Towards a Formal System for Topological Quantum Computation

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We study Topological Quantum Computation (TQC) from the perspective of computability theory with the aim of definining a formal system which is able to capture the computational features of TQC. We discuss the mathematical model for TQC, namely Modular Tensor Categories, and their suitability for the construction of a domain of denotational objects similar to the Scott domain of the λ-calculus. This leads us to believe that a formalism similar to the classical lambda calculus can be defined also for TQC.

1 Introduction

Quantum Computing has changed the way of interpreting the Church-Turing thesis. While there seems to be a unique notion of computability, it is still an open problem whether or not the notion of *efficiently* computatble can be referred to a (classical) Turing machine. The promise of quantum computing is an enormous speed-up in processing classical information using quantum media. However, qubits (the quantum embodiment of bits in quantum mechanics) are very fragile systems that only work in complete isolation from the outside world and therefore difficult to realise. Recently, the 2016 Nobel prize in physics has evidenced the possibility of realising a more robust version of quantum computing that solves the fragility of the qubits by using topological invariants of quantum systems. Thus, this new way of computing quantum mechanically is called *Topological Quantum Computation* (TQC).

The subject of the 2016 Nobel prize winning result are some generalisations of bosons and fermions in two dimensions, called *anyons*, and their manifestation as some topological quantum fields. One way to model anyon systems is to consider the fusion and braiding structure of all elementary excitations in the plain, i.e. to identify the anyon system with an equivalent *unitary modular tensor category* (UMTC) [11, 8]. In TQC, space-time trajectories of anyons are represented by braids; algebraically these correspond to morphisms in the UMTC associated to the anyon system, which can then be thought of as circuits for computation.

In this paper we aim to show that UMTC can be used to construct a model for TQC à la Scott (see [1]), namely a (non-syntactical) λ-model where the domain contains both functions and function arguments. This will be the base to define TQC as a λ-calculus where terms represent anyons and term re-writing their braiding.

2 Modular tensor categories

The two basic notions in TQC are anyons and braiding. Abstractly, the position of n identical particles living in a space X form a Hilbert space of quantum states of such n particles. If the particles are anyons

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then moving from one state to another only depends on the homotopy of the trajectories giving rise to different representations of the state space (statistics of the n particles). Space-time trajectories of n anyons form the n strands braid group $\mathscr{B}_n$. These two basic notions are well represented by modular tensor categories, i.e. semisimple ribbon categories with only a finite number of isomorphism classes of simple objects (and with some additional properties related to the modularity condition, cf [2]).

Essentially, tensor categories are linear monoidal categories with two bifunctors $\oplus$ and $\otimes$ which are the categorification of the sum and multiplication of a ring. They define two coherently co-existing structures: a rigid braided monoidal category with bifunctor $\otimes$, and an abelian category with bifunctor $\oplus$. Objects in an MTC $\mathscr{C}$ are of the form $\bigoplus_j S_j$, where S_j are called *simple* objects and represent those labels that did not undergo any splitting (fusion): any morphism between two such objects are either an isomorphism (if they are the same) or the zero morphism. Morphisms $f : \bigoplus_j S_j \to \bigoplus_k S'_k$ are matrices of morphisms between the summands.

The fact that $\mathscr{C}$ is both abelian and tensor monoidal gives its homsets the structure of complex vector spaces (in fact Hilbert spaces), that we can represent as follows. Suppose that $i \otimes j = \sum_k N^k_{i,j} k$ is a fusion rule of the anyon system. Then

$$Hom(S_{k_1}, S_i \otimes S_j) \oplus \ldots \oplus Hom(S_{k_n}, S_i \otimes S_j) \tag{1}$$

is the vector space representing for each fusion result k the vector space of all fusions of i and j resulting in k. Multiple applications of the fusion rules give rise to more complex vector spaces such as

$$Hom(S_{l_1}, A_1), \text{ with } A_1 = Hom(S_{l_2}, A_2) \ldots A_m = Hom(S_{l_m}, S_i \otimes S_j).$$

Note that each homset can represent a direct sum of vector spaces as in (1), each with its own dimension given by the multiplicity of the corresponding result in the fusion rule. Physically these vector spaces represent so-called *fusion spaces*.

As a braided monoidal category, $\mathscr{C}$ has a tensor product that allows for modeling a compound system of charges while braiding allows us to model their movements. The two operators must interact according to some axioms expressed in the category by natural isomorphisms. One such axiom is the hexagon diagram which says for example that $(A \otimes B) \otimes C)$ is the same as $B \otimes (C \otimes A)$ [12, 9, 2].

2.1 Reflexivity of the objects

In a UMTC an object X is determined by the complex Hilbert space of morphisms $Hom(X,Y)$ for all Y in the category. Therefore it is natural to look at objects in such categories as certain special quantum states. Morphisms between objects are then an abstraction of the quantum processes between those objects and therefore of unitary transformations between Hilbert spaces.

Based on this fact we can easily switch from UMTC to the category of finite dimensional Hilbert spaces for reasoning about the dynamic of anyonic computation. In particular, the homset

$$V^{ij}_k \simeq Hom(S_k, S_i \otimes S_j)$$

is the splitting (or dually fusion) space for the anyons S_k, S_i, S_j, whose vectors are the splitting (or dually fusion) states

$$Hom(S_k, S_k) \longrightarrow Hom(S_k, S_i \otimes S_j).$$

An important result from linear algebra states that every Hilbert space H is a *reflexive domain*; in functional analysis this means that there is an isomorphism between H and its double dual H^{**}, that is the space of all linear bounded functionals on the dual space H^*, which in turn is the space of all linear bounded functionals on H. Based on this we have shown in [3] that it is possible to construct a retract for any H, that is a domain D and two linear maps $F : D \to [D \to D]$ and $G : [D \to D] \to D$ such that $F \circ G$ is the identity in $[D \to D]$. As a result, we can conclude that in any modular tensor category we can construct a λ-model for TQC.

3 Computing with anyons

In TQC, information is encoded in multi-anyon quantum states and computation is carried out within the ground state manifold $V_{n,x,t}$ with x a non-abelian anyon type, n the number of anyons of that type, and t is the total charge [4]. This computation essentially consists in the exchanges of the anyons of the system as a process evolving in time, i.e. to *braiding* the threads (a.k.a. world-lines) starting from each anyon of the system. Particle trajectories are braided according to rules specifying how pairs (or bipartite subsystems) behave under exchange. Topological transformations that leave invariant braided trajectories are turned into algebraic constraints (compatibility conditions), namely the *pentagon equation* (relating 5 fusions) and the Yang-Baxter or *hexagon equation* (relating 3 braidings and 3 fusions). The ground state manifold $V_{n,x,t}$ is also a non-trivial unitary representation of the n-strand braid group $\mathscr{B}_n$, which intuitively enforces the idea of a space where states and computation on them (arguments and functions on them) live together. An initial state in a computation is given by creating anyons pairs from the vacuum to encode the input. This corresponds to implementing a morphism in $Hom(1, X^{\oplus n})$ for some simple object X of type x. The second step is a braiding followed at the end of the computation by measurement which is achieved by fusing anyons together to observe the possible outcomes. This corresponds to the implementation of a morphism in $Hom(X^{\oplus n}, 1)$. The computing result is a probability distribution on anyon types obtained by repeating the same process polynomially many times.

4 Concluding Remarks

We have presented some initial ideas for the definition of a categorical model for TQC that lends itself to an interpretation of this quantum computation paradigm as a 'calculus of functions' similar to the classical λ-calculus. The general workplan is a revisitation of topological quantum computation from the perspective of computability theory by defining a logical formalism equivalent to the existing physical and mathematical models for TQC.

An important question is about the universality of our calculus and its model. Universality by braiding can be achieved only if the fusion rules of the anyon system allow for the construction of braiding patterns which are able to reproduce all the unitary matrices representing quantum circuits. While the Fibonacci anyons have been shown to have this property [6], other anyon systems such as the Ising anyons are not suitable for universal quantum compiling. Our model must therefore refer to the appropriate anyon system. For example, it is known that suitable models are the $SU(2)_k$ anyon models (for k integer), i.e. "q-deformed" versions of the usual $SU(2)$ for $q = e^{i\frac{2\pi}{k+2}}$ (roughly the special unitary 2×2 matrices group where integers n are replaced by $\frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$) [10, 7]. These describe $SU(2)_k$ Chern-Simons theories [9] and give rise to the Jones polynomial of knot theory. Their braiding statistics are known to be universal for TQC for all k except $k = 1, 2, 4$ [5].

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