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Handling Substitutions via Duality

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Abstract

Many interesting problems concerning intuitionistic and intermediate propositional logics (as well as other nonclassical and modal propositional logics) are related to properties of substitutions: among them, besides unification, we have rule admissibility, characterization of projective formulae, definability of maximum and minimum fixpoints, finite periodicity theorems, etc. Since most of these questions can be stated in category-theoretic terms, they are sensible to an approach via duality techniques. The available duality for finitely presented Heyting algebras involves both sheaves (giving the appropriate geometric framework) and bounded bisimulations (handling the combinatorics of definability aspects): we show how to use such duality to attack and solve the above problems in a uniform way. [Recent and new results come from joint work [18] with Luigi Santocanale]

Duality techniques have a long tradition in algebraic logic: given the well-known correspondence between logical calculi and suitable algebraic varieties, it is possible to transfer logical problems into the algebraic context and, inside such a context, such problems can be further reformulated in geometric/topological terms by relying on duality theorems. Among such dualities, we recall Stone spaces duality for Boolean algebras, Priestley spaces duality for distributive lattices, Esakia spaces duality for Heyting algebras, modal spaces duality for modal algebras, etc. (see textbooks like [8, 21, 6, 28] for relevant information).

Duality approaches are particularly suitable for questions involving substitutions, because substitutions can bee seen as algebraic homomorphisms. Unification theory is a typical example in this sense: unification type is a categorical invariant [1, 12] and as such it can be transposed to dual contexts, as fruitfully exemplified for instance in [12, 15, 7] for some first cases of locally finite varieties. Unification theory may become a powerful tool in order to analyze rule admissibility [2, 13, 14, 9]. There are many other questions involving substitutions that are sensible to duality techniques: we quote for instance the problem of the finite convergence to a fixpoint for monotone formulae, see [23] for a survey. Other classical topics, like interpolation and uniform interpolation [24], apparently do not seem to directly refer to substitutions, but they still involve a categorical structure whose meaning clearly appears after dualization [19, 27, 26, 20].

It should be noticed however that the above problems require specific dualities for the restricted subcategory of finitely presented (or of finitely generated free) algebras, rather than a duality for the category of all algebras: in logical terms, whereas arbitrary algebras correspond

- via the Lindenbaum-Tarski construction - to *theories* inside a given logic, finitely presented algebras correspond to *finitely axiomatized theories* in such a logic. Duality theorems for such restricted subcategories can be obtained basically via two techniques, namely finite step-frames constructions [10, 11, 3, 5, 4] and bounded bisimulations [19, 29]. We shall refer to the duality theorem from [19], whose main feature is that of embedding dual categories of finitely presented algebras inside a *sheaf topos*: in this setting, the geometric environment shows how to find relevant mathematical structures (products, equalizers, images,...) using their standard definitions in sheaves and presheaves; on the other hand, the combinatorial aspects show that such constructions are definable, thus meaningful from the logical side. In this sense our ingredients of combinatorial nature (Ehrenfeucht-Fraïssé games and bounded bisimulations) replace the topological ingredients which are common in the algebraic logic literature (working with arbitrary algebras instead of finitely presented ones).

The plan of the talk is the following:

- we show how to exploit finite dualities in order to determine the unification type in locally finite varieties [15];
- we recall some dualities working in non locally finite cases [19];
- we revisit unification results from [13, 14, 17] in these duality contexts;
- as a new recent application, we give a semantic proof [18] of Ruitenburg's Theorem [25].

We recall here the statement of the latter. Let us call an infinite sequence

$$a_1, a_2, \ldots, a_i, \ldots$$

ultimately periodic iff there are N and k such that for all $s_1, s_2 \geq N$, we have that $s_1 \equiv s_2 \mod k$ implies $a_{s_1} = a_{s_2}$. If (N, k) is the smallest (in the lexicographic sense) pair for which this happens, we say that N is an *index* and k a *period* for the ultimately periodic sequence $\{ a_i \}_i$. Thus, for instance, an ultimately periodic sequence with index N and period 2 looks as follows

$$a_1, \ldots, a_N, a_{N+1}, a_N, a_{N+1}, \ldots$$

A typical example of an ultimately periodic sequence is the sequence of the iterations $\{ f^i \}_i$ of an endo-function f of a finite set. Whenever infinitary data are involved, ultimate periodicity comes often as a surprise.

Ruitenburg's Theorem is in fact a surprising result stating the following: take a formula $A(x, \underline{y})$ of intuitionistic propositional calculus (IPC) (by the notation $A(x, \underline{y})$ we mean that the only propositional letters occurring in A are among $x, \underline{y}$ - with $\underline{y}$ being, say, the tuple $y_1, \ldots, y_n$) and consider the sequence $\{ A^i(x, \underline{y}) \}_{i \geq 1}$ so defined:

$$A^1 :\equiv A, \quad \ldots, \quad A^{i+1} :\equiv A(A^i/x, \underline{y}) \tag{1}$$

where the slash means substitution; then, *taking equivalence classes under provable bi-implication in* (IPC)*, the sequence* $\{ [A^i(x, \underline{y})] \}_{i \geq 1}$ *is ultimately periodic with period 2.* The latter means that there is N such that

$$\vdash_{IPC} A^{N+2} \leftrightarrow A^N \quad . \tag{2}$$

An interesting consequence of this result is that *least (and greatest) fixpoints of monotonic formulae are definable in* (IPC) [22, 23, 16]: this is because the sequence (1) becomes increasing when evaluated on $\perp/x$ (if A is monotonic in x), so that the period is decreased to 1. Thus the index of the sequence becomes a finite upper bound for the fixpoints approximation convergence.

Ruitenburg's Theorem was shown in [25] via a, rather involved, purely syntactic proof. The proof has been recently formalized inside the proof assistant COQ by T. Litak (see https://git8.cs.fau.de/redmine/projects/ruitenburg1984). Here we supply a purely semantic proof, using duality and bounded bisimulation machinery (details are available in the preprint [18]).

References

[1] M. H. Albert. Category equivalence preserves unification type. *Algebra Universalis*, 36:457–466, 1996.

[2] F. Baader and S. Ghilardi. Unification in modal and description logics. *Logic Journal of the IGPL*, 19(6):705–730, 2011.

[3] N. Bezhanishvili and M. Gehrke. Finitely generated free Heyting algebras via Birkhoff duality and coalgebra. *Logical Methods in Computer Science*, 7(2), 2011.

[4] N. Bezhanishvili and S. Ghilardi. The bounded proof property via step algebras and step frames. *Ann. Pure Appl. Logic*, 165(12):1832–1863, 2014.

[5] N. Bezhanishvili, S. Ghilardi, and M. Jibladze. Free modal algebras revisited: the step-by-step method. In *Leo Esakia on Duality in Modal and Intuitionistic Logics*, Trends in Logic. Springer, 2014.

[6] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Cambridge University Press, 2001.

[7] Leonardo Cabrer. Unification on subvarieties of pseudocomplemented distributive lattices. *Notre Dame J. Form. Log.*, 57(4):477–502, 2016.

[8] A. Chagrov and M. Zakharyaschev. *Modal Logic*. The Clarendon Press, 1997.

[9] W. Dzik and P. Wojtylak. Modal consequence relations extending S4.3: an application of projective unification. *Notre Dame J. Form. Log.*, 57(4):523–549, 2016.

[10] S. Ghilardi. Free Heyting algebras as bi-Heyting algebras. *Math. Rep. Acad. Sci. Canada XVI.*, 6:240–244, 1992.

[11] S. Ghilardi. An algebraic theory of normal forms. *Annals of Pure and Applied Logic*, 71:189–245, 1995.

[12] S. Ghilardi. Unification through projectivity. *J. Log. Comput.*, 7(6):733–752, 1997.

[13] S. Ghilardi. Unification in intuitionistic logic. *J. Symb. Log.*, 64(2):859–880, 1999.

[14] S. Ghilardi. Best solving modal equations. *Ann. Pure Appl. Logic*, 102(3):183–198, 2000.

[15] S. Ghilardi. Unification, finite duality and projectivity in varieties of Heyting algebras. *Ann. Pure Appl. Logic*, 127(1-3):99–115, 2004.

[16] S. Ghilardi, M. João Gouveia, and L. Santocanale. Fixed-point elimination in the intuitionistic propositional calculus. In *Foundations of Software Science and Computation Structures, FOSSACS 2016, Proceedings*, pages 126–141, 2016.

[17] S. Ghilardi and L. Sacchetti. Filtering unification and most general unifiers in modal logic. *J. Symb. Log.*, 69(3):879–906, 2004.

[18] S. Ghilardi and L. Santocanale. Ruitenburg's theorem via duality and bounded bisimulations. *CoRR*, abs/1804.06130, 2018.

[19] S. Ghilardi and M. Zawadowski. *Sheaves, Games and Model Completions*. Kluwer, 2002.

[20] T. Kowalski and G. Metcalfe. Uniform interpolation and coherence. *CoRR*, abs/1803.09116, 2018.

[21] M. Kracht. *Tools and techniques in modal logic*, volume 142 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., 1999.

[22] S. I. Mardaev. Least fixed points in Grzegorczyk's logic and in the intuitionistic propositional logic. *Algebra and Logic*, 32(5):279–288, 1993.

[23] S. I. Mardaev. Definable fixed points in modal and temporal logics : A survey. *Journal of Applied Non-Classical Logics*, 17(3):317–346, 2007.

[24] A. M. Pitts. On an interpretation of second order quantification in first order intuitionistic propositional logic. *J. Symb. Log.*, 57(1):33–52, 1992.

[25] W. Ruitenburg. On the period of sequences $(a^n(p))$ in intuitionistic propositional calculus. *The Journal of Symbolic Logic*, 49(3):892–899, September 1984.

[26] S. J. van Gool, G. Metcalfe, and C. Tsinakis. Uniform interpolation and compact congruences. *Ann. Pure Appl. Logic*, 168(10):1927–1948, 2017.

[27] S. J. van Gool and L. Reggio. An open mapping theorem for finitely copresented Esakia spaces. *Topology and its Applications*, 240:69–77, 2018.

[28] Y. Venema. Algebras and coalgebras. In P. Blackburn, J. van Benthem, and F. Wolter, editors, *Handbook of Modal Logic*, pages 331–426. Elsevier, 2007.

[29] A. Visser. Uniform interpolation and layered bisimulation. In *Gödel '96 (Brno, 1996)*, volume 6 of *Lecture Notes Logic*, pages 139–164. Springer, Berlin, 1996.