\ldots,a_n)}$ be as in Section 7. By Proposition 10, let f be a balanced (k, m, n)-correspondence. For all m-vectors $\vec{s}$, for all $(\beta_1, \ldots, \beta_n) \in \{0, 1\}^n$ and for all positive integers $i \leq n$, let $f^{-1}_{\vec{s}}(\beta_1, \ldots, \beta_n)$, $\Delta_{\vec{s},i}$ and $c_{\vec{s},i}$ be as in Section 6. Let $(b_1, \ldots, b_n) \in TER^n_{m,n}$ be an n-tuple of terms properly obtained from $(a_1, \ldots, a_n)$. Let τ be the substitution such that for all variables y, if $y \notin \{x_1, \ldots, x_n\}$ then $\tau(y) = y$ and for all positive integers $i \leq n$, $\tau(x_i) = b_i$. Remark that τ is in $\Sigma_{m,n}$. Moreover, by Proposition 14, τ is a unifier of φ. Let υ be a substitution properly obtained from $(a_1, \ldots, a_n)$. By Proposition 12, $\tau \circ \upsilon \simeq \sigma''$. Hence, $\tau \preceq \sigma''$. By the construction of τ, one can deduce that $\tau \preceq \sigma$. □

Proposition 16. *Let* $\varphi \in FOR$. *If* φ *is unifiable then* φ *is either finitary, or unitary.*

Proof. By Propositions 2, 3 and 15. □

Proposition 17. $\mathcal{UNIF}$ *is in* $EXPSPACE$.

Proof. Let $\varphi \in FOR$. Let $m, n \geq 0$ be such that φ's constants form a subset of $\{p_1, \ldots, p_m\}$ and φ's variables form a subset of $\{x_1, \ldots, x_n\}$. By Proposition 15, the reader may easily verify that φ is unifiable iff there exists a unifier σ of φ in $\Sigma_{m,n}$. Each σ in $\Sigma_{m,n}$ is completely described by the terms $\sigma(x_i) \in TER_{m,n}$, i ranging over $\{1, \ldots, n\}$. Hence, by Proposition 7, each σ in $\Sigma_{m,n}$ is completely described by the disjunctions of conjunctions $\bigcup\{p_1^{\vec{s}(1)} \cap \ldots \cap p_m^{\vec{s}(m)} \cap x_1^{\alpha_1} \cap \ldots \cap x_n^{\alpha_n} : \vec{s}$ is an m-vector and $x_1^{\alpha_1} \cap \ldots \cap x_n^{\alpha_n} \in mon_{\vec{s}}(n, \sigma(x_i))\}$, i ranging over $\{1, \ldots, n\}$. Obviously, the size of these disjunctions of conjunctions is at most exponential in $m + n$. Since the validity problem is in $PSPACE$ [Kontchakov *et al.* (2008), Kontchakov *et al.* (2010), Kontchakov *et al.* (2014)], therefore $\mathcal{UNIF}$ is in $EXPSPACE$. □

9 Conclusion

In this paper, we have adapted to the problem of unifiability with constants in TLs the line of reasoning developed by Balbiani and Gencer [4] within the simpler context of the problem of unifiability without constants in Boolean Region Connection Calculus. Much remains to be done. Firstly, about the choice of the mereotopological space $RCP(\mathbb{R}^2)$. It remains to see whether the line of reasoning developed in this paper will still apply to $RC(\mathbb{R}^2)$ and $RCS(\mathbb{R}^2)$. What happens if we consider mereotopological spaces over the topological spaces $(\mathbb{R}^n, \tau_{\mathbb{R}^n})$, i.e. the real space $\mathbb{R}^n$ of dimension n together with its ordinary topology $\tau_{\mathbb{R}^n}$, when $n \geq 3$? Secondly, about the computability of the unifiability problem in TLs. By Proposition 17, this problem is decidable. Nevertheless, its exact complexity is still unknown. In this respect, we believe that arguments developed in [1] could be used. Thirdly, about adding to the language the predicate of connectedness or the predicate of internal connectedness considered in [8, 9, 10]. The line of reasoning developed in this paper up to Proposition 16 will still apply to these extended languages. Nevertheless, in that case, as proved in [8, 9, 10], the validity problem becomes undecidable.

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