Coherence of monoids by insertions

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Abstract

We introduce string data structures as combinatorial descriptions of structured words on totally ordered alphabets. The data can be described by words through a reading map and can be constructed by using an insertion algorithm. The insertion map defines a product on datum. We show that the associativity of this product, the cross-section property of the data structure, and the confluence of the rewriting system defined by the insertion map are equivalent properties. We make explicit a coherent presentation of the monoid presented by the data structure, made of generators, rewriting rules describing the insertion of letters in words and relations among the insertion algorithms.

1 Introduction

A data structure describes a way to organize and to store a collection of structured data. It defines primitive operations such as constructors, insertion and reading maps on the data. In this work, we study string rewriting systems (SRS) whose normal forms can be described using a data structure and whose normalisation strategies are induced by insertion algorithms. Such data structures appear in many contexts in combinatorial algebra, combinatorics and fundamental computer science and describe combinatorial structures such as arrays, tableaux, staircases or binary search trees. They are used to describe combinatorially equivalence relations in free monoids. In particular array structures can be used to study plactic, Chinese, hypoplactic and sylvester monoids.

For instance, the structure of plactic monoid emerged from the works of Schensted [13] and Knuth [9] on the combinatorial study of Young tableaux and it has found several applications in combinatorics and representation theory [11, 4]. The study of plactic monoids (of type A) using SRS on Knuth generators is not straightforward, in particular in rank greater than 4 they do not admit finite completion with respect to the lexicographic order, [10]. Finite completions can be obtained by adding new generators in the quasi-center of the monoid. In particular, by adding column or row generators, the completion procedure ends producing a convergent presentation of plactic monoids, [2, 1]. Such convergent presentations can be used to make explicit coherent presentations of plactic monoids giving all the relations among the relations of the presentations, [7]. The confluence property is essential to obtain such coherence results.

The confluence of the column presentation for plactic monoids is a consequence of the commutation of Schensted's insertion algorithms in Young tableaux: the right insertion (or insertion by rows) and the left insertion (or insertion by columns). In this work, we make explicit this confluence result in a general algebraic framework. We introduce the notion of string data structures as combinatorial descriptions of structured words on totally ordered alphabets. The data can be described by words through a reading map and can be constructed using an insertion algorithm. The insertion map defines a product on datum. We show that the associativity of this product, the cross-section property of the data structure, and the confluence of the rewriting system defined by the insertion map are equivalent properties. We make explicit a coherent presentation of the monoid presented by the data structure, made of generators, rewriting rules describing the insertion of letters in words and relations among the insertion algorithms.

In a first part, we introduce the notion of string data structure. We show that the commutation of left and right insertion algorithms on a data structure induces an associative product on the data. We define an SRS associated to a data structure, whose rules are defined by the insertion map, and we show that the associativity of the product on the data structure yields the confluence of this SRS. In a second part, using the notion of generating set of a string data structure, we construct an SRS on a reduced set of data and we show that the associativity of the data structure induces the confluence of this reduced SRS. In addition, we make explicit a coherent presentation of the monoid presented by a data structure in terms of the normalisation strategy induced by the insertion algorithm on the data structure. We recall in Appendices the Schensted's algorithms, the notion of coherent presentation and we give the proofs of the main results presented in this abstract.

2 String data structures, confluence and cross-section

String data structures. A *string data structure*, SDS for short, $\mathbb{S}$ on a totally ordered alphabet A is a quadruple $(\mathrm{D}_A, \ell, I, R)$ made of a set D_A, a reading ℓ of words on A, a *one-element insertion map* I and a *reading map* R defined as follows:

i) the inclusions $A \subseteq R(\mathrm{D}_A) \subseteq A^*$ hold, where A^* denotes the free monoid on A,

ii) the map $\ell : A^* \to A^*$ sends each word $x_1 \ldots x_k$ in A^* on a word $x_{\sigma_{(1)}} \ldots x_{\sigma_{(k)}}$ in A^*, where σ is a permutation on $\{1, \ldots, k\}$,

iii) $I : \mathrm{D}_A \times A \to \mathrm{D}_A$ inserts an element of A into an element of D_A such that any restriction $I(-, x)$ is injective for $x \in A$. By iteration, one defines an *insertion map* $I^* : \mathrm{D}_A \times A^* \to \mathrm{D}_A$ that inserts a word in A^* into an element of D_A wrt ℓ, that is $I^*(d, x_1 \ldots x_n) = I^*(I(d, y_1), y_2 \ldots y_n)$, for any $d \in \mathrm{D}_A$ and $x_1 \ldots x_n \in A^*$, where $y_1 \ldots y_n = \ell(x_1 \ldots x_n)$,

iv) $R : \mathrm{D}_A \to A^*$ is injective and satisfies $I^*(\emptyset, \ell(-))R = \mathrm{Id}_{\mathrm{D}_A}$ and $R(\emptyset)$ is the empty word.

The map $I^*(\emptyset, \ell(-)) : A^* \to \mathrm{D}_A$ is called the *constructor* of the SDS $\mathbb{S}$. The maps I, R and $I^*(\emptyset, \ell(-))$ will be also denoted by $I_{\mathbb{S}}$, $R_{\mathbb{S}}$ and $C_{\mathbb{S}}$. We will use the *right-to-left* (resp. *left-to-right*) reading of words denoted by ℓ_{r} (resp. ℓ_{l}). A *right* (resp. *left*) *SDS* is an SDS whose insertion map is said *right* (resp. *left*), that is inserting a word into an element of D_A with respect to ℓ_{l} (resp. ℓ_{r}). Two one-element insertion maps $I, J : \mathrm{D}_A \times A \to \mathrm{D}_A$ *commute* if the relation $J(I(d, x), y) = I(J(d, y), x)$ holds for every $d \in \mathrm{D}_A$ and $x, y \in A$. An *opposite* of a right (resp. left) SDS $(\mathrm{D}_A, \ell_{\mathrm{l}}, I, R)$ (resp. $(\mathrm{D}_A, \ell_{\mathrm{r}}, I, R)$) is a left (resp. right) SDS $(\mathrm{D}_A, \ell_{\mathrm{r}}, J, R)$ (resp. $(\mathrm{D}_A, \ell_{\mathrm{l}}, J, R)$) such that I and J commute.

For example, a *(Young) tableau* on the finite set $[n] := \{1, \ldots, n\}$ is a collection of boxes in left-justified rows, filled with elements of $[n]$, where the entries weakly increase along each row and strictly increase down each column. Denote by Yt_n the set of tableaux on $[n]$. Schensted, [13], introduced the *right (or row)* (resp. *left (or column)*) insertion S_r (resp. S_l) : $\mathrm{Yt}_n \times [n] \to \mathrm{Yt}_n$, see Appendix A. Let $R_{col} : \mathrm{Yt}_n \to [n]^*$ be the map reading the columns of a tableau from left to right and from bottom to top. This defines two SDSs $\mathcal{Y}_n^{row} = (\mathrm{Yt}_n, \ell_{\mathrm{l}}, S_r, R_{col})$ and $\mathcal{Y}_n^{col} = (\mathrm{Yt}_n, \ell_{\mathrm{r}}, S_l, R_{col})$ on the structure of tableau.

An SDS $\mathbb{S} = (\mathrm{D}_A, \ell, I, R)$ is *associative* if the product $\star_{\mathbb{S}} : \mathrm{D}_A \times \mathrm{D}_A \to \mathrm{D}_A$ defined by setting $d \star_{\mathbb{S}} d' = I^*(d, \ell(R(d')))$, for any $d, d' \in \mathrm{D}_A$ is associative. That is, the relation $(d \star_{\mathbb{S}} d') \star_{\mathbb{S}} d'' = d \star_{\mathbb{S}} (d' \star_{\mathbb{S}} d'')$ holds for any $d, d', d'' \in \mathrm{D}_A$. For instance, the SDS $(\mathrm{Yt}_n, \ell_{\mathrm{l}}, S_l, R_{col})$ is not associative:

$$\left(\begin{smallmatrix} 1 \\ 4 \\ 6 \end{smallmatrix} \star_{\mathbb{S}} \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \right) \star_{\mathbb{S}} \begin{smallmatrix} 1 \end{smallmatrix} = \begin{smallmatrix} 1 & 2 & 3 \\ 4 & & \\ 6 & & \end{smallmatrix} \star_{\mathbb{S}} \begin{smallmatrix} 1 \end{smallmatrix} = \begin{smallmatrix} 1 & 1 & 3 \\ 2 & & \\ 4 & & \\ 6 & & \end{smallmatrix} \neq \begin{smallmatrix} 1 & 1 & 2 & 3 \\ 4 & & & \\ 6 & & & \end{smallmatrix} = \begin{smallmatrix} 1 \\ 4 \\ 6 \end{smallmatrix} \star_{\mathbb{S}} \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} = \begin{smallmatrix} 1 \\ 4 \\ 6 \end{smallmatrix} \star_{\mathbb{S}} \left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \star_{\mathbb{S}} \begin{smallmatrix} 1 \end{smallmatrix} \right)$$

Theorem 1. *Let $\mathbb{S}$ be a right (resp. left) SDS. If there is a left (resp. right) SDS $\mathbb{T}$ opposite to $\mathbb{S}$, then the SDSs $\mathbb{S}$ and $\mathbb{T}$ commute, that is $d \star_{\mathbb{S}} d' = d' \star_{\mathbb{T}} d$, for any $d, d' \in \mathrm{D}_A$, and are associative.*

Structure monoid of an SDS. Let $\mathbb{S} = (\mathrm{D}_A, \ell, I, R)$ be an SDS. Denote by $|$ the product of the free monoid on D_A. The *structure monoid* associated to the SDS $\mathbb{S}$ is the monoid, denoted by $\mathbf{M}(\mathbb{S})$, and presented by the following SRS

$$\mathcal{R}(\mathbb{S}) = \langle\, \mathrm{D}_A \mid \gamma_{d,d'} : d|d' \to d \star_{\mathbb{S}} d' \;\text{ for any } d, d' \text{ in } \mathrm{D}_A \,\rangle,$$

called the *standard presentation* induced by the SDS $\mathbb{S}$. Since every application of a rewriting rule of $\mathcal{R}(\mathbb{S})$ yields a strictly smaller preceding word with respect to the deglex order on D_A^*, the SRS $\mathcal{R}(\mathbb{S})$ is terminating. Moreover, if the SDS $\mathbb{S}$ is associative, then the SRS $\mathcal{R}(\mathbb{S})$ is convergent. The *reading of the standard presentation* of the SDS $\mathbb{S}$ is the SRS defined by

$$\mathcal{R}(A, \mathbb{S}) = \langle\, A \mid \gamma_{d,d'} : R_{\mathbb{S}}(d)R_{\mathbb{S}}(d') \to R_{\mathbb{S}}(d \star_{\mathbb{S}} d') \;\text{ for any } d, d' \text{ in } \mathrm{D}_A \,\rangle.$$

Any critical pair of $\mathcal{R}(A, \mathbb{S})$ has the form

$$\begin{array}{ccc} & \xrightarrow{\gamma_{d,d'} R_{\mathbb{S}}(d'')} R_{\mathbb{S}}(d \star_{\mathbb{S}} d')R_{\mathbb{S}}(d'') \xrightarrow{\gamma_{d \star_{\mathbb{S}} d', d''}} R_{\mathbb{S}}((d \star_{\mathbb{S}} d') \star_{\mathbb{S}} d'') \\ R_{\mathbb{S}}(d)R_{\mathbb{S}}(d')R_{\mathbb{S}}(d'') & \\ & \xrightarrow[R_{\mathbb{S}}(d)\gamma_{d',d''}]{} R_{\mathbb{S}}(d)R_{\mathbb{S}}(d' \star_{\mathbb{S}} d'') \xrightarrow[\gamma_{d, d' \star_{\mathbb{S}} d''}]{} R_{\mathbb{S}}(d \star_{\mathbb{S}} (d' \star_{\mathbb{S}} d'')) \end{array}$$

for every $d, d', d'' \in \mathrm{D}_A$. As a consequence, if $\mathbb{S}$ is associative, then $\mathcal{R}(A, \mathbb{S})$ is locally confluent.

Compatibility of an SDS. An associative SDS $\mathbb{S} = (\mathrm{D}_A, \ell, I, R)$ is *compatible* with an equivalence relation $\sim$ on A^* if for any $d \in \mathrm{D}_A$ and $w, w' \in A^*$, $w \sim w'$ implies $I^*(d, w) = I^*(d, w')$, and for any $w \in A^*$, one has $R_{\mathbb{S}}C_{\mathbb{S}}(w) \sim w$. If $\mathbb{S} = (\mathrm{D}_A, \ell, I, R)$ is an associative SDS compatible with the relation $\sim$, thus I^* induces a unique map $\widetilde{I^*}$, such that the diagram on the right commutes, where $\pi : A^* \to A^*/\sim$ denotes the quotient map. Hence, the constructor $C_{\mathbb{S}}$ induces a map $\overline{C}_{\mathbb{S}} : A^*/\sim \to \mathrm{D}_A$ defined by $\overline{C_{\mathbb{S}}}(\pi(w)) = \widetilde{I^*}(\emptyset, \pi(\ell(w)))$, for any $w \in A^*$. Moreover, we have $\overline{C}_{\mathbb{S}}\pi R_{\mathbb{S}} = Id_{\mathrm{D}_A}$. Hence, the map $\overline{C}_{\mathbb{S}}$ is bijective.

$$\begin{array}{ccc} \mathrm{D}_A \times A^* & \xrightarrow{I^*} & \mathrm{D}_A \\ {\scriptstyle \mathrm{Id} \times \pi}\downarrow & \nearrow_{\widetilde{I^*}} & \\ \mathrm{D}_A \times A^*/\sim & & \end{array}$$

Proposition 1. *Let $\mathbb{S}$ be a right associative SDS compatible with the equivalence relation $\sim_{\mathbb{S}}$ induced by $\mathcal{R}(A, \mathbb{S})$. The map $C_{\mathbb{S}} : A^* \to \mathrm{D}_A$ induces a monoid isomorphism $\overline{C}_{\mathbb{S}}$ between $A^*/\sim_{\mathbb{S}}$ and $(\mathrm{D}_A, \star_{\mathbb{S}})$, with the inverse induced by the reading map $R_{\mathbb{S}}$.*

Let $\sim$ be an equivalence relation on a free monoid K^* on a set K. Recall that a subset $S \subset K^*$ *satisfies the cross-section property for the monoid $K^*/\sim$* if each equivalence class with respect to $\sim$ contains exactly one element of S. Let $\mathbb{S}$ be a right associative SDS compatible with the equivalence relation $\sim_{\mathbb{S}}$ induced by $\mathcal{R}(A, \mathbb{S})$. By Proposition 1, the monoids $(\mathrm{D}_A, \star_{\mathbb{S}})$ and $A^*/\sim_{\mathbb{S}}$ are isomorphic. One says that $\mathcal{R}(\mathbb{S})$ and $\mathcal{R}(A, \mathbb{S})$ are *Tietze-equivalent*, that is present the same monoid. In particular, if $\mathcal{R}(A, \mathbb{S})$ is terminating, then the set of normal forms wrt $\mathcal{R}(\mathbb{S})$ satisfies the cross-section property for $\mathbf{M}(\mathbb{S})$ if and only if the set $\mathrm{Nf}(A, \mathbb{S})$ of normal forms wrt $\mathcal{R}(A, \mathbb{S})$ satisfies the cross-section property for $\mathbf{M}(\mathbb{S})$.

Theorem 2. *Let $\mathbb{S}$ be a right associative SDS such that the SRS $\mathcal{R}(A, \mathbb{S})$ is terminating. Then the SRSs $\mathcal{R}(\mathbb{S})$ and $\mathcal{R}(A, \mathbb{S})$ are Tietze-equivalent and the set $\mathrm{Nf}(A, \mathbb{S})$ satisfies the cross-section property for the monoid $\mathbf{M}(\mathbb{S})$.*

For instance, the *plactic monoid* $\mathbf{P}_n$ of rank n, [11], is presented by the *Knuth presentation* whose set of generators is $[n]$ submitted to relations $zxy = xzy$ for $x \leqslant y < z$ and $yzx = yxz$ for $x < y \leqslant z$. Schensted showed that S_r and S_l commute, [13]. Then, by Theorem 1 the SDS $\mathcal{Y}_n^{row}$ is associative and the SRS $\mathcal{R}(\mathcal{Y}_n^{row})$ is convergent. One shows that the Knuth presentation is Tietze-equivalent to the reading of the SRS $\mathcal{R}([n], \mathcal{Y}_n^{row})$. By [9], see also [12], the SDS $\mathcal{Y}_n^{row}$ is compatible with the equivalence relation induced by the Knuth presentation. Then, by Proposition 1, $\mathcal{R}(\mathcal{Y}_n^{row})$ is a convergent presentation of the monoid $\mathbf{P}_n$. Hence, the set Yt_n satisfies the cross-section property for $\mathbf{P}_n$.

3 Coherent presentations and SDS

Change of generators. Let $\mathbb{S} = (\mathrm{D}_A, \ell, I, R_{\mathbb{S}})$ be an SDS. One considers a binary relation $|$ on D_A compatible with $R_{\mathbb{S}}$, that is $R_{\mathbb{S}}(d|d') = R_{\mathbb{S}}(d)R_{\mathbb{S}}(d')$ for any $d, d' \in \mathrm{D}_A$, where $d|d'$ denotes $(d, d') \in |$. A *generating set* with respect to such a binary relation is a subset Q of D_A such that $A \subseteq R_{\mathbb{S}}(Q)$, and any element d in D_A can be written $d = c_1|c_2|\ldots|c_k$, with $c_1, \ldots, c_k \in Q$. From such generating set Q of $\mathbb{S}$, one can define an SDS $\mathbb{S}_Q = (\mathrm{D}_A, \ell_Q, I_Q, R_Q)$ on Q, where

i) the map $\ell_Q : Q^* \to Q^*$ induces a permutation on the letters of each words on Q,

ii) $I_Q : \mathrm{D}_A \times Q \to \mathrm{D}_A$ is a one-element insertion map defined by $I_Q(d, c) = I^*(d, R_{\mathbb{S}}(c))$, for any $c \in Q$ and $d \in \mathrm{D}_A$, that induces an insertion map $I_Q^* : \mathrm{D}_A \times Q^* \to \mathrm{D}_A$ wrt ℓ_Q,

iii) $R_Q : \mathrm{D}_A \to Q^*$ is the reading map associated to $|$, that is, for any $d \in \mathrm{D}_A$, $R_Q(d) = c_1|c_2|\ldots|c_k$ is the decomposition of d with respect to $|$.

A reduced presentation. Consider an SDS $\mathbb{S} = (\mathrm{D}_A, \ell, I, R_{\mathbb{S}})$ and a generating set Q of $\mathbb{S}$ with respect to a binary relation $|$ compatible with $R_{\mathbb{S}}$. One defines the following SRS

$$\mathcal{R}(Q, \mathrm{D}_A, \mathbb{S}) = \langle\, Q \mid \gamma_{c,c'} : c|c' \to R_Q(c \star_{\mathbb{S}} c') \;\text{ for any } c, c' \in Q \text{ such that } c|c' \notin \mathrm{D}_A \,\rangle,$$

called the *reduced SRS* of $\mathbb{S}$. We will denote by $\mathrm{Nf}(Q, \mathbb{S})$ the set of normal forms wrt $\mathcal{R}(Q, \mathrm{D}_A, \mathbb{S})$. The SRS $\mathcal{R}(Q, \mathrm{D}_A, \mathbb{S})$ may be non terminating, in particular when the number of generators in Q is not decreasing with the application of rules $\gamma_{c,c'}$. An additional condition is thus necessary on $\mathbb{S}$ to assure the termination of $\mathcal{R}(Q, \mathrm{D}_A, \mathbb{S})$.

Lemma 1. *Let $\mathbb{S} = (\mathrm{D}_A, \ell, I, R_{\mathbb{S}})$ be an associative SDS and Q be a generating set of $\mathbb{S}$ with respect to a binary relation $|$ compatible with $R_{\mathbb{S}}$. If the SRS $\mathcal{R}(Q, \mathrm{D}_A, \mathbb{S})$ is terminating, then the SRSs $\mathcal{R}(\mathbb{S})$ and $\mathcal{R}(Q, \mathrm{D}_A, \mathbb{S})$ are Tietze-equivalent.*

For instance, consider the SDS $\mathcal{Y}_n^{row}$ and let Col_n be the set of tableaux with only one column. Denote by $|$ the concatenation of columns in Yt_n. Every d in Yt_n can be written $d = c_1|c_2|\ldots|c_k$, where $c_1, \ldots, c_k$ are the columns of d from left to right. We have $R_{col}(d) = R_{col}(c_1)R_{col}(c_2)\ldots R_{col}(c_k)$. Then, the concatenation $|$ is a binary relation compatible with R_{col} and the set Col_n is a generating set wrt $|$. Let $R_{\mathrm{Col}_n} : \mathrm{Yt}_n \to \mathrm{Col}_n^*$ be the map that decomposes a tableau as the concatenation of its columns from left to right. The SDS $\mathcal{Y}_n^{row}$ is associative and one shows that the SRS $\mathcal{R}(\mathrm{Col}_n, \mathrm{Yt}_n, \mathcal{Y}_n^{col})$ is terminating. Then by Lemma 1 the SRSs $\mathcal{R}(\mathcal{Y}_n^{col})$ and $\mathcal{R}(\mathrm{Col}_n, \mathrm{Yt}_n, \mathcal{Y}_n^{col})$ are Tietze-equivalent. Hence, the SRS $\mathcal{R}(\mathrm{Col}_n, \mathrm{Yt}_n, \mathcal{Y}_n^{col})$ is a finite convergent presentation of the monoid $\mathbf{P}_n$. By this way we recover the result of [1, 2].

Coherence from insertion. Recall that a *normalisation strategy* for an SRS R specifies a way to apply the rules in a deterministic way. It is defined as a mapping σ of every words u

in X^* to a rewriting step from u to a chosen normal form $\widehat{u}$. We distinguish two canonical strategies to reduce words: the *leftmost* one $\sigma^\top$ and the *rightmost* one $\sigma^\perp$, according to the way we apply first the rewriting rule that reduces the leftmost or the rightmost subword. Given an associative SDS $\mathbb{S} = (\mathrm{D}_A, \ell, I, R)$ and an associated reduced SRS $\mathcal{R}(Q, \mathrm{D}_A, \mathbb{S})$, we say that a normalization strategy σ of $\mathcal{R}(Q, \mathrm{D}_A, \mathbb{S})$ *computes* the constructor $C_\mathbb{S}$ if it is normalizing, and it reduces any word $c_1|c_2|\ldots|c_n$ in Q^* to $R_Q(c_1 \star_\mathbb{S} c_2 \star_\mathbb{S} \ldots \star_\mathbb{S} c_n)$, that is

$$\sigma_{c_1|c_2|\ldots|c_n} : c_1|c_2|\ldots|c_n \to R_Q(c_1 \star_\mathbb{S} c_2 \star_\mathbb{S} \ldots \star_\mathbb{S} c_n) \quad \text{for any } c_1, \ldots, c_n \in Q.$$

Theorem 3. *Let $\mathbb{S}$ be an associative SDS and Q be a generating set of $\mathbb{S}$ such that the SRS $\mathcal{R}(Q, \mathrm{D}_A, \mathbb{S})$ is terminating. If there exists a normalization strategy of $\mathcal{R}(Q, \mathrm{D}_A, \mathbb{S})$ that computes $C_\mathbb{S}$, then the set $\mathrm{Nf}(Q, \mathbb{S})$ satisfies the cross-section property for $\mathbf{M}(\mathbb{S})$.*
In particular, if the leftmost normalization strategy $\sigma^\top$ computes $C_\mathbb{S}$, then the SRS $\mathcal{R}(Q, \mathrm{D}_A, \mathbb{S})$ can be extended into a coherent convergent presentation by adjunction

$$\begin{array}{c} c|c'|c''| \xrightarrow{\sigma^\top_{cc'c''}} R_Q(c \star_\mathbb{S} c' \star_\mathbb{S} c'') \\ c|c'|c''| \xrightarrow{c|\gamma_{c',c''}} c|R_Q(c' \star_\mathbb{S} c'') \xrightarrow{\sigma^\top_{c|R_Q(c' \star_\mathbb{S} c'')}} R_Q(c \star_\mathbb{S} c' \star_\mathbb{S} c'') \end{array} \quad \textit{for every } c, c', c'' \textit{ in } Q.$$

With hypothesis of Theorem 3, consider $\sigma^\top$ (resp. $\sigma^\perp$) the leftmost (resp. rightmost) normalisation strategy with respect to $\mathcal{R}(Q, \mathrm{D}_A, \mathbb{S})$ for a right SDS $\mathbb{S}$. Suppose that there is an SDS $\mathbb{T}$ opposite to $\mathbb{S}$. If the strategy $\sigma^\top$ computes $C_\mathbb{S}$, then $\mathcal{R}(Q, \mathrm{D}_A, \mathbb{S})$ can be extended into a coherent convergent presentation by adjunction of the homotopy generator on the right for every c, c' and c'' in Q, where $\sigma^\top$ (resp. $\sigma^\perp$) corresponds to the application of the right (resp. left) insertion of $\mathbb{S}$ (resp. $\mathbb{T}$).

$$c|c'|c''| \;\underset{\sigma^\perp_{cc'c''}}{\overset{\sigma^\top_{cc'c''}}{\rightrightarrows}}\; R_Q(c \star_\mathbb{S} c' \star_\mathbb{S} c'')$$

Theorem 3 can be used to construct coherent presentations of plactic monoids, see Appendix G.

4 Conclusion and work in progress

We have introduced the notion of SDS to study the confluence of SRS whose rules are defined by insertion algorithm. We show that the fact that a right SDS and a left SDS that present a monoid are opposite and the confluence property of the standard SRS presenting the monoid are equivalent properties. We apply our construction on the *Chinese monoid* of rank n, [3], generated by the set $[n]$ and subject to the relations $zyx = zxy = yzx$, for $x \leqslant y \leqslant z$. By constructing an SDS associated to the insertion algorithm in Chinese staircases, we deduce the confluence of the reduced presentation of the Chinese monoid and we extend this presentation into a finite coherent presentation of the monoid, see Appendix F. Finally, the *sylvester monoid* of rank n, [8], generated by $[n]$ and subject to the relations $cavb = acvb$, for all $a \leqslant b < c$ and $v \in [n]^*$, can be described using the notion of binary search trees. We expect that our methods should conduce to a coherent presentation of the sylvester monoid induced by the insertion algorithm in a binary search tree.

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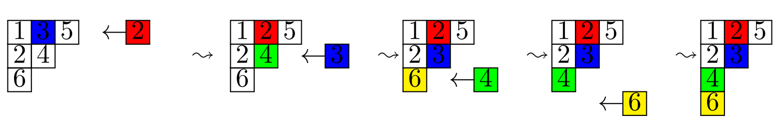
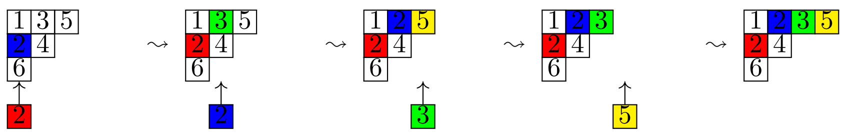
A Schensted's algorithms

Schensted introduced two algorithms to insert an element x in $[n]$ into a tableau t of Yt_n, [13]. The *right (or row) insertion algorithm* $S_r : \mathrm{Yt}_n \times [n] \longrightarrow \mathrm{Yt}_n$ computes a tableau $S_r(t, x)$ as follows. If x is at least as large as the last element of the top row of t, then put x to the right of this row. Otherwise, let y be the smallest element of the top row of t such that $y > x$. Then x replaces y in this row and y is bumped into the next row where the process is repeated. The procedure terminates when the element which is bumped is at least as large as the last element of the next row. Then it is placed at the right of that row. For instance, the four steps to compute $S_r\left(\begin{smallmatrix}1&3&5\\2&4\\6\end{smallmatrix}, 2\right)$ are:



The *left (or column) insertion algorithm* $S_l : \mathrm{Yt}_n \times [n] \longrightarrow \mathrm{Yt}_n$ computes a tableau $S_l(t, x)$ as follows. If x is larger than the first element of the first (leftmost) column of t, then put x to the bottom of this column. Otherwise, let y be the smallest element of the first column of t such that $y \geqslant x$. Then x replaces y in this column and y is bumped into the next column where the process is repeated. The procedure terminates when the element which is bumped is greater than all the elements of the next column. Then it is placed at the bottom of that column. For instance, the four steps to compute $S_l\Big(\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2} 6\\\cline{1-1}\end{array}, 2\Big)$ are:



B Proof of Theorem 1

Consider a right SDS $\mathbb{S} = (\mathrm{D}_A, \ell_{\mathrm{l}}, I, R)$. Let $\mathbb{T} = (\mathrm{D}_A, \ell_{\mathrm{r}}, J, R)$ be an opposite SDS of $\mathbb{S}$. One shows that for any d, d' and d'' in D_A the following equalities hold

$$C_{\mathbb{S}}(R(d)R(d')R(d'')) = (d \star_{\mathbb{S}} d') \star_{\mathbb{S}} d'' \quad \text{and} \quad C_{\mathbb{T}}(R(d)R(d')R(d'')) = (d'' \star_{\mathbb{T}} d') \star_{\mathbb{T}} d. \tag{1}$$

Prove first that the equality $C_{\mathbb{S}}(w) = C_{\mathbb{T}}(w)$ holds for any w in A^*. We proceed by induction. By definition, we have $C_{\mathbb{S}}(x) = C_{\mathbb{T}}(x)$, for any x in A. Suppose that $C_{\mathbb{S}}(x_1 \dots x_k) = C_{\mathbb{T}}(x_1 \dots x_k)$, for any $x_1 \dots x_k$ in A^*. Then we obtain

$$\begin{aligned} C_{\mathbb{S}}(x_1 \dots x_k x_{k+1}) &= I(C_{\mathbb{S}}(x_1 \dots x_k), x_{k+1}) \\ &= I(C_{\mathbb{T}}(x_1 \dots x_k), x_{k+1}) \\ &= I(J(C_{\mathbb{T}}(x_2 \dots x_k), x_1), x_{k+1}) \\ &= J(I(C_{\mathbb{T}}(x_2 \dots x_k), x_{k+1}), x_1) \\ &= J(I(C_{\mathbb{S}}(x_2 \dots x_k), x_{k+1}), x_1) \\ &= J(C_{\mathbb{S}}(x_2 \dots x_k x_{k+1}), x_1) \\ &= J(C_{\mathbb{T}}(x_2 \dots x_k x_{k+1}), x_1) \\ &= C_{\mathbb{T}}(x_1 x_2 \dots x_k x_{k+1}). \end{aligned}$$

In particular, we have $C_{\mathbb{S}}(R(d)R(d')) = C_{\mathbb{T}}(R(d)R(d'))$, for any $d, d' \in \mathrm{D}_A$. Moreover, we have $C_{\mathbb{S}}(R(d)R(d')) \overset{(1)}{=} d \star_{\mathbb{S}} d'$ and $C_{\mathbb{T}}(R(d)R(d')) \overset{(1)}{=} d' \star_{\mathbb{T}} d$, for any $d, d' \in \mathrm{D}_A$. Then we deduce that $\mathbb{S}$ and $\mathbb{T}$ commute.

For any $d, d', d'' \in \mathrm{D}_A$, the equality $C_{\mathbb{S}}(R(d)R(d')R(d'')) = C_{\mathbb{T}}(R(d)R(d')R(d''))$ holds, and we have $C_{\mathbb{S}}(R(d)R(d')R(d'')) \overset{(1)}{=} (d \star_{\mathbb{S}} d') \star_{\mathbb{S}} d''$ and $C_{\mathbb{T}}(R(d)R(d')R(d'')) \overset{(1)}{=} (d'' \star_{\mathbb{T}} d') \star_{\mathbb{T}} d$. Then, the equality $(d \star_{\mathbb{S}} d') \star_{\mathbb{S}} d'' = (d'' \star_{\mathbb{T}} d') \star_{\mathbb{T}} d$ holds for any $d, d', d'' \in \mathrm{D}_A$. Since $\mathbb{S}$ and $\mathbb{T}$ commute, we obtain

$$(d'' \star_{\mathbb{T}} d') \star_{\mathbb{T}} d = d \star_{\mathbb{S}} (d'' \star_{\mathbb{T}} d') = d \star_{\mathbb{S}} (d' \star_{\mathbb{S}} d'').$$

Thus, we obtain $(d \star_{\mathbb{S}} d') \star_{\mathbb{S}} d'' = d \star_{\mathbb{S}} (d' \star_{\mathbb{S}} d'')$, for any d, d', d'' in D_A. Hence, the SDS $\mathbb{S}$ is associative. Similarly, one proves that if there is a right SDS opposite to a left SDS $\mathbb{S}$, then the SDS $\mathbb{S}$ is associative.

C Proof of Theorem 2

Consider a right associative SDS $\mathbb{S} = (\mathrm{D}_A, \ell_{\mathrm{l}}, I, R_{\mathbb{S}})$. One shows that for any d in D_A and $x_1 \dots x_p$ in A^* the following equality holds

$$I^*(d, \ell_{\mathrm{l}}(x_1 \dots x_p)) = I^*(I^*(d, x_1 \dots x_k), x_{k+1} \dots x_p). \tag{2}$$

To prove Theorem 2 it is sufficient to show that $\mathbb{S}$ is compatible with the equivalence relation $\sim_{\mathbb{S}}$ induced by $\mathcal{R}(A,\mathbb{S})$. Let us show that $R_{\mathbb{S}}C_{\mathbb{S}}(w) \sim_{\mathbb{S}} w$, for any $w \in A^*$. Every $w = x_1 \dots x_p$ in A^* can be written $w = R_{\mathbb{S}}(\iota_{\mathbb{S}}(x_1)) \dots R_{\mathbb{S}}(\iota_{\mathbb{S}}(x_p))$, where $\iota_{\mathbb{S}}$ denotes the inclusion map of A into $R_{\mathbb{S}}(\mathrm{D}_A)$. Since $\mathbb{S}$ is associative and $\mathcal{R}(A,\mathbb{S})$ is terminating, $\mathcal{R}(A,\mathbb{S})$ is convergent. Then the application of the rewriting rules of $\mathcal{R}(A,\mathbb{S})$ on w yield to the normal form $R_{\mathbb{S}}(\iota_{\mathbb{S}}(x_1)\star_{\mathbb{S}}\dots\star_{\mathbb{S}}\iota_{\mathbb{S}}(x_p))$ which is equal to $R_{\mathbb{S}}(C_{\mathbb{S}}(w))$. Hence, we obtain $R_{\mathbb{S}}C_{\mathbb{S}}(w) \sim_{\mathbb{S}} w$.

Suppose that for w and w' in A^* we have $w \sim_{\mathbb{S}} w'$. Let us show that, for any d in D_A, we have $I^*(d,w) = I^*(d,w')$. Note that for any w in A^* and d in D_A, the following equality holds

$$C_{\mathbb{S}}(R_{\mathbb{S}}(d)w) = I^*(\emptyset, \ell_1(R_{\mathbb{S}}(d)w)) = I^*(\emptyset, \ell_1(R_{\mathbb{S}}(d))\,\ell_1(w)) \overset{(2)}{=} I^*(I^*(\emptyset, \ell_1(R_{\mathbb{S}}(d)), \ell_1(w)) = I^*(d,w).$$

Since $w \sim_{\mathbb{S}} w'$, we have $R_{\mathbb{S}}(d)w \sim_{\mathbb{S}} R_{\mathbb{S}}(d)w'$. Then by the unique normal form property of the SRS $\mathcal{R}(A,\mathbb{S})$, we have $R_{\mathbb{S}}(C_{\mathbb{S}}(R_{\mathbb{S}}(d)w)) = R_{\mathbb{S}}(C_{\mathbb{S}}(R_{\mathbb{S}}(d)w'))$, for any d in D_A. Since $R_{\mathbb{S}}$ is injective, we obtain $C_{\mathbb{S}}(R_{\mathbb{S}}(d)w) = C_{\mathbb{S}}(R_{\mathbb{S}}(d)w')$. Hence, for any d in D_A, we have $I^*(d,w) = I^*(d,w')$. As a consequence, we obtain that the SDS $\mathbb{S}$ is compatible with the equivalence relation $\sim_{\mathbb{S}}$.

D Coherent presentations of monoids

We recall from [5] the notion of coherent presentation of monoids. Let R be an SRS on an alphabet X. For every rewriting rule β of R we will denote respectively by $s_1(\beta)$ and $t_1(\beta)$ the source and the target of β. We will denote by $R^\top$ the $(2,1)$-category freely generated by the SRS R, that is the free 2-category enriched in groupoid generated by the set of rules R, see [6]. The 2-cells of the $(2,1)$-category $R^\top$ corresponds to elements of the equivalence relation generated by R. A 2-*sphere* of $R^\top$ is a pair (f,g) of 2-cells in $R^\top$ such that $s_1(f) = s_1(g)$ and $t_1(f) = t_1(g)$.

An *extended presentation* of a monoid $\mathbf{M}$ is an SRS R presenting $\mathbf{M}$ extended by a globular extension Γ of the $(2,1)$-category $R^\top$, that is a set of *homotopy generators* $A : f \Rrightarrow g$ relating parallel 2-cells f and g in $R^\top$, respectively denoted by $s_2(A)$ and $t_2(A)$ and satisfying the globular relations $s_1s_2(A) = s_1t_2(A)$ and $t_1s_2(A) = t_1t_2(A)$. We will denote by $\Gamma^\top$ the free $(3,1)$-category generated by such an extended presentation. A *coherent presentation of a monoid* $\mathbf{M}$ is an extended presentation (R,Γ) of $\mathbf{M}$ such that the cellular extension Γ is a *homotopy basis* of the $(2,1)$-category $R^\top$, that is, for every 2-sphere γ of $R^\top$, there exists a homotopy generator in $R^\top$ with boundary γ.

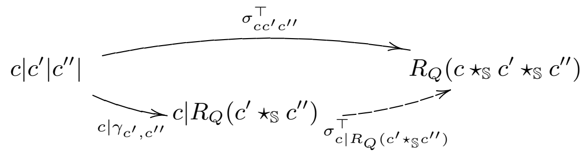
E Proof of Theorem 3

Let $\mathbb{S} = (\mathrm{D}_A, \ell, I, R_{\mathbb{S}})$ be an associative SDS and Q be a generating set of $\mathbb{S}$ such that the SRS $\mathcal{R}(Q,\mathrm{D}_A,\mathbb{S})$ is terminating. Let σ be a normalization strategy of $\mathcal{R}(Q,\mathrm{D}_A,\mathbb{S})$ that computes $C_{\mathbb{S}}$. Let us show that the SRS $\mathcal{R}(Q,\mathrm{D}_A,\mathbb{S})$ is convergent. Any critical pair of $\mathcal{R}(Q,\mathrm{D}_A,\mathbb{S})$ has the form $(\gamma_{c,c'}c'', c\gamma_{c',c''})$, for c,c',c'' in Q. Applying the normalisation σ, we have the following reduction diagram:

$$\begin{array}{ccccc} & \overset{\gamma_{c,c'}c''}{\nearrow} & R_Q(c\star_{\mathbb{S}} c')|c'' & \xrightarrow{\sigma_{R_Q(c\star_{\mathbb{S}}c')|c''}} & R_Q\big((c\star_{\mathbb{S}} c')\star_{\mathbb{S}} c''\big) \\ c|c'|c'' & & & & \\ & \underset{c\gamma_{c',c''}}{\searrow} & c|R_Q(c'\star_{\mathbb{S}} c'') & \xrightarrow{\sigma_{c|R_Q(c'\star_{\mathbb{S}}c'')}} & R_Q\big(c\star_{\mathbb{S}}(c'\star_{\mathbb{S}} c'')\big) \end{array}$$

which is confluent by the associativity of $\star_{\mathbb{S}}$. Hence, the SRS $\mathcal{R}(Q, \mathrm{D}_A, \mathbb{S})$ is convergent by termination hypothesis. Moreover, by Lemma 1, the SRSs $\mathcal{R}(\mathbb{S})$ and $\mathcal{R}(Q, \mathrm{D}_A, \mathbb{S})$ are Tietze-equivalent. Then the SRS $\mathcal{R}(Q, \mathrm{D}_A, \mathbb{S})$ is a presentation of the structure monoid $\mathbf{M}(\mathbb{S})$ and thus the set $\mathrm{Nf}(Q, \mathbb{S})$ satisfies the cross-section property for $\mathbf{M}(\mathbb{S})$.

In particular, if the leftmost normalization strategy $\sigma^\top$ computes $C_{\mathbb{S}}$, then by [14] the SRS $\mathcal{R}(Q, \mathrm{D}_A, \mathbb{S})$ can be extended into a coherent convergent presentation by adjunction of



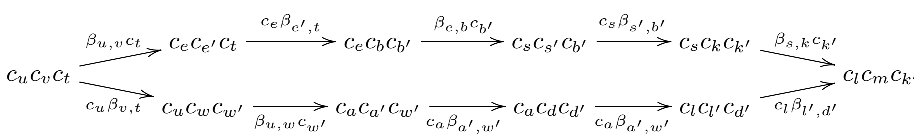
for every c, c', c'' in Q.

F Chinese coherent presentations

The *reduced presentation* of the Chinese monoid is the SRS whose set of generators is

$$Q_n = \{c_{yx} \mid 1 \leqslant x < y \leqslant n\} \cup \{c_{xx} \mid 1 < x < n\} \cup \{c_1, \ldots, c_n\},$$

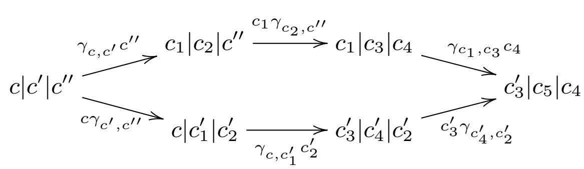
where $c_1, \ldots, c_n$ represent the initial generators $1, \ldots, n$, and whose rewriting rules are of the form $\gamma_{u,v} : c_u c_v \to c_w c_{w'}$, where $c_w c_{w'}$ is obtained by inserting c_v into c_u using the right insertion defined in [3]. We show that this presentation is a finite convergent presentation of the Chinese monoid and it can be extended into a coherent presentation by adjunction of



where the rewriting rules $\beta_{-,-}$ denote either a rewriting rule of the reduced presentation or an identity.

G Plactic coherent presentations

As an application of Theorem 3, consider the SDSs $\mathcal{Y}_n^{col}$ and $\mathcal{Y}_n^{row}$. By definition, the leftmost normalisation strategy $\sigma^\top$ with respect to $\mathcal{R}(\mathrm{Col}_n, \mathrm{Yt}_n, \mathcal{Y}_n^{col})$ computes $C_{\mathcal{Y}_n^{col}}$. Then the SRS $\mathcal{R}(\mathrm{Col}_n, \mathrm{Yt}_n, \mathcal{Y}_n^{col})$ is extended into a finite coherent presentation by adjunction of, [7]:



where $c|c', c'|c'' \notin \mathrm{Yt}_n$, $R_{\mathrm{Col}_n}(c \star_{\mathcal{Y}_n^{col}} c') = c_1|c_2$, $R_{\mathrm{Col}_n}(c_2 \star_{\mathcal{Y}_n^{col}} c'') = c_3|c_4$, $R_{\mathrm{Col}_n}(c_1 \star_{\mathcal{Y}_n^{col}} c_3) = c'_3|c_5$, $R_{\mathrm{Col}_n}(c'' \star_{\mathcal{Y}_n^{row}} c') = c'_1|c'_2$, $R_{\mathrm{Col}_n}(c'_1 \star_{\mathcal{Y}_n^{row}} c) = c'_3|c'_4$ and $R_{\mathrm{Col}_n}(c'_2 \star_{\mathcal{Y}_n^{row}} c'_4) = c_5|c_4$.